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FRACTIONAL AND OTHER DERIVATIVES IN UNIVALENT FUNCTION THEORY

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Abstract

A considerably large variety of linear operators (such as the familiar operators of fractional derivatives, the Ruscheweyh derivative, the Sălăgean derivative, and so on) can be found to have been applied rather frequently in the theory of analytic and univalent functions. The main purpose of this lecture is to present several instances of usefulness of some of the aforementioned derivative operators in univalent function theory.

1. Introduction and Definitions

Let $\mathcal{A}(p,k)$ denote the class of functions f *normalized* by

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k;\ p,k \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \mathcal{U}(1),$$

where, for *latter* convenience,

$$\mathcal{U}(r) := \{z : z \in \mathbb{C} \ \text{ and } \ |z| < r \ \ (r > 0)\}. \tag{1.2}$$

(See, for details, [8], [11], and [33].) Also let

$$\mathcal{A}(p) := \mathcal{A}(p,p+1), \ \mathcal{A} := \mathcal{A}(1), \ \text{ and } \ \mathcal{A}_k := \mathcal{A}(1,k+1). \tag{1.3}$$

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For analytic functions f and g given by

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \quad\text{and}\quad g(z)=\sum_{n=0}^{\infty}b_nz^n, \tag{1.4}$$

we denote by $f*g$ the *Hadamard product* (or *convolution*) of f and g, defined (as usual) by

$$(f*g)(z):=\sum_{n=0}^{\infty}a_nb_nz^n=:(g*f)(z). \tag{1.5}$$

For $\alpha_j\in\mathbb{C}$ $(j=1,\dots,l)$ and

$$\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\ \left(j=1,\dots,m;\ \mathbb{Z}_0^-:=\{0,-1,-2,\dots\}\right),$$

the *generalized hypergeometric function*

$${}_lF_m(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m;\ z)$$

(*with* l numerator and m denominator parameters) is defined here by the infinite series:

$$\begin{aligned}{}_lF_m(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m;\ z)\\ :=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{z^n}{n!}\\ \left(l\leqq m+1;\ l,m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ z\in\mathcal{U}\right),\end{aligned} \tag{1.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the familiar Gamma functions, by

$$(\lambda)_n:=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}).\end{cases} \tag{1.7}$$

Corresponding to a function

$$\begin{aligned}h_p(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m;\ z)\\ :=z^p\,{}_lF_m(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m;\ z),\end{aligned} \tag{1.8}$$

we first consider here a linear operator

$$H_p(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m):\mathcal{A}(p)\to\mathcal{A}(p),$$

which is defined by the Hadamard product (or convolution) (see, for details, Dziok and Srivastava [9, p. 3 *et seq.*]):

$$\begin{aligned} &H_p^{(l,m)}\left(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m\right)f(z) \\ &\qquad := h_p\left(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m;\ z\right) * f(z)\,. \end{aligned} \tag{1.9}$$

Thus, for a function f of the form (1.1), it is easily observed that

$$\begin{aligned} &H_p^{(l,m)}\left(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m\right)f(z) \\ &\qquad = z^p + \sum_{n=k}^{\infty} \Gamma_n\, a_n z^n, \end{aligned} \tag{1.10}$$

where, for convenience,

$$\Gamma_n := \frac{(\alpha_1)_{n-p}\cdots(\alpha_l)_{n-p}}{(n-p)!\,(\beta_1)_{n-p}\cdots(\beta_m)_{n-p}}. \tag{1.11}$$

Furthermore, after some calculations, we find from the definition (1.9) that

$$\begin{aligned} &\alpha_1 H_p^{(l,m)}\left(\alpha_1+1,\alpha_2,\dots,\alpha_l;\ \beta_1,\dots,\beta_m\right)f(z) \\ &\qquad = z\frac{d}{dz}\left\{H_p^{(l,m)}\left(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m\right)f(z)\right\} \\ &\qquad\quad + (\alpha_1-p)\,H_p^{(l,m)}\left(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m\right)f(z)\,. \end{aligned} \tag{1.12}$$

The linear (convolution) operator

$$H_p^{(l,m)}\left(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m\right)$$

includes, as its special cases, various other linear operators which were considered in many earlier works on the subject of analytic and univalent functions. Some of these special cases are being presented here.

I. The linear operator $\mathcal{F}(\alpha,\beta,\gamma)$:

$$\mathcal{F}(\alpha,\beta,\gamma)\,f(z) = H_1^{(2,1)}(\alpha,\beta;\gamma)\,f(z)\,, \tag{1.13}$$

which was considered by Hohlov [12].

II. The linear operator $\mathcal{L}(\alpha,\gamma)$:

$$\mathcal{L}(\alpha,\gamma)\,f(z) = H_1^{(2,1)}(\alpha,1;\gamma)\,f(z) = \mathcal{F}(\alpha,1;\gamma)\,f(z)\,, \tag{1.14}$$

which was considered by Carlson and Shaffer [5].

III. The Ruscheweyh derivative operator $\mathfrak{D}^{\lambda}$: $\mathcal{A} \to \mathcal{A}$, defined by the Hadamard product (or convolution) (*cf.* [23]):

$$\mathfrak{D}^{\lambda} f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) = H_1^{(2,1)}(\lambda+1, 1; 1) f(z) \tag{1.15}$$
$$(\lambda \geqq -1;\ f \in \mathcal{A}),$$

which readily implies that

$$\mathfrak{D}^{n} f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} = H_1^{(2,1)}(n+1, 1; 1) f(z) \tag{1.16}$$
$$(n \in \mathbb{N}_0;\ f \in \mathcal{A}).$$

IV. The generalized Bernardi-Libera-Livingston linear integral operator $\mathcal{J}_{\nu} : \mathcal{A} \to \mathcal{A}$, defined by (*cf.* [4], [16], and [17])

$$\mathcal{J}_{\nu} f(z) := \frac{\nu+1}{z^{\nu}} \int_0^z t^{\nu-1} f(t)\, dt = H_1^{(2,1)}(\nu+1, 1; \nu+2) f(z) \tag{1.17}$$
$$(\nu > -1;\ f \in \mathcal{A}).$$

V. The Srivastava-Owa fractional derivative operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$, defined by (*cf.*, *e.g.*, [31]; see also [28], [29], and [30])

$$\begin{aligned} \Omega^{\lambda} f(z) &:= \Gamma(2-\lambda)\, z^{\lambda} D_z^{\lambda} f(z) = H_1^{(2,1)}(2, 1; 2-\lambda) f(z) \\ &= \mathcal{L}(2, 2-\lambda) f(z) \\ &(\lambda \notin \mathbb{N} \setminus \{1\};\ f \in \mathcal{A}), \end{aligned} \tag{1.18}$$

where $D_z^{\lambda} f(z)$ denotes the fractional derivative of $f(z)$ of order λ, which is defined as follows (see, for example, [18] and [32]).

Definition 1. The *fractional integral of order* λ is defined, for a function f, by

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\, d\zeta \quad (\lambda < 0), \tag{1.19}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin $(z = 0)$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The *fractional derivative of order* λ is defined, for a function f, by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} \, d\zeta \quad (0 \leqq \lambda < 1), \tag{1.20}$$

where f is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1 above.

Definition 3. Under the hypotheses of Definition 2, the *fractional derivative of order* $n+\lambda$ is defined, for a function f, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leqq \lambda < 1;\ n \in \mathbb{N}_0). \tag{1.21}$$

Yet another useful derivative operator, which we shall require in our presentation here, is the Sălăgean derivative operator $\mathcal{D}^n$ of order n, which is defined by ($cf.$ [25])

$$\mathcal{D}^0 f(z) := f(z) \quad (z \in \mathcal{U};\ f \in \mathcal{A}), \tag{1.22}$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) := z f'(z) \qquad (z \in \mathcal{U};\ f \in \mathcal{A}), \tag{1.23}$$

and (in general)

$$\mathcal{D}^n f(z) := \mathcal{D}\left(\mathcal{D}^{n-1} f(z)\right) \qquad (z \in \mathcal{U};\ n \in \mathbb{N};\ f \in \mathcal{A}). \tag{1.24}$$

2. Applications Involving Subclasses of Analytic and Multivalent Functions

Various applications of several special cases of the convolution operator [*cf.* Equation (1.9)]:

$$H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l;\ \beta_1, \ldots, \beta_m)$$

in the study of many interesting subclasses of the class $\mathcal{A}(p,k)$, introduced in Section 1, can be found to be scattered throughout the literature on *Geometric Function Theory*. The recent works of (among others) Saitoh [24], Owa *et al.* [19], Chen *et al.* [7], Fukui *et al.* [10], Li *et al.* [15], and Srivastava *et al.* (*cf.*, *e.g.*, [28], [29], [30], and [31]) may be cited in this connection. In particular, motivated essentially by the work of Kim and Srivastava [14], Dziok and Srivastava [9] introduced and studied systematically a class $\mathcal{V}_k^p(l,m;\ A,B)$ of functions f of the form [*cf.* Equation (1.1)]:

$$f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n \tag{2.1}$$

$$(p < k;\ p,k \in \mathbb{N};\ a_n \geqq 0;\ n = k, k+1, k+2, \dots),$$

which also satisfy the following condition:

$$\alpha_1 \frac{H_p^{(l,m)}(\alpha_1+1,\alpha_2,\dots,\alpha_l;\ \beta_1,\dots,\beta_m)\,f(z)}{H_p^{(l,m)}(\alpha_1,\dots,\alpha_l;\ \beta_1,\dots,\beta_m)\,f(z)} + p - \alpha_1$$
$$\prec p\,\frac{1+Az}{1+Bz} \qquad (-B \leqq A < B;\ 0 \leqq B \leqq -1) \tag{2.2}$$

in terms of *subordination* between analytic functions.

From among many interesting properties and characteristics of the general class $\mathcal{V}_k^p(l,m;\ A,B)$, we choose to recall here the following results (see, for details, [9]).

Theorem 1. *A function f of the form* (2.1) *belongs to the class* $\mathcal{V}_k^p(l,m;\ A,B)$ *if and only if*

$$\sum_{n=k}^{\infty} C_n\, a_n \leqq M \tag{2.3}$$

$$(C_n := \{(B+1)\,n - (A+1)\,p\}\,\Gamma_n;\ M := (B-A)\,p),$$

where Γ_n is defined by (1.11).

Theorem 2. *Let a function f of the form* (2.1) *belong to the class* $\mathcal{V}_k^p(l,m;\ A,B)$. *If the sequence $\{C_n\}$ is nondecreasing, then*

$$r^p - \frac{M}{C_k} \leqq |f(z)| \leqq r^p + \frac{M}{C_k} r^k \qquad (r := |z|;\ z \in \mathcal{U}) \tag{2.4}$$

Furthermore, if the sequence $\left\{\frac{C_n}{n}\right\}$ is nondecreasing, then

$$p r^{p-1} - \frac{kM}{C_k} r^{k-1} \leqq |f'(z)| \leqq \frac{kM}{C_k} r^{k-1} \qquad (r := |z|;\ z \in \mathcal{U}), \tag{2.5}$$

where C_n and M are defined with (2.3). *Each of these results is sharp, with the extremal function f_k given by*

$$f_k(z) = z^p - \frac{M}{C_k} z^k \qquad (z \in \mathcal{U}). \tag{2.6}$$

Theorem 3. *Let C_n and M be defined with* (2.3) *and let us put*

$$f_{k-1}(z) = z^p \quad \text{and} \quad f_n(z) = z^p - \frac{M}{C_n} z^n \quad (n = k, k+1, k+2, \dots). \tag{2.7}$$

Then a function f belongs to the class $\mathcal{V}_k^p(l, m;\ A, B)$ if and only if it is of the form:

$$f(z) = \sum_{n=k-1}^{\infty} \gamma_n f_n(z) \quad (z \in \mathcal{U}), \tag{2.8}$$

where

$$\sum_{n=k-1}^{\infty} \gamma_n = 1 \quad (\gamma_n \geqq 0;\ n = k-1, k, k+1, \dots). \tag{2.9}$$

Theorem 4. *The radii of starlikeness and convexity for the class $\mathcal{V}_k^p(l, m;\ A, B)$ are given by*

$$\inf_{n \geqq k} \left(\frac{p}{n} \frac{C_n}{M} \right)^{1/(n-p)} \quad \text{and} \quad \inf_{n \geqq k} \left(\frac{p^2}{n^2} \frac{C_n}{M} \right)^{1/(n-p)},$$

respectively, where C_n and M are defined with (2.3). *The result is sharp.*

3. Univalence Criteria Involving Ruscheweyh and Sălăgean Derivatives

Making use of some known results due to Pommerenke [21] involving the *Löwner chain*:

$$L(z, t) = A_1(t) z + A_2(t) z^2 + A_3(t) z^3 + \cdots \quad (A_1(t) \neq 0) \tag{3.1}$$

and the *Löwner differential equation*:

$$\frac{\partial L(z,t)}{\partial t} = z \, \frac{\partial L(z,t)}{\partial z} \, \phi(z, t), \tag{3.2}$$

where $\phi(z, t)$ is a function regular in $\mathcal{U}$ for each $t \in [0, \infty)$ such that

$$\Re(\phi(z, t)) > 0 \quad (z \in \mathcal{U};\ 0 \leqq t < \infty),$$

Kanas and Srivastava [13] gave several criteria for univalence involving the Ruscheweyh derivative operator $\mathfrak{D}^\lambda$ defined by (1.15) *and* the Sălăgean derivative operator $\mathcal{D}^n$ defined by (1.22), (1.23), and (1.24). Some of these univalence criteria are presented here in Theorem 5 and Theorem 6 below.

Theorem 5. *Let α be a complex number such that $|\alpha| \leqq 1$ $(\alpha \neq -1)$, and suppose that $f \in \mathcal{A}$. If each of the inequalities*:

$$\left| \frac{f'(z)}{[\mathfrak{D}^n f(z)]'} - \frac{1}{1+\alpha} \right| \leqq \frac{1}{|1+\alpha|} \tag{3.3}$$

and

$$\left| |z|^2 \left((1+\alpha) \frac{f'(z)}{[\mathfrak{D}^n f(z)]'} - 1 \right) + \left(1 - |z|^2\right) \left(\frac{z\,[\mathfrak{D}^n f(z)]''}{[\mathfrak{D}^n f(z)]'} \right) \right| \leqq 1 \tag{3.4}$$

holds true for $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Theorem 6. *Let α be a complex number such that $|\alpha| \leqq 1$ $(\alpha \neq -1)$, and suppose that $f \in \mathcal{A}$. If each of the inequalities*:

$$\left| \frac{f'(z)}{[\mathcal{D}^n f(z)]'} - \frac{1}{1+\alpha} \right| \leqq \frac{1}{|1+\alpha|} \tag{3.5}$$

and

$$\left| |z|^2 \left((1+\alpha) \frac{f'(z)}{[\mathcal{D}^n f(z)]'} - 1 \right) + \left(1 - |z|^2\right) \left(\frac{z\,[\mathcal{D}^n f(z)]''}{[\mathcal{D}^n f(z)]'} \right) \right| \leqq 1 \tag{3.6}$$

holds true for $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Each of these results (Theorem 5 and Theorem 6) would simplify considerably when we set $n = 1$ (*cf.* Kanas and Srivastava [13, p. 268, Corollary 2.2]). Furthermore, in view of the relationships exhibited by (1.16) and (1.22), a familiar univalence criterion due to Lars Valerian Ahlfors (1907-1996) [1] follows immediately from Theorem 5 *as well as* Theorem 6 in their special case when $n = 0$.

4. Analytic Function Classes Using the Sălăgean Derivative

For a function $f \in \mathcal{A}_k$ given by (1.1) *with* (of course) $p = 1$ and k replaced by $k+1$, it follows from the definition in (1.22), (1.23), and (1.24) that

$$\mathcal{D}^n f(z) = z + \sum_{j=k+1}^{\infty} j^n a_j z^j \qquad (n \in \mathbb{N}_0). \tag{4.1}$$

With the help of the Sălăgean derivative operator $\mathcal{D}^n$, we say that a function $f \in \mathcal{A}_k$ is in the class $\mathcal{A}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ if and only if

$$\left|\frac{F_{n,\lambda}(z)-1}{\gamma F_{n,\gamma}(z)+1-(1+\gamma)\alpha}\right|<\beta \tag{4.2}$$

$$(z \in \mathcal{U};\ n \in \mathbb{N}_0;\ 0 \leqq \lambda \leqq 1;\ 0 \leqq \alpha < 1;\ 0 < \beta \leqq 1;\ 0 \leqq \gamma \leqq 1),$$

where, for convenience,

$$F_{n,\lambda}(z) := \frac{(1-\lambda)\,z\,[\mathcal{D}^n f(z)]' + \lambda z\,[\mathcal{D}^{n+1} f(z)]'}{(1-\lambda)\,\mathcal{D}^n f(z) + \lambda \mathcal{D}^{n+1} f(z)} =: \frac{\phi_{n,\lambda}(z)}{\psi_{n,\lambda}(z)}.$$

Let $\mathcal{T}_k$ denote the subclass of $\mathcal{A}_k$ consisting of functions of the form [*cf.* Equation (2.1)]:

$$f(z) = z - \sum_{j=k+1}^{\infty} a_j z^j \tag{4.3}$$

$$(a_j \geqq 0;\ j = k+1, k+2, k+3, \ldots;\ k \in \mathbb{N})$$

and define the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ by

$$\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) = \mathcal{A}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \cap \mathcal{T}_k.$$

We note that, by specializing the parameters k, λ, α, β, γ, and n, we can obtain the following subclasses studied by various authors.

(i) $\mathcal{T}_{0,k}^{\lambda}(\alpha,1,1) = \mathcal{P}(k,\lambda,\alpha)$ (Altintaş [2])

(ii) $\mathcal{T}_{0,1}^{0}(\alpha,1,1) = \mathcal{T}^*(\alpha)$ and $\mathcal{T}_{0,1}^{1}(\alpha,1,1) = \mathcal{T}_{1,1}^{0}(\alpha,1,1) = \mathcal{C}(\alpha)$ (Silverman [27])

(iii) $\mathcal{T}_{0,k}^{0}(\alpha,1,1) = \mathcal{T}_{\alpha}(k)$ and $\mathcal{T}_{0,k}^{1}(\alpha,1,1) = \mathcal{T}_{1,k}^{0}(\alpha,1,1) = \mathcal{C}_{\alpha}(k)$ (Chatterjea [6] and Srivastava *et al.* [34])

(iv) $\mathcal{T}_{n,k}^{\lambda}(\alpha,1,1) = \mathcal{P}(k,\lambda,\alpha,n)$ (Aouf and Srivastava [3]),

where $\mathcal{P}(k,\lambda,\alpha,n)$ represents the class of functions $f \in \mathcal{A}_k$ which satisfy the inequality [3, p. 763, Equation (1.5)]:

$$\Re\left(\frac{(1-\lambda)\,z\,[\mathcal{D}^n f(z)]' + \lambda z\,[\mathcal{D}^{n+1} f(z)]'}{(1-\lambda)\,\mathcal{D}^n f(z) + \lambda \mathcal{D}^{n+1} f(z)}\right) > \alpha$$

$$(z \in \mathcal{U};\ n \in \mathbb{N}_0;\ 0 \leqq \lambda \leqq 1;\ 0 \leqq \alpha < 1).$$

For the general analytic function class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ defined by (4.4), we now present several coefficient (and distortion) inequalities and many other basic properties (and characteristics), which were proven recently by Srivastava *et al.* [35].

Theorem 7. *Let the function f be defined by* (4.3). *Then $f \in \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{j=k+1}^{\infty} j^n (1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}\, a_j \leqq \beta(1+\gamma)(1-\alpha). \tag{4.4}$$

The result is sharp, the extremal function being given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}} z^j \tag{4.5}$$
$$(j \geqq k+1;\; k \in \mathbb{N}).$$

Corollary 1. *Let the function f defined by* (4.3) *be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. Then*

$$a_j \leqq \frac{\beta(1+\gamma)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}} \tag{4.6}$$
$$(j \geqq k+1;\; k \in \mathbb{N}).$$

The equality in (4.6) *is attained for the function f given by* (4.5).

Remark 1. Since

$$1-\lambda+\lambda j \leqq 1-\mu+\mu j \quad (j \geqq k+1;\; k \in \mathbb{N};\; 0 \leqq \lambda \leqq \mu \leqq 1),$$

we have the inclusion property:

$$\mathcal{T}_{n,k}^{\mu}(\alpha,\beta,\gamma) \subseteq \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \quad (0 \leqq \lambda \leqq \mu \leqq 1). \tag{4.7}$$

Furthermore, for $0 \leqq \alpha_1 \leqq a_2 < 1$, it is easily verified that

$$\frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha_1)}{1-\alpha_1} \leqq \frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha_2)}{1-\alpha_2},$$

so that, with the aid of Theorem 7, we obtain the inclusion property:

$$\mathcal{T}_{n,k}^{\lambda}(\alpha_2,\beta,\gamma) \subseteq \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \quad (0 \leqq \alpha_1 \leqq \alpha_2 < 1). \tag{4.8}$$

Theorem 8. *For each $n \in \mathbb{N}_0$,*

$$\mathcal{T}_{n+1,k}^{\lambda}(\alpha,\beta,\gamma) \subset \mathcal{T}_{n,k}^{\lambda}(\xi,\beta,\gamma), \tag{4.9}$$

where

$$\xi := \frac{(1+\beta\gamma)(k+\alpha)+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(k+1)+\beta(1+\gamma)(1-\alpha)}. \tag{4.10}$$

The result is sharp, the extremal function being given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}} z^{k+1}. \tag{4.11}$$

Remark 2. Since $\xi > \alpha$, it follows from Remark 1 that

$$\mathcal{T}_{n,k}^{\lambda}(\xi,\beta,\gamma) \subset \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \quad (n \in \mathbb{N}_0)$$

and hence that

$$\mathcal{T}_{n+1,k}^{\lambda}(\alpha,\beta,\gamma) \subset \mathcal{T}_{n,k}^{\lambda}(\xi,\beta,\gamma) \subset \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \quad (n \in \mathbb{N}_0),$$

where ξ is defined by (4.10).

Theorem 9. *Let* $0 \leqq \alpha_j < 1$ $(j=1,2)$ *and* $0 < \beta_j \leqq 1$ $(j=1,2)$. *Then*

$$\mathcal{T}_{n,k}^{\lambda}(\alpha_1,\beta_1,1) = \mathcal{T}_{n,k}^{\lambda}(\alpha_2,\beta_2,1) \quad (n \in \mathbb{N}_0) \tag{4.12}$$

if and only if

$$\frac{\beta_1(1-\alpha_1)}{1+\beta_1} = \frac{\beta_2(1-\alpha_2)}{1+\beta_2}. \tag{4.13}$$

In particular, if $0 \leqq \alpha < 1$ *and* $0 < \beta \leqq 1$, *then*

$$\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,1) = \mathcal{T}_{n,k}^{\lambda}\left(\frac{1-\beta+2\alpha\beta}{1+\beta},1,1\right) = \mathcal{P}\left(k,\lambda,\frac{1-\beta+2\alpha\beta}{1+\beta},n\right) \tag{4.14}$$

$$(n \in \mathbb{N}_0).$$

Theorem 10. *Let* $0 \leqq \alpha < 1$, $0 < \beta_j \leqq 1$, *and* $0 \leqq \gamma_j \leqq 1$ $(j=1,2)$. *Then*

$$\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta_1,\gamma_1) = \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta_2,\gamma_2) \quad (n \in \mathbb{N}_0) \tag{4.15}$$

if and only if

$$\frac{\beta_1(1+\gamma_1)}{1-\beta_1} = \frac{\beta_2(1+\gamma_2)}{1-\beta_2}.$$

In particular, if $0 < \beta \leqq 1$ *and* $0 \leqq \gamma \leqq 1$, *then*

$$\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) = \mathcal{T}_{n,k}^{\lambda}\left(\alpha,\frac{\beta(1+\gamma)}{2-\beta+\beta\gamma},1\right) \quad (n \in \mathbb{N}_0). \tag{4.16}$$

Let $f(z)$ be defined by (4.3) and let

$$g(z) = z - \sum_{j=k+1}^{\infty} b_j z^j$$

$$(b_j \geqq 0;\ j = k+1, k+2, k+3, \ldots\ ;\ k \in \mathbb{N}). \tag{4.17}$$

Then the *modified* Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined here by

$$(f \bullet g)(z) := z - \sum_{j=k+1}^{\infty} a_j b_j z^j \tag{4.18}$$

$$(a_j \geqq 0;\ b_j \geqq 0;\ j = k+1, k+2, k+3, \ldots\ ;\ k \in \mathbb{N}).$$

In terms of the *modified* Hadamard product (or convolution), by employing the technique used earlier by Schild and Silverman [26], we have

Theorem 11. *Let the function f defined by* (4.3) *and the function g defined by* (4.17) *belong to the class $\mathcal{T}_{n,k}^{\lambda}(\eta, \beta, \gamma)$. Then the modified Hadamard product $f \bullet g$ defined by* (4.18) *belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\eta, \beta, \gamma)$, where*

$$\eta := \frac{\begin{array}{c}(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}^2 \\ -\beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)k + \beta(1+\gamma)\}\end{array}}{(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}^2 - \{\beta(1+\gamma)(1-\alpha)\}^2}. \tag{4.19}$$

The result is sharp, the extremal function being given by

$$\begin{aligned} f(z) &= g(z) \\ &= z - \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}} z^{k+1} \qquad (k \in \mathbb{N}). \end{aligned} \tag{4.20}$$

Theorem 12. *If each of the functions f and g belongs to the same class $\mathcal{T}_{n,k}^{\lambda}(\alpha, \beta, \gamma)$, then $(f \bullet g)(z)$ belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\rho, 1, 1)$ or, equivalently,*

$\mathcal{P}(k,\lambda,n)$, *where*

$$\rho := \frac{(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 - (k+1)\{\beta(1+\gamma)(1-\alpha)\}^2}{(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 - \{\beta(1+\gamma)(1-\alpha)\}^2}.$$

The result is the best possible for the functions $f(z)$ *and* $g(z)$ *defined by* (4.20).

Theorem 13. *Let the function* f *defined by* (4.3) *and the function* g *defined by* (4.17) *be in the same class* $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. *Then the function* $h(z)$ *defined by*

$$h(z) := z - \sum_{j=k+1}^{\infty} \left(a_j^2 + b_j^2\right) z^j \tag{4.21}$$

belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\sigma,\beta,\gamma)$, *where*

$$\sigma := \frac{(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 - 2\beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)k+\beta(1+\gamma)\}}{(k+1)^n (1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 - 2\{\beta(1+\gamma)(1-\alpha)\}^2}.$$

The result is sharp for the functions $f(z)$ *and* $g(z)$ *defined by* (4.20).

Theorem 14. *Let the function* f *defined by* (4.3) *be in the class* $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, *and let* c *be a real number such that* $c > -1$. *Then the function* $F(z)$ *defined by* [*cf.* *Equation* (1.17)]

$$F(z) := \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \qquad (c > -1;\ f \in \mathcal{A}_k) \tag{4.22}$$

belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\kappa,\beta,\gamma)$, *where*

$$\kappa := \frac{(1+\beta\gamma)\{k+(c+1)\alpha\}+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(k+c+1)+\beta(1+\gamma)(1-\alpha)}.$$

The result is sharp for the function $f(z)$ *defined by* (4.11).

Theorem 15. *If* $f \in \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, *then the function* $F(z)$ *defined by* (4.22) *belongs to the class* $\mathcal{T}_{n,k}^{\lambda}(\mu,1,1)$ *or, equivalently,* $\mathcal{P}(k,\lambda,\mu,n)$, *where*

$$\mu := \frac{(1+\beta\gamma)(c+k+1) - c\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(c+k+1)+\beta(1+\gamma)(1-\alpha)}. \tag{4.23}$$

The result is sharp, the extremal function $f(z)$ *being given by* (4.11).

Theorem 16. *Let the function* $F(z)$ *given by*

$$F(z) = z - \sum_{j=k+1}^{\infty} d_j z^j \qquad (d_j \geqq 0;\ j = k+1, k+2, k+3, \ldots;\ k \in \mathbb{N}) \tag{4.24}$$

be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha, \beta, \gamma)$, *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *defined by* (4.22) *is univalent in* $|z| < R$, *where*

$$R := \inf_{j \geqq k+1} \left(\frac{j^{n-1}(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}(c+1)}{\beta(1+\gamma)(1-\alpha)(c+j)} \right)^{1/(j-1)}. \tag{4.25}$$

The result is sharp, the extremal function being given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)(c+j)}{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}(c+1)} z^j \tag{4.26}$$

$$(j \geqq k+1;\ k \in \mathbb{N}).$$

Each of the following distortion inequalities (Theorem 17, Theorem 18, Corollary 2, and Corollary 3) involves the fractional calculus operators which we introduced in Section 1 by means of Definitions 1, 2, and 3.

Theorem 17. *Let the function* f *defined by* (4.3) *be in the class* $\mathcal{T}_{n,k}^{\lambda}(\alpha, \beta, \gamma)$. *Then*

$$\left|D_z^{-\mu}\left(\mathcal{D}^i f(z)\right)\right| \geqq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \cdot \left(1 - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right) \tag{4.27}$$

$$(|z| = r < 1;\ \mu > 0;\ i \in \{0, 1, \ldots, n\})$$

and

$$\left|D_z^{-\mu}\left(\mathcal{D}^i f(z)\right)\right| \leqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}$$

$$\cdot\left(1+\frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)}r^k\right) \tag{4.28}$$

$$(|z|=r<1;\ \mu>0;\ i\in\{0,1,\ldots,n\}).$$

Each of the assertions (4.27) *and* (4.28) *is sharp, the extremal function being given by*

$$\mathcal{D}^i f(z)=z-\frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^{n-1}\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}(1+\lambda k)}z^{k+1}. \tag{4.29}$$

By setting $i=0$ in Theorem 17, we obtain

Corollary 2. *Let the function f defined by* (4.3) *be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. Then*

$$\left|D_z^{-\mu} f(z)\right| \geqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}$$

$$\cdot\left(1-\frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)}r^k\right) \tag{4.30}$$

$$(|z|=r<1;\ \mu>0)$$

and

$$\left|D_z^{-\mu} f(z)\right| \leqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}$$

$$\cdot\left(1+\frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)}r^k\right) \tag{4.31}$$

$$(|z|=r<1;\ \mu>0).$$

The estimates in (4.30) *and* (4.31) *are sharp, the extremal function being given by* (4.29) *with* $i = 0$, *that is, by*

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^{n-1}\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}(1+\lambda k)} z^{k+1}. \quad (4.32)$$

Theorem 18. *Let the function f defined by* (4.3) *be in the class* $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. *Then*

$$\left|D_z^{\mu}\left(\mathcal{D}^i f(z)\right)\right| \geqq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \cdot \left(1 - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right) \quad (4.33)$$

$$(|z| = r < 1;\ 0 \leqq \mu < 1;\ i \in \{0,1,\ldots,n-1\})$$

and

$$\left|D_z^{\mu}\left(\mathcal{D}^i f(z)\right)\right| \leqq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \cdot \left(1 + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right) \quad (4.34)$$

$$(|z| = r < 1;\ 0 \leqq \mu < 1;\ i \in \{0,1,\ldots,n-1\})$$

Each of the assertions (4.33) *and* (4.34) *is sharp, the extremal function being given by* (4.29).

By letting $i = 0$ in Theorem 18, we have

Corollary 3. *Let the function f defined by* (4.3) *be in the class* $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. *Then*

$$\left|D_z^{\mu} f(z)\right| \geqq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \cdot \left(1 - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{n}(1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}\Gamma(k+2-\mu)} r^k\right) \quad (4.35)$$

$$(|z| = r < 1;\ 0 \leqq \mu < 1)$$

and

$$|D_z^{\mu} f(z)| \leqq \frac{r^{1-\mu}}{\Gamma(2-\mu)}$$

$$\cdot \left(1 + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2-\mu)} r^k\right) \tag{4.36}$$

$$(|z| = r < 1;\ 0 \leqq \mu < 1)\,.$$

The estimates in (4.35) *and* (4.36) *are sharp, the extremal function being given by* (4.32).

Remark 3. Many of the results of this section can suitably be extended to hold true for such generalized fractional calculus operators as those with the Gauss hypergeometric function kernel, which were considered earlier by Srivastava *et al.* [36] (see also [3], [22], and [32]).

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