<table>
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<th>Title</th>
<th>Applications of differential subordinations for univalent functions (Study on Inverse Problems in Univalent Function Theory)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>斎藤, 齋</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001(1192): 129-139</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64768">http://hdl.handle.net/2433/64768</a></td>
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<td>Right</td>
<td>部門学術論文</td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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CONVOLUTIONS OF CERTAIN ANALYTIC FUNCTIONS

SHINJI SAITA AND SHIGEYOSHI OWA

Abstract. Two subclasses $M^*(\alpha)$ and $N^*(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $U$ are introduced. The object of the present paper is to derive some convolution properties for functions $f(z)$ belonging to the classes $M^*(\alpha)$ and $N^*(\alpha)$.

1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$U = \{ z \in \mathbb{C} : |z| < 1 \}.$$
Let $M(\omega)$ be the subclass of $A$ consisting of functions $f(x)$ which satisfy

$$\text{Re}\left\{ \frac{xf'(x)}{f(x)} \right\} < \lambda \quad (\chi \in \mathbb{U})$$

for some $\lambda$ ($\lambda > 1$). And let $N(\omega)$ be the subclass of $A$ consisting of functions $f(x)$ which satisfy

$$\text{Re}\left\{ 1 + \frac{x^2f''(x)}{f(x)} \right\} < \lambda \quad (\chi \in \mathbb{U})$$

for some $\lambda$ ($\lambda > 1$). Recently, Uralegaddi, Ganigi and Sarangi [3] (also, Nishiwaki and Owa [11]) have showed that

(i) if $f(x) \in A$ satisfies

$$\sum_{n=0}^{\infty} (n-\alpha)|a_n| < \lambda - 1$$

for some $\lambda$ ($1 < \lambda < \frac{3}{2}$), then $f(x) \in M(\omega)$,

(ii) if $f(x) \in A$ satisfies

$$\sum_{n=0}^{\infty} n(n-\alpha)|a_n| < \lambda - 1$$

for some $\lambda$ ($1 < \lambda < \frac{3}{2}$), then $f(x) \in N(\omega)$.

By means of the above, we define the subclass $M^*(\omega)$ of $M(\omega)$ consisting of $f(x)$ which satisfy (1.2), and the subclass $N^*(\omega)$ of $N(\omega)$ consisting of $f(x)$ which satisfy (1.3).
For functions $f_d(x) \in A$ given by

$$f_d(x) = x + \sum_{n=2}^{\infty} A_{n,d} x^n,$$

the convolution (or Hadamard product) $(f_1 * f_2)(x)$ of $f_1(x)$ and $f_2(x)$ is defined by

$$f_1 * f_2)(x) = x + \sum_{n=2}^{\infty} A_{n,1} A_{n,2} x^n.$$

In the present paper, we consider the convolutions of functions $f_d(x)$ belonging to $M^*(\omega)$ and $N^*(\omega)$.

2. Convolutions of functions in the class $M^*(\omega)$

Applying the same technique for convolutions due to Owa [21], we first derive

**Theorem 1.** If $f_d(x) \in M^*(d_d)$ for $d=1, 2, 3, \ldots, m$, then $(f_1 * f_2 \ast \cdots \ast f_m)(x) \in M^*(\beta)$, where

$$\beta = 1 + \frac{\prod_{d=1}^{m} (d_d-1)}{\prod_{d=1}^{m} (2-d_d) + \prod_{d=1}^{m} (d_d - 1)}$$

The result is sharp for functions

$$f_d(x) = x + \frac{d_d - 1}{2 - d_d} x^2 \quad (d=1, 2, 3, \ldots, m).$$
Proof. We use the mathematical induction for the proof.

Let \( f_1(x) \in M^*(d_1) \) and \( f_2(x) \in M^*(d_2) \). Then

\[
\sum_{n=2}^{\infty} \frac{(n-d_j)}{d_j-1} |a_{n,j}| \leq d_j - 1 \quad (j=1, 2)
\]

gives us that

(2.3) \[
\sum_{n=2}^{\infty} \sqrt{\frac{n-d_j}{d_j-1}} |a_{n,j}| \leq 1 \quad (j=1, 2)
\]

Therefore, an application of Cauchy–Schwarz inequality implies that

\[
\left( \sum_{n=2}^{\infty} \frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)} |a_{n,1}||a_{n,2}| \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{n=2}^{\infty} \frac{(n-d_1)}{d_1-1} |a_{n,1}| \right) \left( \sum_{n=2}^{\infty} \frac{(n-d_2)}{d_2-1} |a_{n,2}| \right) \leq 1
\]

Thus, if

(2.4) \[
\sum_{n=2}^{\infty} \frac{(n-d_j)}{d_j-1} |a_{n,j}||a_{n,j}| \]

\[
= \sum_{n=2}^{\infty} \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)} |a_{n,j}|^2 |a_{n,j}|^2} \leq \sum_{n=2}^{\infty} \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)} |a_{n,j}|^2 |a_{n,j}|^2},
\]

or if
(2.5) \( \sqrt{|a_{n,i}||a_{n,\bar{i}}|} \leq (\frac{\delta - 1}{n - \delta}) \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}} \)

for \( n \geq 2 \), then \( f(x) \in M^*(\delta) \).

Furthermore, since (2.3) gives that

\( \sqrt{|a_{n,j}|} \leq \sqrt{\frac{d_j - 1}{n - d_j}} \quad (j = 1, 2) \)

if

(2.6) \( \sqrt{\frac{(d_1-1)(d_2-1)}{(n-d_1)(n-d_2)}} \leq \left( \frac{\delta - 1}{n - \delta} \right) \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)}} \),

or if

(2.7) \( \frac{n - \delta}{\delta - 1} \leq \frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)} \)

for \( n \geq 2 \), then we have \( f(x) \in M^*(\delta) \).

It follows from (2.7) that

(2.8) \( \delta \geq 1 + \frac{(n-1)(d_1-1)(d_2-1)}{(n-d_1)(n-d_2)+(d_1-1)(d_2-1)} \equiv \kappa(n) \)

for \( n \geq 2 \). Note that \( \kappa(n) \) is decreasing for \( n \geq 2 \).

Consequently, we have
\[ \mathcal{F} \leq \mathcal{L}(2) \]

\[ = 1 + \frac{(d_1-1)(d_2-1)}{(2-d_1)(2-d_2)+(d_1-1)(d_2-1)} \]

which shows that \( f(x) \in M^*(\mathcal{F}) \), where

\[ \mathcal{F} = 1 + \frac{(d_1-1)(d_2-1)}{(2-d_1)(2-d_2)+(d_1-1)(d_2-1)} \]

Next, we suppose that

\[ (f_1 * f_2 * \ldots * f_m)(x) \in M^*(\beta') \]

where

\[ \beta' = 1 + \frac{\prod_{i=1}^{m} (d_i-1)}{\prod_{i=1}^{m} (2-d_i) + \prod_{i=1}^{m} (d_i-1)} \]

Then, by means of the previous proof, we have

\[ (f_1 * f_2 * \ldots * f_m * f_{m+1})(x) \in M^*(\beta) \]

where

\[ \beta = 1 + \frac{(\beta'-1)(d_{m+1}-1)}{(2-\beta')(2-d_{m+1}) + (\beta'-1)(d_{m+1}-1)} \]
Since

\[(\beta' - 1) (d_{m+1} - 1) = \frac{\prod_{j=1}^{m+1} (d_j - 1)}{\prod_{j=1}^{m} (2 - d_j) + \prod_{j=1}^{m} (d_j - 1)} \]

and

\[(2 - \beta') (2 - d_{m+1}) = \frac{\prod_{j=1}^{m+1} (2 - d_j)}{\prod_{j=1}^{m} (2 - d_j) + \prod_{j=1}^{m} (d_j - 1)} \]

(2.13) gives

\[(2.14) \quad \beta = 1 + \frac{\prod_{j=1}^{m+1} (d_j - 1)}{\prod_{j=1}^{m} (2 - d_j) + \prod_{j=1}^{m} (d_j - 1)} \]

Further, consider the functions \( f_j(x) \) defined by (2.2). For such functions \( f_j(x) \), we have

\[(f_1 \ast f_2 \ast \ldots \ast f_m)(x) = x + \left( \prod_{j=1}^{m} \frac{d_j - 1}{2 - d_j} \right) \chi^a = \chi + A_2 \chi^a \]

with \( A_2 = \prod_{j=1}^{m} \frac{d_j - 1}{2 - d_j} \).

It follows that

\[(2.15) \quad \sum_{n=2}^{\infty} \frac{n - \beta}{n - \beta - 1} |A_n| = 1. \]

This completes the proof of Theorem 1.

Letting \( d_j = \alpha \) for \( j = 1, 2, 3, \ldots, m \) in Theorem 1, we have
Corollary 1. If \( f_d(x) \in M^*(\alpha) \) for \( d=1, 2, 3, \ldots, m \),

then \( (f_1 \ast f_2 \ast \ldots \ast f_m)(x) \in M^*(\beta) \), where

\[
(2.16) \quad \beta = 1 + \frac{(\alpha - 1)^m}{(2-\alpha)^m + (\alpha - 1)^m}.
\]

The result is sharp for functions

\[
(2.17) \quad f_d(x) = x + \frac{d-1}{2-\alpha} x^2 \quad (d=1, 2, 3, \ldots, m)
\]

3. Convolutions of functions in the class \( N^*(\alpha) \)

For convolutions of \( f(x) \in N^*(\alpha) \), we have

Theorem 2. If \( f_d(x) \in N^*(\alpha_d) \) for \( d=1, 2, 3, \ldots, m \),

then \( (f_1 \ast f_2 \ast \ldots \ast f_m)(x) \in N^*(\beta) \), where

\[
(3.1) \quad \beta = 1 + \frac{\prod_{d=1}^{m} (\alpha_d - 1)}{2^{m-1} \prod_{d=1}^{m} (2-\alpha_d) + \prod_{d=1}^{m} (\alpha_d - 1)}
\]

The result is sharp for functions

\[
(3.2) \quad f_d(x) = x + \frac{\alpha_d - 1}{2(2-\alpha_d)} x^2 \quad (d=1, 2, 3, \ldots, m)
\]
Proof. As in the proof of Theorem 1, for \( f_1(x) \in N^*(d_1) \) and \( f_2(x) \in N^*(d_2) \), the following inequality:

\[
\sum_{n=2}^{\infty} \frac{n(n-1)}{\delta - 1} |a_n| ||a_{n,2}|| \leq 1
\]

implies \((f_1 \ast f_2)(x) \in N^*(\delta)\). Spending the same manner of the proof in Theorem 1, we obtain that

\[
\delta \equiv 1 + \frac{(n-1)(d_1-1)(d_2-1)}{n(n-d_1)(n-d_2) + (d_1-1)(d_2-1)}
\]

for \( n \geq 2 \). The right hand side of (3.4) takes its maximum value for \( n = 2 \), because it is a decreasing function of \( n \geq 2 \). Therefore, we have \((f_1 \ast f_2)(x) \in N^*(\delta)\) with

\[
\delta = 1 + \frac{(d_1-1)(d_2-1)}{2(2-d_1)(2-d_2) + (d_1-1)(d_2-1)}
\]

Moreover, suppose that

\((f_1 \ast f_2 \ast \cdots \ast f_m)(x) \in N^*(\beta)'\),

where

\[
\beta' = 1 + \frac{\prod_{j=1}^{m} (d_j - 1)}{2^{m-1} \prod_{j=1}^{m} (2-d_j) + \prod_{j=1}^{m} (d_j - 1)}
\]
Then we get

$$(f_1 \ast f_2 \ast \ldots \ast f_m \ast \check{f}_{m+1})(x) \in N^*(\beta),$$

where

$$(3.7) \quad \beta = 1 + \frac{(\beta' - 1)(d_{m+1} - 1)}{2(2-\beta')(2-d_{m+1}) + (\beta' - 1)(d_{m+1} - 1)}$$

$$= 1 + \frac{\prod_{j=1}^{m+1} (d_j - 1)}{2^{m-1} \prod_{j=1}^{m} (2-d_j) + \prod_{j=1}^{m} (d_j - 1)}$$

Further, taking $f_j(x)$ given by (3.2), we know that the result of Theorem 2 is sharp.

Finally, letting $d_j = d$ for $j = 1, 2, 3, \ldots, m$ in Theorem 2,

Corollary 2. If $f_j(x) \in N^*(d)$ for $j = 1, 2, 3, \ldots, m$, then $(f_1 \ast f_2 \ast \ldots \ast f_m)(x) \in N^*(\beta)$, where

$$(3.8) \quad \beta = 1 + \frac{(d-1)^m}{2^{m-1}(2-d)^m + (d-1)^m}.$$ The result is sharp for

$$(3.9) \quad f_j(x) = x + \frac{d-1}{2(2-d)} x^2 \quad (j = 1, 2, 3, \ldots, m).$$
Remark Uralegaddi and Desai [4] have showed some convolution properties of univalent functions with positive coefficients. Our result in the present paper are generalization theorems of the results by Uralegaddi and Desai [4].

References


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