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CONVOLUTIONS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. Two subclasses $M^*(\alpha)$ and $N^*(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $U$ are introduced. The object of the present paper is to derive some convolution properties for functions $f(z)$ belonging to the classes $M^*(\alpha)$ and $N^*(\alpha)$.

1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form

\begin{equation}
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

which are analytic in the open unit disk

$$U = \{ z \in \mathbb{C} : |z|<1 \}.$$
Let $M(\omega)$ be the subclass of $A$ consisting of functions $f(x)$ which satisfy
\[
\Re \left\{ \frac{xf(x)}{f(x)} \right\} < \lambda \quad (\chi \in \Xi)
\]
for some $\lambda$ ($\lambda > 1$). And let $N(\omega)$ be the subclass of $A$ consisting of functions $f(x)$ which satisfy
\[
\Re \left\{ 1 + \frac{xf''(x)}{f(x)} \right\} < \lambda \quad (\chi \in \Xi)
\]
for some $\lambda$ ($\lambda > 1$). Recently, Uraraegadi, Ganigi and Sarangi [3] (also, Nishiwaki and Owa [11]) have showed that

(i) if $f(x) \in A$ satisfies
\[
\sum_{n=1}^{\infty} (n-\lambda)|a_n| \leq \lambda - 1
\]
for some $\lambda$ ($1 < \lambda < 1/3$), then $f(x) \in M(\omega)$,

(ii) if $f(x) \in A$ satisfies
\[
\sum_{n=1}^{\infty} n(n-\lambda)|a_n| \leq \lambda - 1
\]
for some $\lambda$ ($1 < \lambda < 1/3$), then $f(x) \in N(\omega)$.

By means of the above, we define the subclass $M^*(\omega)$ of $M(\omega)$ consisting of $f(x)$ which satisfy (1.2), and the subclass $N^*(\omega)$ of $N(\omega)$ consisting of $f(x)$ which satisfy (1.3).
For functions \( f_j(x) \in A \) given by

\[ f_j(x) = x + \sum_{n=1}^{\infty} a_{n,j} x^n, \]

the convolution (or Hadamard product) \((f_1 \ast f_2)(x)\) of \( f_1(x) \) and \( f_2(x) \) is defined by

\[ (f_1 \ast f_2)(x) = x + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} x^n. \]

In the present paper, we consider the convolutions of functions \( f_\delta(x) \) belonging to \( M^*(\omega) \) and \( N^*(\omega) \).

2. Convolutions of functions in the class \( M^*(\omega) \)

Applying the same technique for convolutions due to Owa [21], we first derive

**Theorem 1.** If \( f_j(x) \in M^*(\omega_j) \) for \( j = 1, 2, 3, \ldots, m \),

then \((f_1 \ast f_2 \ast \ldots \ast f_m)(x) \in M^*(\beta)\), where

\[ \beta = 1 + \frac{\prod_{j=1}^{m} (\omega_j - 1)}{\prod_{j=1}^{m} (2 - \omega_j) + \prod_{j=1}^{m} (\omega_j - 1)} \]

The result is sharp for functions

\[ f_j(x) = x + \frac{\omega_j - 1}{2 - \omega_j} x^2 \quad (j = 1, 2, 3, \ldots, m). \]
Proof. We use the mathematical induction for the proof.

Let \( f_1(x) \in M^{*}(d_1) \) and \( f_2(x) \in M^{*}(d_2) \). Then

\[
\sum_{n=2}^{\infty} \frac{1}{n-1} |a_{n,j}| \leq d_j - 1 \quad (j=1,2)
\]

gives us that

\[
(2.3) \quad \sum_{n=2}^{\infty} \sqrt{\left(\frac{n-1}{d_j-1}\right)|a_{n,j}|} \leq 1 \quad (j=1,2)
\]

Therefore, an application of Cauchy-Schwarz inequality implies that

\[
\left( \sum_{n=2}^{\infty} \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)} |a_{n,1}| |a_{n,2}|} \right)^2 \leq \\
\left( \sum_{n=2}^{\infty} \frac{n-d_1}{d_1-1} |a_{n,1}| \right) \left( \sum_{n=2}^{\infty} \frac{n-d_2}{d_2-1} |a_{n,2}| \right) \leq 1
\]

Thus, if

\[
(2.4) \quad \sum_{n=2}^{\infty} \left(\frac{n-d}{(d-1)}\right) |a_{n,1}| |a_{n,2}|
\]

\[
\leq \sum_{n=2}^{\infty} \sqrt{\frac{(n-d_1)(n-d_2)}{(d_1-1)(d_2-1)} |a_{n,1}| |a_{n,2}|}
\]

or if
\[(2.5) \quad \sqrt{|a_{n,1}|a_{n,2}|} \leq \left( \frac{\delta - 1}{n - \delta} \right) \sqrt{\frac{(n - d_1)(n - d_2)}{(d_1 - 1)(d_2 - 1)}} \]

for \( n \geq 2 \), then \( f(x) \in M^*(\delta) \).

Furthermore, since (2.3) gives that
\[
\sqrt{|a_{n,j}|} \leq \sqrt{\frac{d_j - 1}{n - d_j}} \quad (j = 1, 2)
\]

if
\[(2.6) \quad \sqrt{\frac{(d_1 - 1)(d_2 - 1)}{(n - d_1)(n - d_2)}} \leq \left( \frac{\delta - 1}{n - \delta} \right) \sqrt{\frac{(n - d_1)(n - d_2)}{(d_1 - 1)(d_2 - 1)}} \]

or if
\[(2.7) \quad \frac{n - \delta}{\delta - 1} \leq \frac{(n - d_1)(n - d_2)}{(d_1 - 1)(d_2 - 1)} \]

for \( n \geq 2 \), then we have \( f(x) \in M^*(\delta) \).

It follows from (2.7) that
\[(2.8) \quad \delta \geq 1 + \frac{(n - 1)(d_1 - 1)(d_2 - 1)}{(n - d_1)(n - d_2) + (d_1 - 1)(d_2 - 1)} = \mathcal{H}(n) \]

for \( n \geq 2 \). Note that \( \mathcal{H}(n) \) is decreasing for \( n \geq 2 \). Consequently, we have
(2.9) \[ S \geq \mathcal{H}(2) \]

\[ = 1 + \frac{(d_1-1)(d_2-1)}{(2-d_1)(2-d_2) + (d_1-1)(d_2-1)} \]

which shows that \( f(x) \in M^*(S) \), where

(2.10) \[ S = 1 + \frac{(d_1-1)(d_2-1)}{(2-d_1)(2-d_2) + (d_1-1)(d_2-1)} \]

Next, we suppose that

(2.11) \[ (f_1 * f_2 * \cdots * f_m)(x) \in M^*(\beta') \],

where

(2.12) \[ \beta' = 1 + \frac{\prod_{\delta=1}^{m} (d_{\delta}-1)}{\prod_{\delta=1}^{m} (2-d_{\delta}) + \prod_{\delta=1}^{m} (d_{\delta}-1)} \]

Then, by means of the previous proof, we have

\[ (f_1 * f_2 * \cdots * f_m * f_{m+1})(x) \in M^*(\beta) \]

where

(2.13) \[ \beta = 1 + \frac{(\beta'-1)(d_{m+1}-1)}{(2-\beta')(2-d_{m+1}) + (\beta'-1)(d_{m+1}-1)} \]
since
\[
(\beta'-1) (\alpha_{m+1}-1) = \frac{\prod_{d=1}^{m+1} (d_j-1)}{\prod_{d=1}^{m+1} (2-d_j) + \prod_{d=1}^{m+1} (d_j-1)}
\]
and
\[
(2-\beta') (2-\alpha_{m+1}) = \frac{\prod_{d=1}^{m+1} (2-d_j)}{\prod_{d=1}^{m+1} (2-d_j) + \prod_{d=1}^{m+1} (d_j-1)}
\]

(2.13) gives
\[
\beta = 1 + \frac{\prod_{d=1}^{m+1} (d_j-1)}{\prod_{d=1}^{m+1} (2-d_j) + \prod_{d=1}^{m+1} (d_j-1)}
\]

Further, consider the functions \( f_+ (x) \) defined by (2.2). For such functions \( f_+ (x) \), we have
\[
(f_1 \ast f_2 \ast \cdots \ast f_m) (x) = x + \left( \prod_{d=1}^{m} \frac{d_j-1}{2-d_j} \right) x^2 = x + A_2 x^2
\]
with
\[
A_2 = \prod_{d=1}^{m} \frac{d_j-1}{2-d_j}
\]

It follows that
\[
(2.15) \quad \sum_{n=2}^{\infty} \frac{n-\beta}{n-1} |A_n| = 1
\]

This completes the proof of Theorem 1.

Letting \( d_j = d \) for \( j=1, 2, 3, \ldots, m \) in Theorem 1, we have
Corollary 1. If \( f_d(x) \in M^*(\alpha) \) for \( d=1,2,3,\ldots,m \), then \( (f_1*f_2*\cdots*f_m)(x) \in M^*(\beta) \), where

\[
(2.16) \quad \beta = 1 + \frac{(d-1)^m}{(2-d)^m + (d-1)^m}.
\]

The result is sharp for functions

\[
(2.17) \quad f_d(x) = x + \frac{d-1}{2-d} x^2 \quad (d=1,2,3,\ldots,m)
\]

3. Convolutions of functions in the class \( N^*(\alpha) \)

For convolutions of \( f(x) \in N^*(\alpha) \), we have

Theorem 2. If \( f_d(x) \in N^*(\alpha_d) \) for \( d=1,2,3,\ldots,m \), then \( (f_1*f_2*\cdots*f_m)(x) \in N^*(\beta) \), where

\[
(3.1) \quad \beta = 1 + \frac{\sum_{d=1}^{m} (\alpha_d-1)}{2^{m-1} \sum_{d=1}^{m} (2-d_d) + \sum_{d=1}^{m} (d_d-1)}
\]

The result is sharp for functions

\[
(3.2) \quad f_d(x) = x + \frac{d_d-1}{2(2-d_d)} x^2 \quad (d=1,2,3,\ldots,m)
\]
Proof. As in the proof of Theorem 1, for \( f_1(x) \in \mathcal{N}^*(d_1) \) and \( f_2(x) \in \mathcal{N}^*(d_2) \), the following inequality:

\[
(3.3) \quad \sum_{n=2}^{\infty} \frac{n(n-\delta)}{\delta - 1} |a_n| |a_{n,2}| \leq 1
\]

implies \((f_1 * f_2)(x) \in \mathcal{N}^*(\delta)\). Spending the same manner of the proof in Theorem 1, we obtain that

\[
(3.4) \quad S = 1 + \frac{(n-1)(d_1-1)(d_2-1)}{n(n-d_1)(n-d_2) + (d_1-1)(d_2-1)}
\]

for \( n \geq 2 \). The right hand side of (3.4) takes its maximum value for \( n = 2 \), because it is a decreasing function of \( n \geq 2 \). Therefore, we have \((f_1 * f_2)(x) \in \mathcal{N}^*(\delta)\) with

\[
(3.5) \quad S = 1 + \frac{(d_1-1)(d_2-1)}{2(2-d_1)(2-d_2) + (d_1-1)(d_2-1)}
\]

Moreover, suppose that

\[
(f_1 * f_2 * \cdots * f_m)(x) \in \mathcal{N}^*(\beta)',
\]

where

\[
(3.6) \quad \beta' = 1 + \frac{\prod_{j=1}^{m} (d_{j}-1)}{2^{m-1} \prod_{j=1}^{m} (2-d_{j}) + \prod_{j=1}^{m} (d_{j}-1)}
\]

Then we get

\[(f_1 * f_2 * \ldots * f_m * f_{m+1})(x) \in N^*(\beta),\]

where

\[
\beta \leq 1 + \frac{(\beta'-1)(d_{m+1}-1)}{2(2-\beta')(2-d_{m+1}) + (\beta'-1)(d_{m+1}-1)}
\]

\[= 1 + \frac{\prod_{j=1}^{m+1} (d_j - 1)}{2^{m-1} \prod_{j=1}^{m+1} (2-d_j) + \prod_{j=1}^{m+1} (d_j - 1)}
\]

Further, taking \(f_j(x)\) given by (3.2), we know that the result of Theorem 2 is sharp.

Finally, letting \(d_j = d\) for \(j = 1, 2, 3, \ldots, m\) in Theorem 2,

Corollary 2. If \(f_j(x) \in N^*(d)\) for \(j = 1, 2, 3, \ldots, m\),

then \((f_1 * f_2 * \ldots * f_m)(x) \in N^*(\beta)\), where

\[
\beta \leq 1 + \frac{(d-1)^m}{2^{m-1}(2-d)^m + (d-1)^m}
\]

The result is sharp for

\[
f_j(x) = x + \frac{d-1}{2(2-d)} x^2 \quad (j = 1, 2, 3, \ldots, m).
\]
Remark Uralegaddi and Desai [4] have showed some convolution properties of univalent functions with positive coefficients. Our result in the present paper are generalization theorems of the results by Uralegaddi and Desai [4].

References


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