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CONVOLUTIONS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. Two subclasses $M^*(\alpha)$ and $N^*(\alpha)$ of analytic functions $f(z)$ with $f(0)=0$ and $f'(0)=1$ in the open unit disk U are introduced. The object of the present paper is to derive some convolution properties for functions $f(z)$ belonging to the classes $M^*(\alpha)$ and $N^*(\alpha)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

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Let $M(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \qquad (z \in U)$$

for some α $(\alpha > 1)$. And let $N(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \qquad (z \in U)$$

for some α $(\alpha > 1)$. Recently, Uralegaddi, Ganigi and Sarangi [3] (also, Nishiwaki and Owa [1]) have showed that

(i) if $f(z) \in A$ satisfies

$$(1.2) \qquad \sum_{n=2}^{\infty} (n-\alpha)|a_n| \leqq \alpha - 1$$

for some α $(1 < \alpha < \frac{3}{2})$, then $f(z) \in M(\alpha)$,

(ii) if $f(z) \in A$ satisfies

$$(1.3) \qquad \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq \alpha - 1$$

for some α $(1 < \alpha < \frac{3}{2})$, then $f(z) \in N(\alpha)$.

By means of the above, we define the subclass $M^*(\alpha)$ of $M(\alpha)$ consisting of $f(z)$ which satisfy (1.2), and the subclass $N^*(\alpha)$ of $N(\alpha)$ consisting of $f(z)$ which satisfy (1.3).

For functions $f_j(x) \in A$ given by

(1.4) $$f_j(x) = x + \sum_{n=2}^{\infty} a_{n,j} x^n,$$

the convolution (or Hadamard product) $(f_1 * f_2)(x)$ of $f_1(x)$ and $f_2(x)$ is defined by

(1.5) $$(f_1 * f_2)(x) = x + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} x^n .$$

In the present paper, we consider the convolutions of functions $f_j(x)$ belonging to $M^*(\alpha)$ and $N^*(\alpha)$.

2. Convolutions of functions in the class $M^*(\alpha)$

Applying the same technique for convolutions due to Owa [2], we first derive

Theorem 1. If $f_j(x) \in M^*(\alpha_j)$ for $j = 1, 2, 3, \cdots, m$, then $(f_1 * f_2 * \cdots * f_m)(x) \in M^*(\beta)$, where

(2.1) $$\beta = 1 + \frac{\prod_{j=1}^{m} (\alpha_j - 1)}{\prod_{j=1}^{m} (2 - \alpha_j) + \prod_{j=1}^{m} (\alpha_j - 1)}$$

The result is sharp for functions

(2.2) $$f_j(x) = x + \frac{\alpha_j - 1}{2 - \alpha_j} x^2 \quad (j = 1, 2, 3, \cdots, m) .$$

Proof We use the mathematical induction for the proof.

Let $f_1(z) \in M^*(\alpha_1)$ and $f_2(z) \in M^*(\alpha_2)$. Then

$$\sum_{n=2}^{\infty}(n-\alpha_j)\,|a_{n,j}| \leqq \alpha_j - 1 \qquad (j=1,2)$$

gives us that

$$(2.3) \qquad \sum_{n=2}^{\infty}\sqrt{\left(\frac{n-\alpha_j}{\alpha_j-1}\right)|a_{n,j}|} \leqq 1 \qquad (j=1,2)$$

Therefore, an application of Cauchy-Schwarz inequality implies that

$$\left|\sum_{n=2}^{\infty}\sqrt{\frac{(n-\alpha_1)(n-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}|a_{n,1}||a_{n,2}|}\right|^2$$

$$\leqq \left(\sum_{n=2}^{\infty}\left(\frac{n-\alpha_1}{\alpha_1-1}\right)|a_{n,1}|\right)\left(\sum_{n=2}^{\infty}\left(\frac{n-\alpha_2}{\alpha_2-1}\right)|a_{n,2}|\right) \leqq 1$$

Thus, if

$$(2.4) \qquad \sum_{n=2}^{\infty}\left(\frac{n-\delta}{\delta-1}\right)|a_{n,1}||a_{n,2}|$$

$$\leqq \sum_{n=2}^{\infty}\sqrt{\frac{(n-\alpha_1)(n-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}|a_{n,1}||a_{n,2}|},$$

or if

$$(2.5) \quad \sqrt{|a_{n,1}||a_{n,2}|} \leqq \left(\frac{\delta-1}{n-\delta}\right)\sqrt{\frac{(n-\alpha_1)(n-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}}$$

for $n \geqq 2$, then $f(z) \in M^*(\delta)$.

Furthermore, since (2.3) gives that

$$\sqrt{|a_{n,j}|} \leqq \sqrt{\frac{\alpha_j-1}{n-\alpha_j}} \qquad (j=1,2)$$

if

$$(2.6) \quad \sqrt{\frac{(\alpha_1-1)(\alpha_2-1)}{(n-\alpha_1)(n-\alpha_2)}} \leqq \left(\frac{\delta-1}{n-\delta}\right)\sqrt{\frac{(n-\alpha_1)(n-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} ,$$

or if

$$(2.7) \quad \frac{n-\delta}{\delta-1} \leqq \frac{(n-\alpha_1)(n-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}$$

for $n \geqq 2$, then we have $f(z) \in M^*(\delta)$.

It follows from (2.7) that

$$(2.8) \quad \delta \geqq 1 + \frac{(n-1)(\alpha_1-1)(\alpha_2-1)}{(n-\alpha_1)(n-\alpha_2)+(\alpha_1-1)(\alpha_2-1)} \equiv h(n)$$

for $n \geqq 2$. Note that $h(n)$ is decreasing for $n \geqq 2$.

Consequently, we have

(2.9) $$\delta \geqq h(2) = 1 + \frac{(\alpha_1-1)(\alpha_2-1)}{(2-\alpha_1)(2-\alpha_2)+(\alpha_1-1)(\alpha_2-1)},$$

which shows that $f(z) \in M^*(\delta)$, where

(2.10) $$\delta = 1 + \frac{(\alpha_1-1)(\alpha_2-1)}{(2-\alpha_1)(2-\alpha_2)+(\alpha_1-1)(\alpha_2-1)}.$$

Next, we suppose that

(2.11) $$(f_1 * f_2 * \cdots * f_m)(z) \in M^*(\beta'),$$

where

(2.12) $$\beta' = 1 + \frac{\prod_{j=1}^{m}(\alpha_j-1)}{\prod_{j=1}^{m}(2-\alpha_j) + \prod_{j=1}^{m}(\alpha_j-1)}.$$

Then, by means of the previous proof, we have

$$(f_1 * f_2 * \cdots * f_m * f_{m+1})(z) \in M^*(\beta),$$

where

(2.13) $$\beta = 1 + \frac{(\beta'-1)(\alpha_{m+1}-1)}{(2-\beta')(2-\alpha_{m+1})+(\beta'-1)(\alpha_{m+1}-1)}$$

since

$$(\beta'-1)(\alpha_{m+1}-1)=\frac{\prod_{j=1}^{m+1}(\alpha_j-1)}{\prod_{j=1}^{m}(2-\alpha_j)+\prod_{j=1}^{m}(\alpha_j-1)}$$

and

$$(2-\beta')(2-\alpha_{m+1})=\frac{\prod_{j=1}^{m+1}(2-\alpha_j)}{\prod_{j=1}^{m}(2-\alpha_j)+\prod_{j=1}^{m}(\alpha_j-1)},$$

(2.13) gives

$$\beta=1+\frac{\prod_{j=1}^{m+1}(\alpha_j-1)}{\prod_{j=1}^{m+1}(2-\alpha_j)+\prod_{j=1}^{m+1}(\alpha_j-1)} \tag{2.14}$$

Further, consider the functions $f_j(z)$ defined by (2.2). For such functions $f_j(z)$, we have

$$(f_1*f_2*\cdots*f_m)(z)=z+\left(\prod_{j=1}^{m}\frac{\alpha_j-1}{2-\alpha_j}\right)z^2=z+A_2z^2$$

with $$A_2=\prod_{j=1}^{m}\frac{\alpha_j-1}{2-\alpha_j}.$$

It follows that

$$\sum_{n=2}^{\infty}\frac{n-\beta}{\beta-1}|A_n|=1. \tag{2.15}$$

This completes the proof of Theorem 1.

Letting $\alpha_j=\alpha$ for $j=1,2,3,\cdots,m$ in Theorem 1, we have

Corollary 1. If $f_j(x) \in M^*(\alpha)$ for $j=1,2,3,\cdots,m$, then $(f_1 * f_2 * \cdots * f_m)(x) \in M^*(\beta)$, where

$$\beta = 1 + \frac{(\alpha-1)^m}{(2-\alpha)^m + (\alpha-1)^m} \qquad (2.16)$$

The result is sharp for functions

$$f_j(x) = x + \frac{\alpha-1}{2-\alpha} x^2 \quad (j=1,2,3,\cdots,m) \qquad (2.17)$$

3. Convolutions of functions in the class $N^*(\alpha)$

For convolutions of $f(x) \in N^*(\alpha)$, we have

Theorem 2. If $f_j(x) \in N^*(\alpha_j)$ for $j=1,2,3,\cdots,m$, then $(f_1 * f_2 * \cdots * f_m)(x) \in N^*(\beta)$, where

$$\beta = 1 + \frac{\prod_{j=1}^{m}(\alpha_j - 1)}{2^{m-1}\prod_{j=1}^{m}(2-\alpha_j) + \prod_{j=1}^{m}(\alpha_j-1)} \qquad (3.1)$$

The result is sharp for functions

$$f_j(x) = x + \frac{\alpha_j - 1}{2(2-\alpha_j)} x^2 \quad (j=1,2,3,\cdots,m) \qquad (3.2)$$

Proof As in the proof of Theorem 1, for $f_1(x) \in N^*(\alpha_1)$ and $f_2(x) \in N^*(\alpha_2)$, the following inequality

(3.3) $$\sum_{n=2}^{\infty} \frac{n(n-\delta)}{\delta - 1} |a_{n,1}||a_{n,2}| \leqq 1$$

implies $(f_1 * f_2)(x) \in N^*(\delta)$. Spending the same manner of the proof in Theorem 1, we obtain that

(3.4) $$\delta \geqq 1 + \frac{(n-1)(\alpha_1 - 1)(\alpha_2 - 1)}{n(n-\alpha_1)(n-\alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)}$$

for $n \geqq 2$. The right hand side of (3.4) takes its maximum value for $n=2$, because it is a decreasing function of $n \geqq 2$. Therefore, we have $(f_1 * f_2)(x) \in N^*(\delta)$ with

(3.5) $$\delta = 1 + \frac{(\alpha_1 - 1)(\alpha_2 - 1)}{2(2-\alpha_1)(2-\alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)} .$$

Moreover, suppose that

$$(f_1 * f_2 * \cdots * f_m)(x) \in N^*(\beta)' ,$$

where

(3.6) $$\beta' = 1 + \frac{\prod_{j=1}^{m} (\alpha_j - 1)}{2^{m-1} \prod_{j=1}^{m} (2-\alpha_j) + \prod_{j=1}^{m} (\alpha_j - 1)}$$

Then we get

$$(f_1 * f_2 * \cdots * f_m * f_{m+1})(z) \in N^*(\beta),$$

where

$$\beta = 1 + \frac{(\beta'-1)(\alpha_{m+1}-1)}{2(2-\beta')(2-\alpha_{m+1}) + (\beta'-1)(\alpha_{m+1}-1)} \tag{3.7}$$

$$= 1 + \frac{\prod_{j=1}^{m+1}(\alpha_j - 1)}{2^{m-1}\prod_{j=1}^{m+1}(2-\alpha_j) + \prod_{j=1}^{m+1}(\alpha_j - 1)}$$

Further, taking $f_j(z)$ given by (3.2), we know that the result of Theorem 2 is sharp.

Finally, letting $\alpha_j = \alpha$ for $j = 1, 2, 3, \cdots, m$ in Theorem 2,

Corollary 2. If $f_j(z) \in N^*(\alpha)$ for $j = 1, 2, 3, \cdots, m$, then $(f_1 * f_2 * \cdots * f_m)(z) \in N^*(\beta)$, where

$$\beta = 1 + \frac{(\alpha-1)^m}{2^{m-1}(2-\alpha)^m + (\alpha-1)^m}. \tag{3.8}$$

The result is sharp for

$$f_j(z) = z + \frac{\alpha - 1}{2(2-\alpha)} z^2 \quad (j = 1, 2, 3, \cdots, m). \tag{3.9}$$

Remark Uralegaddi and Desai [4] have showed some convolution properties of univalent functions with positive coefficients. Our result in the present paper are generalization theorems of the results by Uralegaddi and Desai [4].

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