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On certain conditions for starlikeness

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Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be strongly starlike of order $\alpha$.

1 Introduction.

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. A function $f(z)$ in $A$ is said to be starlike in $U$ if it satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).$$

We denote by $S^*$ the subclass of $A$ consisting of all starlike functions $f(z)$ in $U$. Further a function $f(z)$ belonging to $A$ is said to be strongly starlike of order $\alpha$ in $U$ if it satisfies

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$

for some $\alpha$ $(0 < \alpha \leq 1)$. We denote by $SS^*(\alpha)$ the subclass of $A$ consisting of all strongly starlike functions of order $\alpha$ in $U$.

From the definition for strongly starlike functions of order $\alpha$, we note that $f(z) \in SS^*(\alpha)$ is univalent and starlike in $U$. Recently, Tuneski [2] obtained the following theorem.

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**Theorem A.** Let a function $f(z) \in A$ satisfy

$$\frac{f(z)f''(z)}{f'(z)^2} < 2 - \frac{2}{(1-z)^2} \quad (z \in U),$$

where the symbol "<" means the subordination. Then $f(z) \in S^*$. 

To derive our main theorem, we need the following lemma due to Nunokawa [1].

**Lemma.** Let $p(z)$ be analytic in $U$ with $p(0) = 1$ and $p(z) \neq 0 (z \in U)$. If there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| \leq \frac{\pi}{2} \alpha \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2} \alpha \quad (\alpha > 0),$$

then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,$$

where $k \geq 1$ when $\arg(p(z_0)) = (\pi/2)\alpha$ and $k \leq -1$ when $\arg(p(z_0)) = -(\pi/2)\alpha$.

2 Strongly starlikeness of order $\alpha$

Now we derive

**Theorem.** Let $f(z)$ in $A$ satisfy the following inequalities

$$\pi - \frac{\pi}{2} \alpha - \tan^{-1} \alpha < \arg \left( \frac{f(z)f''(z)}{f'(z)^2} - 1 \right) < \pi + \frac{\alpha}{2} + \tan^{-1} \alpha \quad (z \in U)$$

for some $\alpha (0 < \alpha \leq 1)$. Then $f(z)$ belongs to the class $SS^*(\alpha)$ in $U$. 
Proof. From the assumption in the theorem, we see that $f'(z) \neq 0$ in $U$. Let us define the function $p(z)$ by $p(z) = zf'(z)/f(z)$. Then $p(z)$ satisfies

$$\frac{f(z)f''(z)}{f'(z)^2} = 1 + \frac{zp'(z)}{p(z)^2} - \frac{1}{p(z)}$$

and so

$$\frac{f(z)f''(z)}{f'(z)^2} - 1 = \frac{1}{p(z)} \left( -1 + \frac{zp'(z)}{p(z)} \right).$$

If there exists a point $z_0 \in U$ such that

$$|\arg(p(z_0))| < \frac{\pi}{2}\alpha \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha,$$

then Lemma gives us that

(i) for the case $\arg(p(z_0)) = (\pi/2)\alpha$,

$$\arg \left( \frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1 \right) = \arg \left\{ \frac{1}{p(z_0)} \left( \frac{z_0 p'(z_0)}{p(z_0)} - 1 \right) \right\}$$

$$= -\frac{\pi}{2}\alpha + \arg \left( -1 + \frac{z_0 p'(z_0)}{p(z_0)} \right)$$

$$= -\frac{\pi}{2}\alpha + \arg(-1 + ik\alpha)$$

$$\leq \pi - \frac{\pi}{2}\alpha - \tan^{-1}\alpha.$$

This contradicts our condition in the theorem.

(ii) for the case $\arg(p(z_0)) = -(\pi/2)\alpha$, the application of the same method as in (i) shows that

$$\arg \left( \frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1 \right) \geq \pi + \frac{\pi}{2}\alpha + \tan^{-1}\alpha.$$ 

This also contradicts the assumption of the theorem. Thus we complete the proof of our main theorem.

Putting $\alpha = 1$ in Theorem, we have the following corollary.
Corollary. If \( f(z) \in A \) satisfies
\[
\frac{\pi}{4} < \arg \left( \frac{f(z)f''(z)}{f'(z)^2} - 1 \right) < \frac{7\pi}{4} \quad (z \in U),
\]
then \( f(z) \in S^* \).

References
