Title: Sufficient conditions for Caratheodory functions (Study on Inverse Problems in Univalent Function Theory)

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On certain conditions for starlikeness

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Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be strongly starlike of order $\alpha$.

1 Introduction.

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $A$ is said to be starlike in $U$ if it satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).$$

We denote by $S^*$ the subclass of $A$ consisting of all starlike functions $f(z)$ in $U$. Further a function $f(z)$ belonging to $A$ is said to be strongly starlike of order $\alpha$ in $U$ if it satisfies

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad (z \in U)$$

for some $\alpha$ ($0 < \alpha \leq 1$). We denote by $SS^*(\alpha)$ the subclass of $A$ consisting of all strongly starlike functions of order $\alpha$ in $U$.

From the definition for strongly starlike functions of order $\alpha$, we note that $f(z) \in SS^*(\alpha)$ is univalent and starlike in $U$. Recently, Tuneski [2] obtained the following theorem.

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Theorem A. Let a function \( f(z) \in A \) satisfy
\[
\frac{f(z)f''(z)}{f'(z)^2} < 2 - \frac{2}{(1 - z)^2} \quad (z \in U),
\]
where the symbol "\(<\)" means the subordination. Then \( f(z) \in S^{*} \).

To derive our main theorem, we need the following lemma due to Nunokawa [1].

**Lemma.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) (\( z \in U \)). If there exists a point \( z_0 \in U \) such that
\[
|\arg(p(z))| \leq \frac{\pi}{2} \alpha \quad \text{for} \quad |z| < |z_0|
\]
and
\[
|\arg(p(z_0))| = \frac{\pi}{2} \alpha \quad (\alpha > 0),
\]
then we have
\[
\frac{z_0p'(z_0)}{p(z_0)} = ik\alpha,
\]
where \( k \geq 1 \) when \( \arg(p(z_0)) = (\pi/2)\alpha \) and \( k \leq -1 \) when \( \arg(p(z_0)) = -(\pi/2)\alpha \).

2 Strongly starlikeness of order \( \alpha \)

Now we derive

**Theorem.** Let \( f(z) \) in \( A \) satisfy the following inequalities
\[
\pi - \frac{\pi}{2} \alpha - \tan^{-1}\alpha < \arg\left(\frac{f(z)f''(z)}{f'(z)^2} - 1\right) < \pi + \frac{\alpha}{2} + \tan^{-1}\alpha \quad (z \in U)
\]
for some \( \alpha (0 < \alpha \leq 1) \). Then \( f(z) \) belongs to the class \( SS^{*} (\alpha) \) in \( U \).
Proof. From the assumption in the theorem, we see that $f'(z) \neq 0$ in $U$. Let us define the function $p(z)$ by $p(z) = zf'(z)/f(z)$. Then $p(z)$ satisfies

$$\frac{f(z)f''(z)}{f'(z)^2} = 1 + \frac{zp'(z)}{p(z)^2} - \frac{1}{p(z)}$$

and so

$$\frac{f(z)f''(z)}{f'(z)^2} - 1 = \frac{1}{p(z)} \left( -1 + \frac{zp'(z)}{p(z)} \right).$$

If there exists a point $z_0 \in U$ such that

$$|\arg(p(z_0))| < \frac{\pi}{2}\alpha$$

for $|z| < |z_0|$ and

$$|\arg(p(z_0))| = \frac{\pi}{2}\alpha,$$

then Lemma gives us that

(i) for the case $\arg(p(z_0)) = (\pi/2)\alpha$,

$$\arg\left(\frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1\right) = \arg\left\{\frac{1}{p(z_0)} \left( \frac{z_0p'(z_0)}{p(z_0)} - 1 \right)\right\}$$

$$= -\frac{\pi}{2}\alpha + \arg\left(-1 + \frac{z_0p'(z_0)}{p(z_0)}\right)$$

$$= -\frac{\pi}{2}\alpha + \arg(-1 + ik\alpha)$$

$$\leq \pi - \frac{\pi}{2}\alpha - \tan^{-1}\alpha.$$  

This contradicts our condition in the theorem.

(ii) for the case $\arg(p(z_0)) = -(\pi/2)\alpha$, the application of the same method as in (i) shows that

$$\arg\left(\frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1\right) \geq \pi + \frac{\pi}{2}\alpha + \tan^{-1}\alpha.$$  

This also contradicts the assumption of the theorem. Thus we complete the proof of our main theorem.

Putting $\alpha = 1$ in Theorem, we have the following corollary.
Corollary. If $f(z) \in A$ satisfies

$$\frac{\pi}{4} < \arg \left( \frac{f(z)f''(z)}{f'(z)^2} - 1 \right) < \frac{7\pi}{4} \quad (z \in U),$$

then $f(z) \in S^*$.

References


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