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Kyoto University
On certain conditions for starlikeness

MAMORU NUNOKAWA [布川 護] (群馬大学・教育学部)
SHIGEYOSHI OWA [尾和 重義] (近畿大学・理工学部)

Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be strongly starlike of order \( \alpha \).

1 Introduction.

Let \( A \) be the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). A function \( f(z) \) in \( A \) is said to be starlike in \( U \) if it satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).
\]

We denote by \( S^* \) the subclass of \( A \) consisting of all starlike functions \( f(z) \) in \( U \). Further a function \( f(z) \) belonging to \( A \) is said to be strongly starlike of order \( \alpha \) in \( U \) if it satisfies

\[
\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \alpha \quad (z \in U)
\]

for some \( \alpha \) \( (0 < \alpha \leq 1) \). We denote by \( SS^*(\alpha) \) the subclass of \( A \) consisting of all strongly starlike functions of order \( \alpha \) in \( U \).

From the definition for strongly starlike functions of order \( \alpha \), we note that \( f(z) \in SS^*(\alpha) \) is univalent and starlike in \( U \). Recently, Tuneski [2] obtained the following theorem.

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Theorem A. Let a function $f(z) \in A$ satisfy
\[ \frac{f(z)f''(z)}{f'(z)^2} < 2 - \frac{2}{(1-z)^2} \quad (z \in U), \]
where the symbol "\(<" means the subordination. Then $f(z) \in S^*$.

To derive our main theorem, we need the following lemma due to Nunokawa [1].

Lemma. Let $p(z)$ be analytic in $U$ with $p(0) = 1$ and $p(z) \neq 0 (z \in U)$. If there exists a point $z_0 \in U$ such that
\[ |\arg(p(z))| \leq \frac{\pi}{2} \alpha \quad \text{for} \quad |z| < |z_0| \]
and
\[ |\arg(p(z_0))| = \frac{\pi}{2} \alpha \quad (\alpha > 0), \]
then we have
\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik \alpha, \]
where $k \geq 1$ when $\arg(p(z_0)) = (\pi/2) \alpha$ and $k \leq -1$ when $\arg(p(z_0)) = -(\pi/2) \alpha$.

2 Strongly starlikeness of order $\alpha$

Now we derive

**Theorem.** Let $f(z)$ in $A$ satisfy the following inequalities
\[ \pi - \frac{\pi}{2} \alpha - \tan^{-1} \alpha < \arg \left( \frac{f(z)f''(z)}{f'(z)^2} - 1 \right) < \pi + \frac{\alpha}{2} + \tan^{-1} \alpha \quad (z \in U) \]
for some $\alpha(0 < \alpha \leq 1)$. Then $f(z)$ belongs to the class $SS^*(\alpha)$ in $U$. 
Proof. From the assumption in the theorem, we see that \( f'(z) \neq 0 \) in \( U \). Let us define the function \( p(z) \) by \( p(z) = z f'(z) / f(z) \). Then \( p(z) \) satisfies

\[
\frac{f(z)f''(z)}{f'(z)^2} = 1 + \frac{zp'(z)}{p(z)^2} - \frac{1}{p(z)},
\]

and so

\[
\frac{f(z)f''(z)}{f'(z)^2} - 1 = \frac{1}{p(z)} \left( -1 + \frac{zp'(z)}{p(z)} \right).
\]

If there exists a point \( z_0 \in U \) such that

\[
|\arg(p(z_0))| < \frac{\pi}{2} \alpha \quad \text{for } |z| < |z_0|
\]

and

\[
|\arg(p(z_0))| = \frac{\pi}{2} \alpha,
\]

then Lemma gives us that

(i) for the case \( \arg(p(z_0)) = (\pi/2)\alpha \),

\[
\arg \left( \frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1 \right) = \arg \left\{ \frac{1}{p(z_0)} \left( \frac{z_0p'(z_0)}{p(z_0)} - 1 \right) \right\}
\]

\[
= -\frac{\pi}{2} \alpha + \arg \left( -1 + \frac{z_0p'(z_0)}{p(z_0)} \right)
\]

\[
= -\frac{\pi}{2} \alpha + \arg(-1 + ik\alpha)
\]

\[
\leq \pi - \frac{\pi}{2} \alpha - \tan^{-1} \alpha.
\]

This contradicts our condition in the theorem.

(ii) for the case \( \arg(p(z_0)) = -(\pi/2)\alpha \), the application of the same method as in (i) shows that

\[
\arg \left( \frac{f(z_0)f''(z_0)}{f'(z_0)^2} - 1 \right) \geq \pi + \frac{\pi}{2} \alpha + \tan^{-1} \alpha.
\]

This also contradicts the assumption of the theorem. Thus we complete the proof of our main theorem.

Putting \( \alpha = 1 \) in Theorem, we have the following corollary.
Corollary. If \( f(z) \in A \) satisfies

\[
\frac{\pi}{4} < \arg \left( \frac{f(z)f''(z)}{f'(z)^2} - 1 \right) < \frac{7\pi}{4} \quad (z \in U),
\]

then \( f(z) \in S^* \).

References


Mamoru Nunokawa
Department of Mathematics
University of Gunma
Aramaki, Maebashi, Gunma 371-8510
Japan
e-mail: nunokawa@edu.gunma-u.ac.jp

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail: owa@math.kindai.ac.jp