

Sufficient conditions for Carathéodory functions

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Abstract. For Carathéodory functions $p(z)$ which are analytic in the open unit disk U with $p(0) = 1$, S.S.Miller (Bull. Amer. Math. Soc. 81(1975), 79-81) has shown some sufficient conditions applying the differential inequalities. The object of the present paper is to derive some improvements of results by S.S.Miller.

1 Introduction

Let A be the class of functions $p(z)$ of the form

$$p(z) = 1 + p_1z + p_2z^2 + \dots \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If $p(z)$ in A satisfies $\operatorname{Re} p(z) > 0$ for $z \in U$, then we say that $p(z)$ is the Carathéodory function. For Carathéodory functions, Miller [1] has given

Theorem A. *Let $p(z)$ be in the class A .*

- (i) *If $\operatorname{Re} \{p(z)^2 + zp'(z)\} > 0$ ($z \in U$), then $\operatorname{Re} p(z) > 0$ ($z \in U$).*
- (ii) *If $\operatorname{Re} \{p(z) + \alpha zp'(z)\} > 0$ ($z \in U$) for some α ($\alpha \geq 1$), then $\operatorname{Re} p(z) > 0$ ($z \in U$),*
- (iii) *If $p(z) \neq 0$ ($z \in U$) and $\operatorname{Re} \left\{ p(z) - \frac{zp'(z)}{p(z)^2} \right\} > 0$ ($z \in U$), then $\operatorname{Re} p(z) > 0$ ($z \in U$).*

Let $f(z)$ and $g(z)$ be analytic in U . If there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $f(z) = g(w(z))$, then $f(z)$ is said to be subordinate to $g(z)$ in U .

Mathematics Subject Classification (1991): 30C45

Key Words and Phrases: Analytic function, Carathéodory function, subordination.

We denote this subordination by $f(z) \prec g(z)$. We note that the subordination $f(z) \prec g(z)$ implies that $f(U) \subset g(U)$. Applying the subordination principles, we improve Theorem A by Miller [1]. To prove our results for Carathéodory functions, we have to recall here the following lemma due to Nunokawa [3] (also due to Miller and Mocanu [2]).

Lemma. *Let $p(z) \in A$ and suppose that there exists a point $z_0 \in U$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$ with $p(z_0) \neq 0$. Then we have*

$$z_0 p'(z_0) \leq -\frac{1}{2}(1 + a^2), \quad (1.2)$$

where $p(z_0) = ia$ ($a \neq 0$).

2 Subordination theorems for Carathéodory functions

Our first result for Carathéodory functions is contained in

Theorem 1. *Let $p(z) \in A$ and $w(z)$ be analytic in U with $w(0) = \alpha$ and $w(z) \neq k$ ($k \in \mathbb{R}, z \in U$). If*

$$\alpha p(z)^2 + \beta z p'(z) \prec w(z), \quad (2.1)$$

then $\operatorname{Re} p(z) > 0$ ($z \in U$), where $\beta > 0$, $\alpha \geq -\frac{\beta}{2}$, and $k \leq -\frac{\beta}{2}$.

Proof. Let us suppose that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0 \quad (p(z_0) \neq 0).$$

Then Lemma gives that $p(z_0) = ia$ ($a \neq 0$) and $z_0 p'(z_0) \leq -\frac{1}{2}(1 + a^2)$. It follows that

$$\begin{aligned} \alpha p(z_0)^2 + \beta z_0 p'(z_0) &= -\alpha a^2 + \beta z_0 p'(z_0) \\ &\leq -\frac{1}{2} \{ \beta + (2\alpha + \beta)a^2 \} \\ &\leq -\frac{\beta}{2}. \end{aligned} \quad (2.2)$$

Since $w(0) = \alpha$ and $w(e^{i\theta}) \leq -\frac{\beta}{2}$, the inequality (2.2) contradicts our condition (2.1). Therefore $\operatorname{Re} p(z) > 0$ for all $z \in U$. \square

Remark 1. Theorem 1 is the improvement of (i) of Theorem A by Miller [1].

Corollary 1. If $p(z) \in A$ satisfies

$$\alpha p(z)^2 + \beta zp'(z) \prec \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z} \right)^2 - \frac{\beta}{2}, \quad (2.3)$$

where $\beta > 0$ and $\alpha \geq -\frac{\beta}{2}$, then $\operatorname{Re} p(z) > 0$ ($z \in U$).

Proof. Taking

$$w(z) = \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z} \right)^2 - \frac{\beta}{2} \quad (2.4)$$

in Theorem 1, we see that $w(z)$ is analytic in U , $w(0) = \alpha$ and

$$w(e^{i\theta}) = \frac{2\alpha + \beta}{2} \left(\frac{1+e^{i\theta}}{1-e^{i\theta}} \right)^2 - \frac{\beta}{2} \leq -\frac{\beta}{2}. \quad (2.5)$$

Thus $w(z)$ satisfies the conditions in Theorem 1. \square

Theorem 2. Let $p(z) \in A$ and $w(z)$ be analytic in U with $w(0) = \alpha$ and $w(z) \neq ik$ ($k \in \mathbb{R}, z \in U$). If

$$\alpha p(z) + \beta \frac{zp'(z)}{p(z)} \prec w(z), \quad (2.6)$$

then $\operatorname{Re} p(z) > 0$ ($z \in U$), where $\alpha > 0$, $\beta > 0$, and $k^2 \geq \beta(2\alpha + \beta)$.

Proof. From the subordination (2.6), we have $p(z) \neq 0$ in U , because if $p(z)$ has a zero of order l at $z = z_0 \in U$, then we have $p(z) = (z - z_0)^l q(z)$, where $q(z)$ is analytic in U , $q(z_0) \neq 0$, and l is a positive integer.

Letting $z \rightarrow z_0$ such that

$$\arg(z - z_0) = \arg(z_0) - \frac{\pi}{2},$$

we have

$$\begin{aligned} \lim_{z \rightarrow z_0} \operatorname{Im} \left(\alpha p(z) + \beta \frac{zp'(z)}{p(z)} \right) &= \lim_{z \rightarrow z_0} \operatorname{Im} \left(\alpha p(z) + \frac{\beta z(lq(z) + (z - z_0)q'(z))}{(z - z_0)q(z)} \right) \\ &= +\infty. \end{aligned}$$

This contradicts (2.6) and so we conclude that $p(z) \neq 0$ for all $z \in U$.
We assume that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0.$$

Then using Lemma, we have

$$\begin{aligned} \alpha p(z_0) + \beta \frac{z_0 p'(z_0)}{p(z_0)} &= i\alpha a + \frac{\beta}{ia} z_0 p'(z_0) \\ &= i \left(\alpha a - \frac{\beta}{a} z_0 p'(z_0) \right) \\ &= iv, \end{aligned} \tag{2.7}$$

where v is real, because $z_0 p'(z_0) \leq -\frac{1}{2}(1+a^2)$. Furthermore, we have, if $a > 0$, then

$$\begin{aligned} v &\geq \alpha a + \frac{\beta}{2a}(1+a^2) \\ &\geq \sqrt{\beta(2\alpha + \beta)}, \end{aligned} \tag{2.8}$$

and if $a < 0$, then

$$\begin{aligned} v &\leq -\alpha b - \frac{\beta}{2b}(1+a^2) \quad (b = -a > 0) \\ &\leq -\sqrt{\beta(2\alpha + \beta)}. \end{aligned} \tag{2.9}$$

This contradicts our condition that $w(e^{i\theta}) = ik$ ($|k| \geq \sqrt{\beta(2\alpha + \beta)}$). Thus we conclude that $\operatorname{Re} p(z) > 0$ for all $z \in U$. \square

Using Theorem 2, we have the following corollary.

Corollary 2. *If $p(z) \in A$ satisfies*

$$p(z) + \frac{z p'(z)}{p(z)} < \frac{1 + 4z + z^2}{1 - z^2}, \tag{2.10}$$

then $\operatorname{Re} p(z) > 0$ ($z \in U$).

Proof. Let us consider the case of $\alpha = \beta = 1$ in Theorem 2. Defining the function $w(z)$ by

$$w(z) = \frac{1 + 4z + z^2}{1 - z^2}, \tag{2.11}$$

we know that $w(z)$ is analytic in U , $w(0) = 1$, and

$$w(e^{i\theta}) = \frac{2 + \cos \theta}{\sin \theta} i. \quad (2.12)$$

Letting

$$g(\theta) = \left(\frac{2 + \cos \theta}{\sin \theta} \right)^2 \quad (0 \leq \theta \leq 2\pi), \quad (2.13)$$

we have $g'(\theta) = 0$ when $\cos \theta = -\frac{1}{2}$.

It follows from the above that $g(\theta) \geq 3$, that is, that $w(z) \neq ik$ ($|k| \geq \sqrt{3}$). \square

Next, we derive

Theorem 3. *If $p(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \alpha p(z) - \beta \frac{zp'(z)}{p(z)^2} \right\} > -\frac{\beta}{2} \quad (z \in U) \quad (2.14)$$

for some $\alpha \geq 0$ and $\beta > 0$, then $\operatorname{Re} p(z) > 0$ ($z \in U$).

Proof. Applying the same method as the proof of Theorem 2, the condition (2.14) gives us that $p(z) \neq 0$ in U , because if $p(z)$ has a zero of order l at a point $z = z_0 \in U$, then we have $p(z) = (z - z_0)^l q(z)$, where $q(z)$ is analytic in U , $q(z_0) \neq 0$ and l is a positive integer. Letting $z \rightarrow z_0$ such that

$$\arg(z - z_0) = \frac{\arg(z_0) - \arg(q(z_0))}{l + 1},$$

we see that

$$\begin{aligned} \lim_{z \rightarrow z_0} \left(\alpha p(z) - \beta \frac{zp'(z)}{p(z)^2} \right) &= \lim_{z \rightarrow z_0} \left(\alpha p(z) - \beta \frac{lzq(z) + (z - z_0)zq'(z)}{(z - z_0)^{l+1}q(z)^2} \right) \\ &= -\infty. \end{aligned}$$

This contradicts our condition (2.14) and so we have $p(z) \neq 0$ in U .

By means of Lemma, if there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0,$$

then $p(z_0) = ia$ ($a \neq 0$) and $z_0 p'(z_0) \leq -\frac{1}{2}(1 + a^2)$.

This implies that

$$\operatorname{Re} \left\{ \alpha p(z_0) - \beta \frac{z_0 p'(z_0)}{p(z_0)^2} \right\} \leq -\frac{\beta}{2a^2}(1 + a^2) \leq -\frac{\beta}{2} \quad (2.15)$$

which contradicts our condition (2.14). Thus $\operatorname{Re} p(z) > 0$ for all $z \in U$. \square

Remark 2. Theorem 3 is the improvement of (iii) of Theorem A by Miller [1].

Finally we have

Corollary 3. If $p(z) \in A$ satisfies

$$\alpha p(z) - \beta \frac{zp'(z)}{p(z)^2} < \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z} \right)^2 - \frac{\beta}{2} \quad (2.16)$$

for some $\alpha \geq 0$ and $\beta > 0$, then $\operatorname{Re} p(z) > 0$ ($z \in U$).

Proof. Since the function

$$w(z) = \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z} \right)^2 - \frac{\beta}{2} \quad (2.17)$$

maps the open unit disk U onto the complex domain which has the slit

$$\delta = \left\{ w : \operatorname{Re}(w) < -\frac{\beta}{2} \right\},$$

the proof of Corollary 3 follows from the above. □

References

- [1] S.S.Miller, *Differential inequalities and Carathéodory functions*, Bull. Amer. Math. Soc. **81**(1975),79-81.
- [2] S.S.Miller and P.T.Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65**(1978),289-305.
- [3] M.Nunokawa, *On properties of Non-Carathéodory functions*, Proc. Japan. Acad. **68**(1992),152-153.

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