Coefficient Inequalities for Certain Analytic Functions

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Abstract. For real $\alpha (\alpha > 1)$, subclasses $M(\alpha)$ and $N(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in $U$ are introduced. The object of the present paper is to consider the coefficient inequalities for functions $f(z)$ to be in the classes $M(\alpha)$ and $N(\alpha)$. Further, the bounds of $\alpha$ for functions $f(z)$ to be starlike in $U$ are considered.

1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $M(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in U)$$

for some $\alpha (\alpha > 1)$. And let $N(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in U)$$

for some $\alpha (\alpha > 1)$. Then, we see that $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$. Let us give examples of functions $f(z)$ in the classes $M(\alpha)$ and $N(\alpha)$.

Remark For $1 < \alpha \leq \frac{4}{3}$, the classes $M(\alpha)$ and $N(\alpha)$ were introduced by Uralegaddi, Ganigi and Sarangi [2].

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Example (i) \( f(z) = z(1 - z)^{2(\alpha-1)} \in M(\alpha) \)

(ii) \( g(z) = \frac{1}{2\alpha - 1} \{1 - (1 - z)^{2\alpha-1}\} \in N(\alpha) \)

Proof. Since \( f(z) \in M(\alpha) \) if and only if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha,
\]

we can write

\[
\frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1} = \frac{1 + z}{1 - z},
\]

which is equivalent to

\[
\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{2(\alpha - 1)}{1 - z}.
\]

Integrating both sides of the above equality, we have

\[
f(z) = z(1 - z)^{2(\alpha-1)} \in M(\alpha).
\]

Next, since \( g(z) \in N(\alpha) \) if and only if \( zg'(z) \in M(\alpha) \),

\[
zg'(z) = z(1 - z)^{2(\alpha-1)}.
\]

for function \( g(z) \in N(\alpha) \). It follows that

\[
g(z) = \frac{-1}{2\alpha - 1} (1 - z)^{2\alpha-1} + \frac{1}{2\alpha - 1}
\]

\[
= \frac{1}{2\alpha - 1} \{1 - (1 - z)^{2\alpha-1}\} \in N(\alpha).
\]

2 Coefficient inequalities for the classes \( M(\alpha) \) and \( N(\alpha) \)

We try to derive sufficient conditions for \( f(z) \) which are given by using coefficient inequalities.
**Theorem 1.** If \( f(z) \in A \) satisfies

\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)
\]

for some \( \alpha (\alpha > 1) \), then \( f(z) \in M(\alpha) \).

**Proof.** Let us suppose that

\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)
\]

for \( f(z) \in A \).

It suffices to show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in U).
\]

We have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n||z|^{n-1}}{2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1||a_n||z|^{n-1}}
\]

\[
< \frac{\sum_{n=2}^{\infty} (n-1) |a_n|}{2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1||a_n|}.
\]

The last expression is bounded above by 1 if

\[
\sum_{n=2}^{\infty} (n-1) |a_n| \leq 2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1||a_n|
\]

which is equivalent to our condition

\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)
\]

of the theorem. This completes the proof of the theorem. \( \square \)

By using Theorem 1, we have
Corollary 1. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n\{(n-1) + |n - 2\alpha + 1|\}|a_n| \leq 2(\alpha - 1)
\]
for some \( \alpha(\alpha > 1) \), then \( f(z) \in N(\alpha) \).

Proof. From \( f(z) \in N(\alpha) \) if and only if \( zf'(z) \in M(\alpha) \), replacing \( a_n \) by \( na_n \) in Theorem 1 we have the corollary. \( \square \)

In view of Theorem 1 and Corollary 1, if \( 1 < \alpha \leq 2 \), then \( n - 2\alpha + 1 \geq 0 \) for all \( n \geq 2 \). Thus we have

Corollary 2. (i) If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq \alpha - 1
\]
for some \( \alpha(1 < \alpha \leq 2) \), then \( f(z) \in M(\alpha) \).

(ii) If \( f(z) \in A \), satisfies
\[
\sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leq \alpha - 1
\]
for some \( \alpha(1 < \alpha \leq 2) \), then \( f(z) \in N(\alpha) \).

3 Starlikeness for functions in \( M(\alpha) \) and \( N(\alpha) \)

By Silverman [1], we know that if \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n|a_n| \leq 1,
\]
then \( f(z) \in S^* \), where \( S^* \) denotes the subclasses of \( A \) consisting of all univalent and starlike functions \( f(z) \) in \( U \). Thus we have

Theorem 2. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq \alpha - 1
\]
for some $\alpha(1 < \alpha \leq \frac{4}{3})$, then $f(z) \in S^* \cap M(\alpha)$, therefore, $f(z)$ is starlike in $U$. Further, if $f(z) \in A$ satisfies
\[ \sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1 \]
for some $\alpha(1 < \alpha \leq \frac{3}{2})$, then $f(z) \in S^* \cap N(\alpha)$, therefore, $f(z)$ is starlike in $U$.

Proof. Let us consider $\alpha$ such that
\[ \sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha-1}|a_n| \leq 1. \]
Then we have $f(z) \in S^* \cap M(\alpha)$ by means of Theorem1. This inequality holds true if
\[ n \leq \frac{n-\alpha}{\alpha-1} \quad (n = 2, 3, 4, \ldots). \]
Therefore, we have
\[ 1 < \alpha \leq 2 - \frac{2}{n+1} \quad (n = 2, 3, 4, \ldots), \]
which shows $1 < \alpha \leq \frac{4}{3}$. Next, considering $\alpha$ such that
\[ \sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha-1}|a_n| \leq 1, \]
we have
\[ n \leq \frac{n(n-\alpha)}{\alpha-1} \quad (n = 2, 3, 4, \ldots). \]
which is equivalent to
\[ 1 < \alpha \leq \frac{n+1}{2} \quad (n = 2, 3, 4, \ldots). \]
This implies that $1 < \alpha \leq \frac{3}{2}$. \hfill \Box

Finally, by virtue of the result for convex functions by Silverman [1], we have if $f(z) \in A$ satisfies
\[ \sum_{n=2}^{\infty} n^2|a_n| \leq 1, \]
then $f(z) \in K$, where $K$ denotes the subclass of $A$ consisting of all univalent and convex functions $f(z)$ in $U$. Using the same manner as in the proof of Theorem2, we derive
Theorem 3. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1
\]
for some \( \alpha (1 < \alpha \leq \frac{4}{3}) \), then \( f(z) \in K \cap N(\alpha) \), therefore, \( f(z) \) is convex in \( U \).

4 Bounds of \( \alpha \) for starlikeness

Note that the sufficient for \( f(z) \) to be in the class \( M(\alpha) \) is given by
\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1).
\]

Since, if \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n|a_n| \leq 1,
\]
then \( f(z) \in S^* \) (cf. [1]), it is interesting to find the bounds of \( \alpha \) for starlikeness of \( f(z) \in M(\alpha) \).

To do this, we have to consider the following inequality
\[
\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{2(\alpha - 1)} \sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 1
\]
which is equivalent to
\[
\sum_{n=2}^{\infty} \{|n - 2\alpha + 1| + (3 - 2\alpha)n\} |a_n| \geq 0.
\]

Let us define
\[
F(n) = |n - 2\alpha + 1| + (3 - 2\alpha)n \quad (n \geq 2).
\]

Then, if \( F(n) \) satisfies
\[
\sum_{n=2}^{\infty} F(n)|a_n| \geq 0,
\]
then \( f(z) \) belongs to \( S^* \).

Theorem 4. Let \( f(z) \in A \) satisfy
\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)
\]
for some $\alpha > 1$. Further, let $\delta_k$ be defined by

$$
\delta_k = \sum_{n=k}^{\infty} F(n)|a_n|.
$$

Then,

(i) if $1 < \alpha \leq \frac{3}{2}$, then $f(z) \in S^*$,

(ii) if $\frac{3}{2} \leq \alpha \leq \min \left( \frac{13}{8}, \frac{3 + \delta_3}{2} \right)$, then $f(z) \in S^*$,

(iii) if $\frac{8}{3} \leq \alpha \leq \min \left( \frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12} \right)$, then $f(z) \in S^*$.

Proof. For $1 < \alpha \leq \frac{3}{2}$, we know that

$$
n - 2\alpha + 1 \geq 3 - 2\alpha \geq 0 \quad (n \geq 2),
$$

that is that $F(n) \geq 0 \ (n \geq 2)$. Therefore, we have

$$
\sum_{n=2}^{\infty} F(n)|a_n| \geq 0.
$$

If $\frac{3}{2} \leq \alpha \leq \frac{13}{8}$, then $F(2) = 3 - 2\alpha \leq 0$ and

$$
F(n) = 2n(2 - \alpha) + 1 - 2\alpha \\
\geq 13 - 8\alpha \\
\geq 0
$$

for $n \geq 3$. Further, we know that

$$
|a_n| \leq \frac{2(\alpha - 1)}{(n - 1) + |n - 2\alpha + 1|} \quad (n \geq 2),
$$

which given us that $|a_2| \leq 1$. therefore, we obtain that

$$
\sum_{n=2}^{\infty} F(n)|a_n| = F(2)|a_2| + \sum_{n=3}^{\infty} F(n)|a_n| \\
\geq 3 - 2\alpha + \delta_3 \geq 0
$$

for

$$
\frac{3}{2} \leq \alpha \leq \min \left( \frac{13}{8}, \frac{3 + \delta_3}{2} \right),
$$
Furthermore, if \( \frac{13}{8} \leq \alpha \leq \frac{17}{10} \), then

\[
F(2) = 3 - 2\alpha \leq 0
\]

\[
F(3) = |4 - 2\alpha| + 3(3 - 2\alpha)
= 13 - 8\alpha
\leq 0,
\]

and

\[
F(n) = |n - 2\alpha + 1| + (3 - 2\alpha)n
= 4n + 1 - 2(n + 1)\alpha
\geq \frac{3(n - 4)}{5}
\geq 0
\]

for \( n \geq 4 \). Noting that \( |a_2| \leq 1 \) and \( |a_3| \leq \frac{\alpha - 1}{3 - \alpha} \), we conclude that

\[
\sum_{n=2}^{\infty} F(n)|a_n| = F(2)|a_2| + F(3)|a_3| + \sum_{n=4}^{\infty} F(n)|a_n|
\geq (3 - 2\alpha) + (13 - 8\alpha)\frac{\alpha - 1}{3 - \alpha} + \delta_4
\geq 0.
\]

for \( \alpha \) which satisfies

\[
6\alpha^2 - (12 - \delta_4)\alpha + 4 - 3\delta_4 \leq 0.
\]

This shows that

\[
\frac{8}{3} \leq \alpha \leq \min \left( \frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12} \right),
\]

This completes the proof of Theorem 4.

Finally, by virtue of Theorem 4, we may suppose Conjecture. If \( f(z) \in A \) satisfies

\[
\sum_{n=2}^{\infty} \{(n - 1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)
\]

for some \( 1 < \alpha < 2 \), then \( f(z) \in S^* \). \( \square \)
References


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