<table>
<thead>
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<th>Title</th>
<th>On certain conditions for starlikeness (Study on Inverse Problems in Univalent Function Theory)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1192: 103-106</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64772">http://hdl.handle.net/2433/64772</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Coefficient Inequalities for Certain Analytic Functions

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Abstract. For real $\alpha (\alpha > 1)$, subclasses $M(\alpha)$ and $N(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in $U$ are introduced. The object of the present paper is to consider the coefficient inequalities for functions $f(z)$ to be in the classes $M(\alpha)$ and $N(\alpha)$. Further, the bounds of $\alpha$ for functions $f(z)$ to be starlike in $U$ are considered.

1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $M(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in U)$$

for some $\alpha (\alpha > 1)$. And let $N(\alpha)$ be the subclass of $A$ consisting of functions $f(z)$ which satisfy

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in U)$$

for some $\alpha (\alpha > 1)$. Then, we see that $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$. Let us give examples of functions $f(z)$ in the classes $M(\alpha)$ and $N(\alpha)$.

Remark For $1 < \alpha \leq \frac{4}{3}$, the classes $M(\alpha)$ and $N(\alpha)$ were introduced by Uralegaddi, Ganigi and Sarangi [2].
Example (i) \( f(z) = z(1 - z)^{2(\alpha - 1)} \in M(\alpha) \)

(ii) \( g(z) = \frac{1}{2\alpha - 1} \{1 - (1 - z)^{2\alpha - 1}\} \in N(\alpha) \)

Proof. Since \( f(z) \in M(\alpha) \) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha,
\]

we can write

\[
\frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1} = \frac{1 + z}{1 - z},
\]

which is equivalent to

\[
\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{2(\alpha - 1)}{1 - z}.
\]

Integrating both sides of the above equality, we have

\[
f(z) = z(1 - z)^{2(\alpha - 1)} \in M(\alpha).
\]

Next, since \( g(z) \in N(\alpha) \) if and only if \( zg'(z) \in M(\alpha) \),

\[
zg'(z) = z(1 - z)^{2(\alpha - 1)}.
\]

for function \( g(z) \in N(\alpha) \). It follows that

\[
g(z) = \frac{-1}{2\alpha - 1} (1 - z)^{2\alpha - 1} + \frac{1}{2\alpha - 1}
\]

\[
= \frac{1}{2\alpha - 1} \{1 - (1 - z)^{2\alpha - 1}\} \in N(\alpha).
\]

\[\square\]

2 Coefficient inequalities for the classes \( M(\alpha) \) and \( N(\alpha) \)

We try to derive sufficient conditions for \( f(z) \) which are given by using coefficient inequalities.
Theorem 1. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1| \} |a_n| \leq 2(\alpha - 1)
\]
for some \( \alpha (\alpha > 1) \), then \( f(z) \in M(\alpha) \).

Proof. Let us suppose that
\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1| \} |a_n| \leq 2(\alpha - 1)
\]
for \( f(z) \in A \).
It suffices to show that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in U).
\]
We have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1)|a_n||z|^{n-1}}{2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1||a_n||z|^{n-1}} < \frac{\sum_{n=2}^{\infty} (n-1)|a_n|}{2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1||a_n|}.
\]
The last expression is bounded above by 1 if
\[
\sum_{n=2}^{\infty} (n-1)|a_n| \leq 2(\alpha - 1) - \sum_{n=2}^{\infty} |n - 2\alpha + 1||a_n|
\]
which is equivalent to our condition
\[
\sum_{n=2}^{\infty} \{(n-1) + |n - 2\alpha + 1| \} |a_n| \leq 2(\alpha - 1)
\]
of the theorem. This completes the proof of the theorem. \( \square \)

By using Theorem 1, we have
Corollary 1. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n \{ (n-1) + |n - 2\alpha + 1| \} |a_n| \leq 2(\alpha - 1)
\]
for some \( \alpha (\alpha > 1) \), then \( f(z) \in N(\alpha) \).

Proof. From \( f(z) \in N(\alpha) \) if and only if \( zf'(z) \in M(\alpha) \), replacing \( a_n \) by \( na_n \) in Theorem 1 we have the corollary.

In view of Theorem 1 and Corollary 1, if \( 1 < \alpha \leq 2 \), then \( n - 2\alpha + 1 \geq 0 \) for all \( n \geq 2 \). Thus we have

Corollary 2. (i) If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1
\]
for some \( \alpha (1 < \alpha \leq 2) \), then \( f(z) \in M(\alpha) \).

(ii) If \( f(z) \in A \), satisfies
\[
\sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq \alpha - 1
\]
for some \( \alpha (1 < \alpha \leq 2) \), then \( f(z) \in N(\alpha) \).

3 Starlikeness for functions in \( M(\alpha) \) and \( N(\alpha) \)

By Silverman [1], we know that if \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n |a_n| \leq 1,
\]
then \( f(z) \in S^* \), where \( S^* \) denotes the subclsses of \( A \) consisting of all univalent and starlike functions \( f(z) \) in \( U \). Thus we have

Theorem 2. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1
\]
for some $\alpha(1 < \alpha \leq \frac{4}{3})$, then $f(z) \in S^* \cap M(\alpha)$, therefore, $f(z)$ is starlike in $U$. Further, if $f(z) \in A$ satisfies
\[ \sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leq \alpha - 1 \]
for some $\alpha(1 < \alpha \leq \frac{3}{2})$, then $f(z) \in S^* \cap N(\alpha)$, therefore, $f(z)$ is starlike in $U$.

**Proof.** Let us consider $\alpha$ such that
\[ \sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n - \alpha}{\alpha - 1}|a_n| \leq 1. \]
Then we have $f(z) \in S^* \cap M(\alpha)$ by means of Theorem 1. This inequality holds true if
\[ n \leq \frac{n - \alpha}{\alpha - 1} \quad (n = 2, 3, 4, \ldots). \]
Therefore, we have
\[ 1 < \alpha \leq 2 - \frac{2}{n + 1} \quad (n = 2, 3, 4, \ldots), \]
which shows $1 < \alpha \leq \frac{4}{3}$. Next, considering $\alpha$ such that
\[ \sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n(n - \alpha)}{\alpha - 1}|a_n| \leq 1, \]
we have
\[ n \leq \frac{n(n - \alpha)}{\alpha - 1} \quad (n = 2, 3, 4, \ldots). \]
which is equivalent to
\[ 1 < \alpha \leq \frac{n + 1}{2} \quad (n = 2, 3, 4, \ldots). \]
This implies that $1 < \alpha \leq \frac{3}{2}$. \qed

Finally, by virtue of the result for convex functions by Silverman [1], we have if $f(z) \in A$ satisfies
\[ \sum_{n=2}^{\infty} n^2|a_n| \leq 1, \]
then $f(z) \in K$, where $K$ denotes the subclass of $A$ consisting of all univalent and convex functions $f(z)$ in $U$. Using the same manner as in the proof of Theorem 2, we derive
Theorem 3. If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1
\]
for some \( \alpha (1 < \alpha \leq \frac{4}{3}) \), then \( f(z) \in K \cap N(\alpha) \), therefore, \( f(z) \) is convex in \( U \).

4 Bounds of \( \alpha \) for starlikeness

Note that the sufficient for \( f(z) \) to be in the class \( M(\alpha) \) is given by
\[
\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha - 1).
\]
Since, if \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n|a_n| \leq 1,
\]
then \( f(z) \in S^* \) (cf. [1]), it is interesting to find the bounds of \( \alpha \) for starlikeness of \( f(z) \in M(\alpha) \).

To do this, we have to consider the following inequality
\[
\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{2(\alpha - 1)} \sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 1
\]
which is equivalent to
\[
\sum_{n=2}^{\infty} \{|n-2\alpha+1|+(3-2\alpha)n\} |a_n| \geq 0.
\]
Let us define
\[
F(n) = |n-2\alpha+1|+(3-2\alpha)n \quad (n \geq 2).
\]
Then, if \( F(n) \) satisfies
\[
\sum_{n=2}^{\infty} F(n)|a_n| \geq 0,
\]
then \( f(z) \) belongs to \( S^* \).

Theorem 4. Let \( f(z) \in A \) satisfy
\[
\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha - 1)
\]
for some $\alpha > 1$. Further, let $\delta_k$ be defined by

$$
\delta_k = \sum_{n=k}^{\infty} F(n)|a_n|.
$$

Then,

(i) if $1 < \alpha \leq \frac{3}{2}$, then $f(z) \in S^*$,

(ii) if $\frac{3}{2} \leq \alpha \leq \min \left( \frac{13}{8}, \frac{3 + \delta_3}{2} \right)$, then $f(z) \in S^*$,

(iii) if $\frac{8}{3} \leq \alpha \leq \min \left( \frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12} \right)$, then $f(z) \in S^*$.

Proof. For $1 < \alpha \leq \frac{3}{2}$, we know that

$$
n - 2\alpha + 1 \geq 3 - 2\alpha \geq 0 \quad (n \geq 2),
$$

that is that $F(n) \geq 0 \ (n \geq 2)$. Therefore, we have

$$
\sum_{n=2}^{\infty} F(n)|a_n| \geq 0.
$$

If $\frac{3}{2} \leq \alpha \leq \frac{13}{8}$, then $F(2) = 3 - 2\alpha \leq 0$ and

$$
F(n) = 2n(2 - \alpha) + 1 - 2\alpha
\geq 13 - 8\alpha
\geq 0
$$

for $n \geq 3$. Further, we know that

$$
|a_n| \leq \frac{2(\alpha - 1)}{(n - 1) + |n - 2\alpha + 1|} \quad (n \geq 2),
$$

which given us that $|a_2| \leq 1$. Therefore, we obtain that

$$
\sum_{n=2}^{\infty} F(n)|a_n| = F(2)|a_2| + \sum_{n=3}^{\infty} F(n)|a_n|
\geq 3 - 2\alpha + \delta_3
\geq 0
$$

for

$$
\frac{3}{2} \leq \alpha \leq \min \left( \frac{13}{8}, \frac{3 + \delta_3}{2} \right),
$$
Furthermore, if \( \frac{13}{8} \leq \alpha \leq \frac{17}{10} \), then

\[
F(2) = 3 - 2\alpha \leq 0
\]

\[
F(3) = |4 - 2\alpha| + 3(3 - 2\alpha)
= 13 - 8\alpha
\leq 0,
\]

and

\[
F(n) = |n - 2\alpha + 1| + (3 - 2\alpha)n
= 4n + 1 - 2(n + 1)\alpha
\geq \frac{3(n - 4)}{5}
\geq 0
\]

for \( n \geq 4 \). Noting that \( |a_2| \leq 1 \) and \( |a_3| \leq \frac{\alpha - 1}{3 - \alpha} \), we conclude that

\[
\sum_{n=2}^{\infty} F(n)|a_n| = F(2)|a_2| + F(3)|a_3| + \sum_{n=4}^{\infty} F(n)|a_n|
\geq (3 - 2\alpha) + (13 - 8\alpha)\frac{\alpha - 1}{3 - \alpha} + \delta_4
\geq 0.
\]

for \( \alpha \) which satisfies

\[
6\alpha^2 - (12 - \delta_4)\alpha + 4 - 3\delta_4 \leq 0.
\]

This shows that

\[
\frac{8}{3} \leq \alpha \leq \min \left( \frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12} \right),
\]

This completes the proof of Theorem 4.

Finally, by virtue of Theorem 4, we may suppose Conjecture. If \( f(z) \in A \) satisfies

\[
\sum_{n=2}^{\infty} \{(n - 1) + |n - 2\alpha + 1|\} |a_n| \leq 2(\alpha - 1)
\]

for some \( 1 < \alpha < 2 \), then \( f(z) \in S^* \).

\(\square\)
References


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