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Sufficient conditions for starlikeness

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Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be starlike.

1 Introduction.
Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in $A$ is said to be starlike of order $\alpha$ in $U$ if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U).$$

We denote by $S^*(\alpha)$ the subclass of $A$ consisting of all functions $f(z)$ which are starlike of order $\alpha$ in $U$. We denote by $S^*(0) \equiv S^*$.

Lewandowski, Miller and Zlotkiewicz [1] have shown

**Theorem A.** If $f(z) \in A$ satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} \left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.

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Recently, Ramesha, Kumar and Padmanabhan [5] have given

**Theorem B.** If $f(z) \in A$ satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U)$$

for some $\alpha (\alpha \geq 0)$, then $f(z) \in S^*$.  

On the other hand, Obradović [4] has proved

**Theorem C.** If $f(z) \in A$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{1}{6} \quad (z \in U),$$

then $f(z) \in S^*$.  

Further, more recently, Li and Owa [2] have derived

**Theorem D.** If $f(z) \in A$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{3}{2} \quad (z \in U),$$

then $f(z) \in S^*$.  

To derive our theorems, we need the following lemma due to Miller and Mocanu [3].

**Lemma.** Let $\Omega$ be a set in the complex plane $\mathbb{C}$. Suppose that $\Phi$ is a mapping from $\mathbb{C}^2 \times U$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in U$, and for all real $x, y$ such that $y \leq -(1 + x^2)/2$. If the function $p(z)$ is analytic in $U$ with $p(0) = 1$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$, then $\text{Re}(p(z)) > 0 \quad (z \in U)$.  

2 Conditions for starlikeness

In this section, we derive some sufficient conditions for starlikeness, which are the improvements of the previous theorems. Our first result is contained in

**Theorem 1.** If \( f(z) \in A \) satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2} \quad (z \in U)
\]  

(2.1)

for some \( \alpha (\alpha \geq 0) \), then \( f(z) \in S^* \).

**Proof.** Let us define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1z + p_2z^2 + \cdots
\]  

(2.2)

Making use of the logarithmic differentiations of both sides in (2.2), we know that

\[
\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \alpha zp'(z) + \alpha p(z)^2 + (1 - \alpha)p(z).
\]  

(2.3)

Let \( \Omega = \{ w \in \mathbb{C} : \text{Re}(w) > -\alpha/2 \} \) and

\[
\Phi(z_1, z_2; z) = \alpha z_2 + \alpha z_1^2 + (1 - \alpha)z_1.
\]

Then from (2.1) and (2.3), we have \( \Phi(p(z), zp'(z); z) \in \Omega \) for all \( z \in U \). Further, we have

\[
\text{Re} \{ \Phi(ix, y; z) \} = \alpha y - \alpha x^2
\]

\[
\leq -\frac{\alpha}{2} - \frac{3}{2}\alpha x^2
\]

\[
\leq -\frac{\alpha}{2}.
\]

This shows that \( \Phi(ix, y; z) \in \Omega \). Therefore, by virtue of Lemma, we conclude that \( f(z) \in S^* \).

Letting \( \alpha = 1 \) in Theorem 1, we have
Corollary 1. If \( f(z) \in A \) satisfies
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{1}{2} \quad (z \in U),
\]
then \( f(z) \in S^* \).

Next we derive

Theorem 2. If \( f(z) \in A \) satisfies
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha^2}{4} (1 - \alpha) \quad (z \in U) \quad (2.4)
\]
for some \( 0 \leq \alpha < 2 \), then \( f(z) \in S^*(\alpha/2) \).

\textbf{Proof} Define the function \( p(z) \) by
\[
\frac{zf'(z)}{f(z)} = \left( 1 - \frac{\alpha}{2} \right) p(z) + \frac{\alpha}{2} \quad (z \in U). \quad (2.5)
\]
Then \( p(z) \) is analytic in \( U \) and \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \). Differentiating (2.6) logarithmically, we see that
\[
\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \alpha \left( 1 - \frac{\alpha}{2} \right) zp'(z) + \alpha \left( 1 - \frac{\alpha}{2} \right)^2 p(z)^2
\]
\[
+ \left( 1 - \frac{\alpha}{2} \right) (\alpha^2 + 1 - \alpha)p(z) + \frac{\alpha^3}{4} + \frac{\alpha}{2} (1 - \alpha). \quad (2.6)
\]
Let us define
\[
\Omega = \left\{ w \in \mathbb{C} : \Re(w) > -\frac{(1-\alpha)\alpha^2}{4} \right\}
\]
and
\[
\Phi(z_1, z_2; z) = \alpha \left( 1 - \frac{\alpha}{2} \right) z_2 + \alpha \left( 1 - \frac{\alpha}{2} \right)^2 \frac{z_1^2}{2} + \left( 1 - \frac{\alpha}{2} \right) (\alpha^2 - \alpha + 1)z_1 + \frac{\alpha^3}{4} + \frac{\alpha}{2} (1 - \alpha).
\]
Then by (2.4) and (2.6), we know that \( \Phi(p(z), zp'(z); z) \in \Omega \). Further, for all \( z \in U \) and for all real \( x, y \) such that \( y \leq -(1 + x^2)/2 \), we have
\[
\Re \{ \Phi(ix, y; z) \} = \alpha \left( 1 - \frac{\alpha}{2} \right) y - \alpha \left( 1 - \frac{\alpha}{2} \right)^2 x^2 + \frac{\alpha^3}{4} + \frac{\alpha}{2} (1 - \alpha)
\]
Thus, by applying Lemma, we have $\Re(p(z)) > 0$ for $z \in U$, which, in view of (2.5), is equivalent to $f(z) \in S^*(\alpha/2)$.

If we take $\alpha - 1$ in Theorem 2, then we have

**Corollary 2.** If $f(z) \in A$ satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*(1/2)$.

Finally, we consider

**Theorem 3.** If $f(z) \in A$ satisfies

$$\left| \left( \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} + 1 \right) \right) \right| < \rho \quad (z \in U),$$

where

$$\rho = \left( \frac{827 + 73\sqrt{73}}{288} \right)^{\frac{1}{2}} = 2.2443697 \cdots,$$

then $f(z) \in S^*$.

**Proof.** Let the function $p(z)$ be defined by (2.2). Then it follows that

$$\frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} + 1 \right) = (p(z) - 1) \left( \frac{zp'(z)}{p(z)} + p(z) - 1 \right). \quad (2.7)$$

Letting $\Omega = \{w \in \mathbb{C}: |w| < \rho\}$ and

$$\Phi(z_1, z_2 : z) = (z_1 - 1) \left( \frac{z_2}{z_1} + z_1 - 1 \right),$$
we have $\Phi(p(z), zp'(z); z) \in \Omega$. Further, for all $z \in U$, and for all real $x, y$ with $y \leq -(1 + x^2)/2$, $\Phi(p(z), zp'(z); z)$ satisfies

$$|\Phi(ix, y; z)| = \sqrt{(1 + x^2) \left(1 + \frac{(x^2 - y)^2}{x^2}\right)} \equiv g(x^2, y), \quad (2.8)$$

where $t = x^2 > 0$ and $y \leq -(1 + t)/2$. Since

$$\frac{\partial g(t, y)}{\partial y} = 2 \frac{1 + t}{t} (y - t) < 0,$$

we have

$$g(t, y) \geq g(t, \frac{1 + t}{t}) = \frac{(t + 1)^2(9t + 1)}{4t} \equiv h(t). \quad (2.9)$$

Further, since

$$h'(t) = \frac{(t + 1) \left(t + \frac{\sqrt{73} + 1}{36}\right) \left(t - \frac{\sqrt{73} - 1}{36}\right)}{4t^2},$$

we obtain

$$\min_{t>0} h(t) = h\left(\frac{\sqrt{73} - 1}{36}\right) = \frac{827 + 73\sqrt{73}}{288} = \rho^2. \quad (2.10)$$

This implies that $|\Phi(ix, y; z)| \geq \rho$. It follows from (2.8), (2.9) and (2.10) that $\Phi(ix, y; z) \notin \Omega$. An application of Lemma gives us that $\text{Re}(p(z)) > 0$ for $z \in U$. Thus we conclude that $f(z) \in S^*$. 

References


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