数理解析研究所講究録
1192 巻 2001 年 87-93

Sufficient conditions for starlikeness

Jian-Lin Li and SHIGEYOSHI OWA

Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be starlike.

1 Introduction.

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in A is said to be starlike of order α in U if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U).$$

We denote by $S^*(\alpha)$ the subclass of A consisting of all functions $f(z)$ which are starlike of order α in U. We denote by $S^*(0) \equiv S^*$.
Lewandowski, Miller and Zlotkiewicz [1] have shown

Theorem A. *If $f(z) \in A$ satisfies*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.

Mathematics Subject Classification (1991): 30C45
Key Wards and Phrases: Analytic, starlikeness.

Recently, Ramesha, Kumar and Padmanabhan [5] have given

Theorem B. *If* $f(z) \in A$ *satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > 0 \quad (z \in U)$$

for some $\alpha(\alpha \geqq 0)$, *then* $f(z) \in S^*$.

On the other hand, Obradović [4] has proved

Theorem C. *If* $f(z) \in A$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right| < \frac{1}{6} \quad (z \in U),$$

then $f(z) \in S^*$.

Further, more recently, Li and Owa [2] have derived

Theorem D. *If* $f(z) \in A$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right| < \frac{3}{2} \quad (z \in U),$$

then $f(z) \in S^*$.

To derive our theorems, we need the following lemma due to Miller and Mocanu [3].

Lemma. *Let* Ω *be a set in the complex plane* $\mathbb{C}$. *Suppose that* Φ *is a mapping from* $\mathbb{C}^2\text{x}U$ *to* $\mathbb{C}$ *which satisfies* $\Phi(ix, y; z) \notin \Omega$ *for* $z \in U$, *and for all real* x, y *such that* $y \leqq -(1+x^2)/2$. *If the function* $p(z)$ *is analytic in* U *with* $p(0) = 1$ *and* $\Phi(p(z), zp'(z); z) \in \Omega$ *for all* $z \in U$, *then* $\operatorname{Re}(p(z)) > 0$ $(z \in U)$.

2 Conditions for starlikeness

In this section, we derive some sufficient conditions for starlikeness, which are the improvements of the previous theorems. Our first result is contained in

Theorem 1. *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{\alpha}{2} \quad (z \in U) \tag{2.1}$$

for some α ($\alpha \geqq 0$), then $f(z) \in S^*$.

Proof. Let us define the analytic function $p(z)$ in U by

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1 z + p_2 z^2 + \cdots . \tag{2.2}$$

Making use of the logarithmic differentiations of both sides in (2.2), we know that

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) = \alpha z p'(z) + \alpha p(z)^2 + (1-\alpha)p(z). \tag{2.3}$$

Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re}(w) > -\alpha/2\}$ and

$$\Phi(z_1, z_2; z) = \alpha z_2 + \alpha z_1^2 + (1-\alpha)z_1.$$

Then from (2.1) and (2.3), we have $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$. Further, we have

$$\begin{aligned}\operatorname{Re}\{\Phi(ix, y; z)\} &= \alpha y - \alpha x^2 \\ &\leqq -\frac{\alpha}{2} - \frac{3}{2}\alpha x^2 \\ &\leqq -\frac{\alpha}{2}.\end{aligned}$$

This shows that $\Phi(ix, y; z) \notin \Omega$. Therefore, by virtue of Lemma, we conclude that $f(z) \in S^*$.

Letting $\alpha = 1$ in Theorem 1, we have

Corollary 1. *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{1}{2} \quad (z \in U),$$

then $f(z) \in S^*$.

Next we derive

Theorem 2. *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{\alpha^2}{4}(1-\alpha) \quad (z \in U) \tag{2.4}$$

for some $\alpha (0 \leqq \alpha < 2)$, then $f(z) \in S^*(\alpha/2)$.

Proof Define the function $p(z)$ by

$$\frac{zf'(z)}{f(z)} = \left(1-\frac{\alpha}{2}\right)p(z) + \frac{\alpha}{2} \quad (z \in U). \tag{2.5}$$

Then $p(z)$ is analytic in U and $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Differentiating (2.6) logarithmically, we see that

$$\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) = \alpha\left(1-\frac{\alpha}{2}\right)zp'(z) + \alpha\left(1-\frac{\alpha}{2}\right)^2 p(z)^2$$

$$+ \left(1-\frac{\alpha}{2}\right)(\alpha^2+1-\alpha)p(z) + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1-\alpha). \tag{2.6}$$

Let us define

$$\Omega = \left\{w \in \mathbb{C} : \operatorname{Re}(w) > -\frac{(1-\alpha)\alpha^2}{4}\right\}$$

and

$$\Phi(z_1, z_2; z) = \alpha\left(1-\frac{\alpha}{2}\right)z_2 + \alpha\left(1-\frac{\alpha}{2}\right)^2 z_1^2 + \left(1-\frac{\alpha}{2}\right)(\alpha^2-\alpha+1)z_1 + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1-\alpha).$$

Then by (2.4) and (2.6), we know that $\Phi(p(z), zp'(z); z) \in \Omega$. Further, for all $z \in U$ and for all real x, y such that $y \leqq -(1+x^2)/2$, we have

$$\operatorname{Re}\{\Phi(ix, y; z)\} = \alpha\left(1-\frac{\alpha}{2}\right)y - \alpha\left(1-\frac{\alpha}{2}\right)^2 x^2 + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1-\alpha)$$

$$\leqq \frac{\alpha^2}{4}(\alpha-1) - \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)(3-\alpha)x^2$$
$$\leqq -\frac{\alpha^2}{4}(1-\alpha).$$

Thus, by applying Lemma, we have $\mathrm{Re}(p(z)) > 0$ for $z \in U$, which, in view of (2.5), is equivalent to $f(z) \in S^*(\alpha/2)$.

If we take $\alpha - 1$ in Theorem 2, then we have

Corollary 2. *If $f(z)$ inA satisfies*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)\right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*(1/2)$.

Finally, we consider

Theorem 3. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right| < \rho \quad (z \in U),$$

where

$$\rho = \left(\frac{827+73\sqrt{73}}{288}\right)^{\frac{1}{2}} = 2.2443697\cdots,$$

then $f(z) \in S^*$.

Proof. Let the function $p(z)$ be defined by (2.2). Then it follows that

$$\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right) = (p(z)-1)\left(\frac{zp'(z)}{p(z)}+p(z)-1\right). \qquad (2.7)$$

Letting $\Omega = \{w \in \mathbb{C}: |w| < \rho\}$ and

$$\Phi(z_1, z_2 : z) = (z_1 - 1)\left(\frac{z_2}{z_1} + z_1 - 1\right),$$

we have $\Phi(p(z), zp'(z) : z) \in \Omega$. Further, for all $z \in U$, and for all real x, y with $y \leqq -(1+x^2)/2$, $\Phi(p(z), zp'(z); z)$ satisfies

$$|\Phi(ix, y; z)| = \sqrt{(1+x^2)\left(1+\frac{(x^2-y)^2}{x^2}\right)} \equiv \sqrt{g(x^2, y)}, \qquad (2.8)$$

where $t = x^2 > 0$ and $y \leqq -(1+t)/2$. Since

$$\frac{\partial g(t,y)}{\partial y} = 2\frac{1+t}{t}(y-t) < 0,$$

we have

$$g(t,y) \geqq g\left(t, -\frac{1+t}{t}\right) = \frac{(t+1)^2(9t+1)}{4t} \equiv h(t). \qquad (2.9)$$

Further, since

$$h'(t) = \frac{(t+1)\left(t+\frac{\sqrt{73}+1}{36}\right)\left(t-\frac{\sqrt{73}-1}{36}\right)}{4t^2},$$

we obtain

$$\min_{t>0} h(t) = h\left(\frac{\sqrt{73}-1}{36}\right) = \frac{827+73\sqrt{73}}{288} = \rho^2. \qquad (2.10)$$

This implies that $|\Phi(ix, y; z)| \geqq \rho$. It follows from (2.8), (2.9) and (2.10) that $\Phi(ix, y; z) \notin \Omega$. An application of Lemma gives us that $\mathrm{Re}(p(z)) > 0$ for $z \in U$. Thus we conclude that $f(z) \in S^*$.

References

[1] Z. Lewandowski, S. S. Miller and E. Zlotkiewicz, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc.**56**(1976), 111 - 117.

[2] Jian-Lin Li and S. Owa, *Properties of the Salagean operator*, Georgian Math. J.**5**(1998), 361 - 366.

[3] S. S. Miller and P. T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Diff. Eqns. **67**(1987), 199 - 211.

[4] M. Obradović, *Ruscheweyh derivatives and some classes of univalent functions*, Current Topics in Analytic Function Theory (H. M. Srivastava and S. Owa,Editors), 220 - 233, Word Scientfic Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[5] C. Ramesha, S, Kumar and K. S. Padmanabhan, *A sufficient condition for starlikeness*, Chinese J. Math. **23**(1995), 167 - 171.

Jian-Lin Li
Department of Applied Mathematics
Northwestern Polytechnical University
Xi An, Shaan Xi 710072
People's Republic of China

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan