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<thead>
<tr>
<th>Title</th>
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</tr>
</thead>
<tbody>
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<td>Li, Jian-Lin; Owa, Shigeyoshi</td>
</tr>
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</tr>
</tbody>
</table>

Kyoto University
Sufficient conditions for starlikeness

Jian-Lin Li and SHIGEYOSHI OWA

Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be starlike.

1 Introduction.
Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. A function $f(z)$ in $A$ is said to be starlike of order $\alpha$ in $U$ if it satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U).$$

We denote by $S^*(\alpha)$ the subclass of $A$ consisting of all functions $f(z)$ which are starlike of order $\alpha$ in $U$. We denote by $S^*(0) \equiv S^*$.

Lewandowski, Miller and Zlotkiewicz [1] have shown

**Theorem A.** If $f(z) \in A$ satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.
Recently, Ramesha, Kumar and Padmanabhan [5] have given

**Theorem B.** If \( f(z) \in A \) satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U)
\]

for some \( \alpha (\alpha \geq 0) \), then \( f(z) \in S^* \).

On the other hand, Obradović [4] has proved

**Theorem C.** If \( f(z) \in A \) satisfies

\[
\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{1}{6} \quad (z \in U),
\]

then \( f(z) \in S^* \).

Further, more recently, Li and Owa [2] have derived

**Theorem D.** If \( f(z) \in A \) satisfies

\[
\left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{3}{2} \quad (z \in U),
\]

then \( f(z) \in S^* \).

To derive our theorems, we need the following lemma due to Miller and Mocanu [3].

**Lemma.** Let \( \Omega \) be a set in the complex plane \( \mathbb{C} \). Suppose that \( \Phi \) is a mapping from \( \mathbb{C} \times U \) to \( \mathbb{C} \) which satisfies \( \Phi(ix, y; z) \notin \Omega \) for \( z \in U \), and for all real \( x, y \) such that \( y \leq -(1 + x^2)/2 \). If the function \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \) and \( \Phi(p(z), zp'(z); z) \in \Omega \) for all \( z \in U \), then \( \text{Re}(p(z)) > 0 \) (\( z \in U \)).
2 Conditions for starlikeness

In this section, we derive some sufficient conditions for starlikeness, which are the improvements of the previous theorems. Our first result is contained in

Theorem 1. If \( f(z) \in A \) satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2} \quad (z \in U)
\]

(2.1)

for some \( \alpha (\alpha \geq 0) \), then \( f(z) \in S^* \).

Proof. Let us define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1 z + p_2 z^2 + \cdots
\]

(2.2)

Making use of the logarithmic differentiations of both sides in (2.2), we know that

\[
\frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \alpha z p'(z) + \alpha p(z)^2 + (1 - \alpha) p(z).
\]

(2.3)

Let \( \Omega = \{ w \in \mathbb{C} : \text{Re}(w) > -\alpha/2 \} \) and

\[
\Phi(z_1, z_2; z) = \alpha z_2 + \alpha z_1^2 + (1 - \alpha) z_1.
\]

Then from (2.1) and (2.3), we have \( \Phi(p(z), zp'(z); z) \in \Omega \) for all \( z \in U \). Further, we have

\[
\text{Re} \{ \Phi(ix, y; z) \} = \alpha y - \alpha x^2
\]

\[
\leq -\frac{\alpha}{2} - \frac{3}{2} \alpha x^2
\]

\[
\leq -\frac{\alpha}{2}.
\]

This shows that \( \Phi(ix, y; z) \in \Omega \). Therefore, by virtue of Lemma, we conclude that \( f(z) \in S^* \).

Letting \( \alpha = 1 \) in Theorem 1, we have
Corollary 1. If $f(z) \in A$ satisfies
\[
\text{Re}\left\{\frac{zf'(z)}{f(z)} + 1\right\} > -\frac{1}{2} (z \in U),
\]
then $f(z) \in S^*$. 

Next we derive

**Theorem 2.** If $f(z) \in A$ satisfies
\[
\text{Re}\left\{\frac{zf'(z)}{f(z)} + 1\right\} > -\frac{\alpha^2}{4} (1 - \alpha) (z \in U)
\]
for some $\alpha (0 \leq \alpha < 2)$, then $f(z) \in S^*(\alpha/2)$.

**Proof** Define the function $p(z)$ by
\[
\frac{zf'(z)}{f(z)} = (1 - \frac{\alpha}{2}) p(z) + \frac{\alpha}{2} (z \in U).
\]
Then $p(z)$ is analytic in $U$ and $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Differentiating (2.6) logarithmically, we see that
\[
\frac{zf'(z)}{f(z)} + 1 = \alpha (1 - \frac{\alpha}{2}) z p'(z) + \alpha (1 - \frac{\alpha}{2})^2 p(z)^2
\]
\[+ (1 - \frac{\alpha}{2}) (\alpha^2 + 1 - \alpha) p(z) + \frac{\alpha^3}{4} + \frac{\alpha}{2} (1 - \alpha).
\]
Let us define
\[
\Omega = \left\{ w \in \mathbb{C} : \text{Re}(w) > \frac{(1-\alpha)\alpha^2}{4} \right\}
\]
and
\[
\Phi(z_1, z_2; z) = \alpha \left(1 - \frac{\alpha}{2}\right) z_2 + \alpha \left(1 - \frac{\alpha}{2}\right)^2 z_1^2 + \alpha \left(1 - \frac{\alpha}{2}\right) z_1^2 + \frac{\alpha^3}{4} + \frac{\alpha}{2} (1 - \alpha).
\]
Then by (2.4) and (2.6), we know that $\Phi(p(z), z p'(z); z) \in \Omega$. Further, for all $z \in U$ and for all real $x, y$ such that $y \leq -(1 + x^2)/2$, we have
\[
\text{Re} \{\Phi(ix, y; z)\} = \alpha \left(1 - \frac{\alpha}{2}\right) y - \alpha \left(1 - \frac{\alpha}{2}\right)^2 x^2 + \frac{\alpha^3}{4} + \frac{\alpha}{2} (1 - \alpha).
\]
\[ \leq \frac{\alpha^2}{4} (\alpha - 1) - \frac{\alpha}{2} \left( 1 - \frac{\alpha}{2} \right) (3 - \alpha) \leq \leq -\frac{\alpha^2}{4} (1 - \alpha). \]

Thus, by applying Lemma, we have \( \text{Re}(p(z)) > 0 \) for \( z \in U \), which, in view of (2.5), is equivalent to \( f(z) \in S^{*}(\alpha/2) \).

If we take \( \alpha - 1 \) in Theorem 2, then we have

**Corollary 2.** If \( f(z) \) in \( A \) satisfies

\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U), \]

then \( f(z) \in S^{*}(1/2) \).

Finally, we consider

**Theorem 3.** If \( f(z) \in A \) satisfies

\[ \left| \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho \quad (z \in U), \]

where

\[ \rho = \left( \frac{827 + 73\sqrt{73}}{288} \right)^{\frac{1}{2}} = 2.2443697 \cdots, \]

then \( f(z) \in S^{*} \).

**Proof.** Let the function \( p(z) \) be defined by (2.2). Then it follows that

\[ \frac{zf''(z)}{f'(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) = (p(z) - 1) \left( \frac{zp'(z)}{p(z)} + p(z) - 1 \right). \quad (2.7) \]

Letting \( \Omega = \{ w \in \mathbb{C} : |w| < \rho \} \) and

\[ \Phi(z_1, z_2 : z) = (z_1 - 1) \left( \frac{z_2}{z_1} + z_1 - 1 \right), \]
we have $\Phi(p(z),zp'(z) : z) \in \Omega$. Further, for all $z \in U$, and for all real $x, y$ with $y \leq -(1 + x^2)/2$, $\Phi(p(z),zp'(z); z)$ satisfies

$$|\Phi(ix, y; z)| = \sqrt{(1 + x^2) \left(1 + \frac{(x^2 - y)^2}{x^2}\right)} \equiv g(x^2, y),$$  \hspace{1cm} (2.8)

where $t = x^2 > 0$ and $y \leq -(1 + t)/2$. Since

$$\frac{\partial g(t,y)}{\partial y} = 2 \frac{1 + t}{t} (y - t) < 0,$$

we have

$$g(t,y) \geq g\left(t, \frac{1 + t}{t}\right) = \frac{(t + 1)^2(9t + 1)}{4t} \equiv h(t).$$  \hspace{1cm} (2.9)

Further, since

$$h'(t) = \frac{(t + 1) \left(t + \frac{\sqrt{73} + 1}{36}\right) \left(t - \frac{\sqrt{73} - 1}{36}\right)}{4t^2},$$

we obtain

$$\min_{t > 0} h(t) = h\left(\frac{\sqrt{73} - 1}{36}\right) = \frac{827 + 73\sqrt{73}}{288} = \rho^2.$$  \hspace{1cm} (2.10)

This implies that $|\Phi(ix, y; z)| \geq \rho$. It follows from (2.8), (2.9) and (2.10) that $\Phi(ix, y; z) \notin \Omega$. An application of Lemma gives us that $\text{Re}(p(z)) > 0$ for $z \in U$. Thus we conclude that $f(z) \in S^*$. 

References


Jian-Lin Li  
*Department of Applied Mathematics*  
*Northwestern Polytechnical University*  
*Xi An, Shaan Xi 710072*  
*People's Republic of China*  

Shigeyoshi Owa  
*Department of Mathematics*  
*Kinki University*  
*Higashi-Osaka, Osaka 577-8502*  
*Japan*