数理解析研究所講究録
1192 巻 2001 年 75-86

THE RADIUS OF β-CONVEXITY FOR THE CLASSES OF λ-SPIRALLIKE ORDER α FUNCTIONS

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ABSTRACT. We get sharp bounds for the radius of β-convexity for the classes of λ-spirallike of order α and p-fold λ-spirllike of order α functions.

1. Introduction

Let A denote the class of functions of the form

$$s(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in unit disk $D = \{z : |z| < 1\}$. And let S denote the subclass of A consisting of analytic and univalent function $s(z)$ in unit disk D.

A function $s(z)$ in S is said to be starlike if

$$Re\left\{\frac{zs'(z)}{s(z)}\right\} > 0 \qquad (z \in D). \tag{1.2}$$

We denote by S^* the class of all starlike functions. A function $s(z)$ in S is said to be convex if

$$Re\left\{1 + \frac{zs''(z)}{s'(z)}\right\} > 0 \qquad (z \in D). \tag{1.3}$$

And we denote by K the class of all convex functions. These classes S^*

2000 AMS Subject Classification : 30C45.

Key words and phrases. radius of β-convexity, γ-spirallike order α function and p-fold univalent function.

Definition 1.1. A function $s(z)$ in S is said to be λ-spirallike if

$$Re\left\{e^{i\lambda}z\frac{s'(z)}{s(z)}\right\} > 0 \qquad (z \in D) \tag{1.4}$$

for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$. The class of these functions is denoted by S_λ^*

Definition 1.2. A function $s(z)$ in S is said to be λ-spirallike of order α if

$$Re\left\{e^{i\lambda}z\frac{s'(z)}{s(z)}\right\} > \alpha\cos\lambda \qquad (z \in D) \tag{1.5}$$

for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$ and α $(0 \le \alpha < 1)$. The above classes were introduced by Spacek ([12]). For $\lambda = 0$ in (1.4) the class is a starlike function (1.2).

Definition 1.3. Let F denote a non-empty collection of functions $s(z)$ each of which is univalent in D, and let β be given $0 \le \beta \le 1$. Then the real number

$$R_\alpha(F) = \sup\{R \mid Re\ J(\beta, s(z)) > 0, |z| < R, s(z) \in F\} \tag{1.6}$$

is called the radius of β-convexity of F, where $J(\beta, s(z))$ is defined by the relation,

$$J(\beta, s(z)) = (1-\beta)z\frac{s'(z)}{s(z)} + \beta\left(1 + z\frac{s''(z)}{s'(z)}\right). \tag{1.7}$$

The radius of β-convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). For $\beta = 0$ and $\beta = 1$ in (1.7), we define a starlike function (1.2) and a convex function (1.3), respectively.

Definition 1.4. Consider a function $s(z) = z + a_2z^2 + a_3z^3 + \cdots$ which is univalent in U. Then the function defined by the relation.

$$f(z) = (s(z^p))^{\frac{1}{p}} = z + \sum_{n=1}^{\infty} a_{np+1}z^{np+1} \tag{1.8}$$

is also univalent in U, and $f(z)$ is called p-fold univalent function. If the function $f(z)$ defined by the relation (1.8) satisfies the collection

$$Re\left\{e^{i\lambda}z\frac{f'(z)}{f(z)}\right\} > 0 \qquad (z \in D), \tag{1.9}$$

then the function $f(z)$ is called a p-fold λ-spirallike function in U, for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$ ([1]), and the class of these functions is denoted by $S^*_{\lambda p}$. And also we can define a p-fold λ-spirallike function of order α in U, denoted by $S^*_{\lambda p}(\alpha)$.

The radius of β-convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). There are many open problems about the radius of starlikeness, convexity and β-convexity for the classes S ([1]). So, we get sharp bounds for the radius of β-convexity for the classes of λ-spirallike of order α and p-fold λ-spirllike of order α functions.

2. The radius of β-convexity

Lemma 2.1 ([5]). *If $s(z) \in S^*_\lambda(\alpha)$, then*

$$\left|z\frac{s'(z)}{s(z)} - \frac{1 + \{2(1-\alpha)\cos\lambda e^{-i\lambda} - 1\}r^2}{1-r^2}\right| \leq \frac{2(1-\alpha)r\cos\lambda}{1-r^2}. \tag{2.1}$$

Lemma 2.2 ([10]). *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ *is analytic in* D, *and satisfies the conditions* $Re\, p(z) > 0$, $p(0) = 1$. *Then*

$$\left| z\frac{p'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2}. \tag{2.2}$$

Lemma 2.3 ([7]). *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ *is analytic in* D, *and satisfies the conditions* $Re\, p(z) > 0$, *then*

(i) $$|p_n| \leq 2 \quad \text{for} \quad n \geq 1,$$

(ii) $$|p(z)| \leq \frac{1+|z|}{1-|z|},$$

$$Re\, p(z) \geq \frac{1-|z|}{1+|z|}.$$

Lemma 2.4. *If* $s(z) \in S^*_\lambda(\alpha)$, *then*

(2.3)

(i) for $\lambda \neq 0$,
$$\left| 1 + z\frac{s''(z)}{s'(z)} - \frac{1 + \{2(1-\alpha)\cos\lambda e^{-i\lambda} - 1\}r^2}{1-r^2} \right| \leq \frac{2(1-\alpha)r\{1+r+(1-r)|\sin\lambda|\}\cos\lambda}{(1-r)^2(1+r)|\sin\lambda|}$$

and

(ii) for $\lambda = 0$,
$$\left| 1 + z\frac{s''(z)}{s'(z)} - \frac{1+(1-2\alpha)r^2}{1-r^2} \right| \leq \frac{4r(1-\alpha)\{1+(1-\alpha)r\}}{(1-r^2)\{(1-\alpha)(1+r)+\alpha(1-r)\}}.$$

Proof. (i) for $\lambda \neq 0$, since $s(z) \in S^*_\lambda(\alpha)$, then

$$\frac{e^{i\lambda}\dfrac{zs'(z)}{s(z)} - \alpha\cos\lambda - i\sin\lambda}{(1-\alpha)\cos\lambda} = p(z), \tag{2.4}$$

where $p(z)$ is analytic in D, and satisfies the conditions $Re\ p(z) > 0$, $p(0) = 1$. Logarithmic differentiation yields

$$1 + z\frac{s''(z)}{s'(z)} - z\frac{s'(z)}{s(z)} = \frac{z(1-\alpha)\cos\lambda p'(z)}{(1-\alpha)\cos\lambda p(z) + \alpha\cos\lambda + i\sin\lambda}. \tag{2.5}$$

By Lemma 2.2 and putting $\frac{1}{p(z)} = U + iV$, we have

$$\begin{aligned}
&\left|1 + z\frac{s''(z)}{s'(z)} - z\frac{s'(z)}{s(z)}\right| \\
&= \left|\frac{\frac{zp'(z)}{p(z)}}{1 + \frac{\alpha}{1-\alpha}\frac{1}{p(z)} + i\frac{1}{1-\alpha}\tan\lambda\frac{1}{p(z)}}\right| \\
&= (1-\alpha)\left|\frac{\frac{zp'(z)}{p(z)}}{(1-\alpha) + \alpha\frac{1}{p(z)} + i\tan\lambda\frac{1}{p(z)}}\right| \qquad (2.6)\\
&\le \frac{(1-\alpha)\frac{2r}{1-r^2}}{(1-\alpha) + \alpha\frac{1}{p(z)} + i\tan\lambda\frac{1}{p(z)}} \\
&\le \frac{(1-\alpha)\frac{2r}{1-r^2}}{U|\tan\lambda|}.
\end{aligned}$$

Using Lemma 2.3 and (2.6), we have the following results.

$$\left|1 + z\frac{s''(z)}{s'(z)} - z\frac{s'(z)}{s(z)}\right| \le \frac{2(1-\alpha)r}{(1-r)^2|\tan\lambda|}. \tag{2.7}$$

And by Lemma 2.3 and (2.7), we get

$$\begin{aligned}
&\left|1 + z\frac{s''(z)}{s'(z)} - \frac{1 + \{2(1-\alpha)\cos\lambda e^{i\lambda} - 1\}r^2}{1-r^2}\right| \\
&\le \frac{2(1-\alpha)r\{1 + r + (1-r)|\sin\lambda|\}\cos\lambda}{(1-r)^2(1+r)|\sin\lambda|}.
\end{aligned} \tag{2.8}$$

(i) for $\lambda = 0$, from (2.4) we get

$$\frac{zs'(z)}{s(z)} - \alpha = (1-\alpha)p(z). \tag{2.9}$$

Using Lemma 2.2 and (2.9), by similar method as $\lambda \neq 0$,

$$\left|1 + z\frac{s''(z)}{s'(z)} - z\frac{s'(z)}{s(z)}\right| \leq \frac{2r(1-\alpha)}{\{(1-\alpha)(1+r)+\alpha(1-r)\}(1-r)}. \tag{2.10}$$

From Lemma 2.1 ($\lambda = 0$), we get

$$\begin{aligned}&\left|1 + z\frac{s''(z)}{s'(z)} - \frac{1+(1-2\alpha)r^2}{1-r^2}\right| \\ \leq &\frac{4r(1-\alpha)\{1+(1-\alpha)r\}}{(1-r^2)\{(1-\alpha)(1+r)+\alpha(1-r)\}}.\end{aligned} \tag{2.11}$$

Theorem 2.5. *If $s(z) \in S^*_\lambda(\alpha)$ $(\lambda \neq 0)$, then $s(z)$ is convex in $|z| < R(\lambda, \alpha)$, where $R(\lambda, \alpha)$ is the smallest positive root of the equation*

$$\begin{aligned}T(r) =& r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} - r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| \\ &- (1-|\sin\lambda|)\cos\lambda\} - |\sin\lambda|] + r\{|\sin\lambda| \\ &+ 2(1-\alpha)(1+|\sin\lambda|)\cos\lambda\} - |\sin\lambda|,\end{aligned} \tag{2.12}$$

the result is sharp.

Proof. From Lemma 2.4, we obtain

$$\begin{aligned}&Re\left(1 + z\frac{s''(z)}{s'(z)}\right) \\ \geq &\frac{-r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r)^2(1+r)|\sin\lambda|} \\ &\frac{+r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (1-|\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r)^2(1+r)|\sin\lambda|} \\ &\frac{-r\{|\sin\lambda| + 2(1-\alpha)(1+|\sin\lambda|)\cos\lambda\} + |\sin\lambda|}{(1-r)^2(1+r)|\sin\lambda|}.\end{aligned} \tag{2.13}$$

Since $T(0) < 0$ and $T(1) > 1$, there exists a real root of $T(r) = 0$ in $(0, 1)$. Let $R(\lambda, \alpha)$ be the smallest positive root of $T(r) = 0$ in $(0, 1)$. Then $s(z)$ is convex in $|z| < R(\lambda, \alpha)$. Sharpness is attained for the function,

$$s(z) = \frac{z}{(1-z)^{2(1-\alpha)\cos\lambda\exp(-i\lambda)}}. \tag{2.14}$$

Remark 1. In the case $\lambda = 0$, from Lemma 2.4(ii) we get

$$\begin{aligned} &Re\left(1 + z\frac{s''(z)}{s'(z)}\right) \\ \ge &\frac{1+(1-2\alpha)r^2}{1-r^2} - \frac{4r(1-\alpha)\{1+r-\alpha r\}}{\{(1+r)(1-\alpha)+\alpha(1-r)\}(1-r^2)} \\ = &\frac{(1-2\alpha)^2r^3 - (4\alpha^2-6\alpha+3)r^2 + (2\alpha-3)r + 1}{(1-r^2)\{(1-\alpha)(1+r)+\alpha(1-r)\}}. \end{aligned} \tag{2.15}$$

We have $s(z)$ is convex in $|z| < R(\alpha)$, where $R(\alpha)$ is the smallest positive root of the equation

$$T(r) = (1-2\alpha)^2r^3 - (4\alpha^2 - 6\alpha + 3)r^2 + (2\alpha - 3)r + 1. \tag{2.16}$$

Remark 2. If $\alpha = 0$ in (2.16), we get $r = 2 - \sqrt{3}$. This result is obtained by R. J. Libera [2].

Theorem 2.6. *If $s(z) \in S^*_\lambda(\alpha)$ $(\lambda \neq 0)$, then $s(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta)$, where $R(\lambda, \alpha, \beta)$ is the smallest positive root of the equation*

$$\begin{aligned} T(r) = &r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} - r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| \\ &- (\beta - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|] \\ &+ r\{2(1-\alpha)\cos\lambda(\beta + |\sin\lambda|) + |\sin\lambda|\} - |\sin\lambda|, \end{aligned} \tag{2.17}$$

the result is sharp.

Proof. From inequatlity (2.1) we have

$$(2.18) \qquad Re\, z\frac{s'(z)}{s(z)} \geq \frac{\{2(1-\alpha)\cos^2\lambda - 1\}r^2 - 2(1-\alpha)r\cdot\cos\lambda + 1}{1-r^2}.$$

Let $0 \leq \beta \leq 1$. If we multiply both sides of (2.18) by $(1-\beta)$ and of (2.13) by β

$$(2.19) \qquad \begin{aligned} &Re\, J(\beta, s(z)) \\ &\geq \frac{-r^3|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} + r^2[2(1-\alpha)\{\cos^2\lambda|\sin\lambda|}{(1-r)^2(1+r)|\sin\lambda|} \\ &\quad \frac{-(\beta - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|] - r\{2(1-\alpha)\cos\lambda(\beta + |\sin\lambda|)}{(1-r)^2(1+r)|\sin\lambda|} \\ &\quad \frac{+|\sin\lambda|\} + |\sin\lambda|}{(1-r)^2(1+r)|\sin\lambda|}. \end{aligned}$$

Since $T(0) < 0$ and $T(1) > 0$, there exist a real root of $T(r) = 0$ in $(0,1)$. Let $R(\lambda, \alpha, \beta)$ be the smallest positive root $T(r) = 0$ in $(0,1)$. Then $s(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta)$. We obtain sharp for the extremal function is given by (2.14).

Corollary 2.7. *If $\beta = 1$, then we obtain the radius of convexity for the class of λ-spirallike of order α functions which is given in Theorem 2.5.*

Corollary 2.8. *If $\beta = 0$, then*

$$r = \frac{(1-\alpha)\cos\lambda - \sqrt{1-(1-\alpha^2)\cos^2\lambda}}{2(1-\alpha)\cos^2\lambda - 1}.$$

Remark 3. If $\alpha = 0$ in Corollary 2.8, then $r = \dfrac{1}{|\sin\lambda| + \cos\lambda}$.

It is the radius of starlikness for λ-spirallike functions, which was obtained by M. S. Rebertson [10] and R. J. Libera [2].

3. The radius of β-convexity for p-fold λ-spirallike functions

Theorem 3.1. *If $f(z) \in S^*_{\lambda p}(\alpha)$ $(\lambda \neq 0)$, then $f(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta, p)$, where $R(\lambda, \alpha, \beta, p)$ is the smallest positive root of the equation*

(3.1)
$$\begin{aligned} T(r) =& r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\} - r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| \\ &- (\beta p - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|] + r^p\{2(1-\alpha)\cos\lambda(\beta p \\ &+ |\sin\lambda|) + |\sin\lambda|\} - |\sin\lambda|. \end{aligned}$$

Proof. From the relation (1.8) we obtain

$$1 + z^p\frac{s''(z^p)}{s'(z^p)} = \frac{1}{p}\left(1 + z\frac{f'(z)}{f(z)}\right) + \left(1 - \frac{1}{p}\right)z\frac{f'(z)}{f(z)}.$$

From a simple caculation of (1.8), (2.13) and (2.16) we obtain

$$\begin{aligned} &Re\ J\left(\frac{1}{p}, f(z)\right) \\ &\geq \frac{-r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad \frac{+r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (1-|\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad \frac{-r^p\{|\sin\lambda| + 2(1-\alpha)(1+|\sin\lambda|)\cos\lambda\} + |\sin\lambda|}{(1-r^p)^2(1+r^p)|\sin\lambda|}, \end{aligned} \tag{3.2}$$

$$(3.3)\qquad Re\, z\frac{f'(z)}{f(z)} \geq \frac{\{2(1-\alpha)\cos^2\lambda - 1\}r^{2p} - 2(1-\alpha)r^p\cos\lambda + 1}{1-r^{2p}}.$$

If we multiply both sides of (3.2) by γ and (3.3) by $1-\gamma$, the add the corresponding members, we obtain

$$(3.4)\qquad \begin{aligned} &ReJ\left(\frac{\gamma}{p}, f(z)\right) \\ &\geq \frac{-r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad \frac{+r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (\gamma - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad \frac{-r^p\{2(1-\alpha)\cos\lambda(\gamma + |\sin\lambda|) + |\sin\lambda|\} + |\sin\lambda|}{(1-r^p)^2(1+r^p)|\sin\lambda|} \end{aligned}$$

where $0 \leq \gamma \leq 1$. If we take $\dfrac{\gamma}{p} = \beta$ the inequality (3.2) can be written in the form

$$(3.5)\qquad \begin{aligned} &ReJ\left(\beta, f(z)\right) \\ &\geq \frac{-r^{3p}|\sin\lambda|\{2(1-\alpha)\cos^2\lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad \frac{+r^{2p}[2(1-\alpha)\{\cos^2\lambda|\sin\lambda| - (\beta p - |\sin\lambda|)\cos\lambda\} - |\sin\lambda|]}{(1-r^p)^2(1+r^p)|\sin\lambda|} \\ &\quad \frac{-r^p\{2(1-\alpha)\cos\lambda(\beta p + |\sin\lambda|) + |\sin\lambda|\} + |\sin\lambda|}{(1-r^p)^2(1+r^p)|\sin\lambda|} \end{aligned}$$

where $0 \leq \beta \leq 1$.

Since $T(0) < 0$ and $T(1) > 0$, there exist a real root of $T(r) = 0$ in $(0,1)$. Let $R(\lambda, \alpha, \beta, p)$ be the smallest positive root $T(r) = 0$ in $(0,1)$. Then $f(z)$ is β-convex in $|z| < R(\lambda, \alpha, \beta, p)$. We obtain sharp because the extremal $f(z) = z/(1-z^p)^{2(1-\alpha)\cos\lambda\exp(-i\lambda)/p}$. This shows that the theorem is true.

Corollary 3.2. *If $p = 1$, then we obtain the radius of β-convexity for the class of λ-spirallike of order α functions which is given in Theorem 2.5.*

Corollary 3.3. *If $\alpha = 0$, then we obtain the radius of β-convexity for the class of λ-spirallike functions.*

Corollary 3.4. *For $\beta = 0$ we obtain $r = \sqrt[p]{\frac{(1-\alpha)\cos\lambda - \sqrt{1-(1-\alpha)\cos^2\lambda}}{2(1-\alpha)\cos^2\lambda - 1}}$. This is the radius of starlikeness for the p-fold λ-spirallike function. If we take $p = 1$, $\alpha = 0$ and $\beta = 0$, we obtain $r = (|\sin\lambda| + \cos\lambda)^{-1}$, which was obtained by M. S. Roberston [8] and R. J. Libera [2].*

Corollary 3.5. *In the case $\lambda = 0$, we obtain the radius of β-convexity for the class of p-fold starlike of order α functions. If we take $\alpha = 0$ we obtain the radius of β-convexity for the class of p-fold starlike functions.*

Corollary 3.6. *For $p = 1$, $\beta = 0$, $\lambda = 0$ and $\alpha = 0$, we obtain $r = 2 - \sqrt{3}$, the radius obtained by R. J. Libera [2].*

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