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THE RADIUS OF $\beta$-CONVEXITY FOR THE
CLASSES OF $\lambda$-SPIRALLIKE ORDER $\alpha$ FUNCTIONS

OH SANG, KWON AND SHIGEYOSHI, OWA

ABSTRACT. We get sharp bounds for the radius of $\beta$-convexity for the
classes of $\lambda$-spirallike of order $\alpha$ and $p$-fold $\lambda$-spirallike of order $\alpha$
fuctions.

1. Introduction

Let $A$ denote the class of functions of the form

$$ s(z) = z + \sum_{n=2}^{\infty} a_n z^n $$

which are analytic in unit disk $D = \{ z : |z| < 1 \}$. And let $S$ denote the
subclass of $A$ consisting of analytic and univalent function $s(z)$ in unit
disk $D$.

A function $s(z)$ in $S$ is said to be starlike if

$$ \Re \left\{ \frac{zs'(z)}{s(z)} \right\} > 0 \quad (z \in D). $$

We denote by $S^*$ the class of all starlike functions. A function $s(z)$ in $S$
is said to be convex if

$$ \Re \left\{ 1 + \frac{zs''(z)}{s'(z)} \right\} > 0 \quad (z \in D). $$

And we denote by $K$ the class of all convex functions. These classes $S^*$

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$p$-fold univalent function.

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Definition 1.1. A function $s(z)$ in $S$ is said to be $\lambda$-spirallike if

\[(1.4) \quad \Re \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > 0 \quad (z \in D)\]

for some real $\lambda \left( |\lambda| < \frac{\pi}{2} \right)$. The class of these functions is denoted by $S_{\lambda}^\ast$.

Definition 1.2. A function $s(z)$ in $S$ is said to be $\lambda$-spirallike of order $\alpha$ if

\[(1.5) \quad \Re \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > \alpha \cos \lambda \quad (z \in D)\]

for some real $\lambda \left( |\lambda| < \frac{\pi}{2} \right)$ and $\alpha (0 \leq \alpha < 1)$. The above classes were introduced by Spacek ([12]). For $\lambda = 0$ in (1.4) the class is a starlike function (1.2).

Definition 1.3. Let $F$ denote a non-empty collection of functions $s(z)$ each of which is univalent in $D$, and let $\beta$ be given $0 \leq \beta \leq 1$. Then the real number

\[(1.6) \quad R_\alpha(F) = \sup \{ R | \Re J(\beta, s(z)) > 0, |z| < R, s(z) \in F \}\]

is called the radius of $\beta$-convexity of $F$, where $J(\beta, s(z))$ is defined by the relation,

\[(1.7) \quad J(\beta, s(z)) = (1 - \beta)z \frac{s'(z)}{s(z)} + \beta \left( 1 + z \frac{s''(z)}{s'(z)} \right).\]

The radius of $\beta$-convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). For $\beta = 0$ and $\beta = 1$ in (1.7), we define a starlike function (1.2) and a convex function (1.3), respectively.
**Definition 1.4.** Consider a function \( s(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) which is univalent in \( U \). Then the function defined by the relation

\[
(1.8) \quad f(z) = (s(z^p))^\frac{1}{p} = z + \sum_{n=1}^{\infty} a_{np+1} z^{np+1}
\]

is also univalent in \( U \), and \( f(z) \) is called \( p \)-fold univalent function. If the function \( f(z) \) defined by the relation (1.8) satisfies the collection

\[
(1.9) \quad Re \left\{ e^{i\lambda} z \frac{f'(z)}{f(z)} \right\} > 0 \quad (z \in D),
\]

then the function \( f(z) \) is called a \( p \)-fold \( \lambda \)-spirallike function in \( U \), for some real \( \lambda \) \( \left( |\lambda| < \frac{\pi}{2} \right) \) ([1]), and the class of these functions is denoted by \( S_{\lambda p}^* \). And also we can define a \( p \)-fold \( \lambda \)-spirallike function of order \( \alpha \) in \( U \), denoted by \( S_{\lambda p}^*(\alpha) \).

The radius of \( \beta \)-convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). There are many open problems about the radius of starlikeness, convexity and \( \beta \)-convexity for the classes \( S \) ([1]). So, we get sharp bounds for the radius of \( \beta \)-convexity for the classes of \( \lambda \)-spirallike of order \( \alpha \) and \( p \)-fold \( \lambda \)-spirallike of order \( \alpha \) functions.

**2. The radius of \( \beta \)-convexity**

**Lemma 2.1** ([5]). If \( s(z) \in S_{\lambda}^*(\alpha) \), then

\[
(2.1) \quad \left| z \frac{s'(z)}{s(z)} - \frac{1 + \{2(1 - \alpha) \cos \lambda e^{-i\lambda} - 1\} r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha) r \cos \lambda}{1 - r^2}.
\]
Lemma 2.2 ([10]). If \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) is analytic in \( D \), and satisfies the conditions \( \text{Re} \ p(z) > 0 \), \( p(0) = 1 \). Then

\[
|z \frac{p'(z)}{p(z)}| \leq \frac{2r}{1 - r^2}.
\]

Lemma 2.3 ([7]). If \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) is analytic in \( D \), and satisfies the conditions \( \text{Re} \ p(z) > 0 \), then

(i) \(|p_n| \leq 2 \) for \( n \geq 1 \),

(ii) \(|p(z)| \leq \frac{1 + |z|}{1 - |z|} \), \( \text{Re} \ p(z) \geq \frac{1 - |z|}{1 + |z|} \).

Lemma 2.4. If \( s(z) \in S_\lambda^\alpha(\alpha) \), then

\[
(i) \quad \text{for } \lambda \neq 0, \quad \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + \{2(1 - \alpha) \cos \lambda e^{-i\lambda} - 1\}r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r\{1 + r + (1 - r)|\sin \lambda|\} \cos \lambda}{(1 - r^2)(1 + r)|\sin \lambda|}
\]

and

\[
(ii) \quad \text{for } \lambda = 0, \quad \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{4r(1 - \alpha)\{1 + (1 - \alpha)r\}}{(1 - r^2)((1 - \alpha)(1 + r) + \alpha(1 - r))}.
\]

**Proof.** (i) for \( \lambda \neq 0 \), since \( s(z) \in S_\lambda^\alpha(\alpha) \), then

\[
\frac{e^{i\lambda} z s'(z)}{s(z)} - \alpha \cos \lambda - i \sin \lambda
\]

\[
\frac{(1 - \alpha) \cos \lambda}{(1 - \alpha) \cos \lambda} = p(z),
\]

\[
(2.4)
\]
where \( p(z) \) is analytic in \( D \), and satisfies the conditions \( \text{Re} \, p(z) > 0 \), \( p(0) = 1 \). Logarithmic differentiation yields

\[
1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} = \frac{z(1 - \alpha) \cos \lambda p'(z)}{(1 - \alpha) \cos \lambda p(z) + \alpha \cos \lambda + i \sin \lambda}.
\]

By Lemma 2.2 and putting \( \frac{1}{p(z)} = U + iV \), we have

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| = \left| \frac{zp'(z)}{p(z)} \right| \leq \frac{(1 - \alpha) \overline{1 - r^2}}{11} \frac{2r}{1 - r^2} \frac{1 - \alpha}{U |\tan \lambda|}.
\]

Using Lemma 2.3 and (2.6), we have the following results.

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| \leq \frac{2(1 - \alpha)r}{(1 - r)^2 |\tan \lambda|}.
\]

And by Lemma 2.3 and (2.7), we get

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + \{2(1 - \alpha) \cos \lambda e^{i\lambda} - 1\}r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r \{1 + r + (1 - r)|\sin \lambda| \} \cos \lambda}{(1 - r)^2(1 + r)|\sin \lambda|}.
\]
(i) for $\lambda = 0$, from (2.4) we get

\[(2.9) \quad \frac{zs'(z)}{s(z)} - \alpha = (1 - \alpha)p(z).\]

Using Lemma 2.2 and (2.9), by similar method as $\lambda \neq 0$,

\[(2.10) \quad \left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| \leq \frac{2r(1 - \alpha)}{\{(1 - \alpha)(1 + r) + \alpha(1 - r)\}(1 - r)}.

From Lemma 2.1 ($\lambda = 0$), we get

\[(2.11) \quad \left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{4r(1 - \alpha)(1 + (1 - \alpha)r)}{(1 - r^2)(1 - r)(1 + r + \alpha(1 - r))}.

**Theorem 2.5.** If $s(z) \in S^*_\alpha(\lambda) \ (\lambda \neq 0)$, then $s(z)$ is convex in $|z| < R(\lambda, \alpha)$, where $R(\lambda, \alpha)$ is the smallest positive root of the equation

\[(2.12) \quad T(r) = r^3|\sin \lambda|\{2(1 - \alpha)\cos^2 \lambda - 1\} - r^2[2(1 - \alpha)\cos^2 \lambda|\sin \lambda|

\quad - (1 - |\sin \lambda|)\cos \lambda - |\sin \lambda| + r(|\sin \lambda|

\quad + 2(1 - \alpha)(1 + |\sin \lambda|)\cos \lambda - |\sin \lambda|,

the result is sharp.

**Proof.** From Lemma 2.4, we obtain

\[(2.13) \quad Re \left(1 + z \frac{s''(z)}{s'(z)}\right) \geq \frac{-r^3|\sin \lambda|\{2(1 - \alpha)\cos^2 \lambda - 1\}}{(1 - r)^2(1 + r)|\sin \lambda|}

\quad + r^2[2(1 - \alpha)\cos^2 \lambda|\sin \lambda| - (1 - |\sin \lambda|)\cos \lambda - |\sin \lambda|

\quad + r\{|\sin \lambda| + 2(1 - \alpha)(1 + |\sin \lambda|)\cos \lambda\} + |\sin \lambda|] \over (1 - r)^2(1 + r)|\sin \lambda|.
Since $T(0) < 0$ and $T(1) > 1$, there exists a real root of $T(r) = 0$ in $(0, 1)$. Let $R(\lambda, \alpha)$ be the smallest positive root of $T(r) = 0$ in $(0, 1)$. Then $s(z)$ is convex in $|z| < R(\lambda, \alpha)$. Sharpness is attained for the function,

$$s(z) = \frac{z}{(1 - z)^2(1 - \alpha) \cos \lambda \exp(-i\lambda)}.$$  

(2.14)

**Remark 1.** In the case $\lambda = 0$, from Lemma 2.4(i) we get

$$\text{Re}\left(1 + z \frac{s''(z)}{s'(z)}\right) \geq \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} - \frac{4r(1 - \alpha)(1 + r - \alpha r)}{(1 + r)(1 - \alpha) + \alpha(1 - r)}.$$  

(2.15)

We have $s(z)$ is convex in $|z| < R(\alpha)$, where $R(\alpha)$ is the smallest positive root of the equation

$$T(r) = (1 - 2\alpha)^2r^3 - (4\alpha^2 - 6\alpha + 3)r^2 + (2\alpha - 3)r + 1.$$  

(2.16)

**Remark 2.** If $\alpha = 0$ in (2.16), we get $r = 2 - \sqrt{3}$. This result is obtained by R. J. Libera [2].

**Theorem 2.6.** If $s(z) \in S_\lambda^*(\alpha)$ ($\lambda \neq 0$), then $s(z)$ is $\beta$-convex in $|z| < R(\lambda, \alpha, \beta)$, where $R(\lambda, \alpha, \beta)$ is the smallest positive root of the equation

$$T(r) = r^3|\sin \lambda|\{2(1 - \alpha) \cos^2 \lambda - 1\} - r^2[2(1 - \alpha)\cos^2 \lambda|\sin \lambda|$$  

$$(\beta - |\sin \lambda|)\cos \lambda - |\sin \lambda|$$  

$$+ r\{2(1 - \alpha) \cos \lambda(\beta + |\sin \lambda|) + |\sin \lambda|\} - |\sin \lambda|,$$  

(2.17)
the result is sharp.

**Proof.** From inequality (2.1) we have

\[
\text{Re } z \frac{s'(z)}{s(z)} \geq \frac{\{2(1-\alpha)\cos^2 \lambda - 1\}r^2 - 2(1-\alpha)r \cdot \cos \lambda + 1}{1 - r^2}.
\]

Let \(0 \leq \beta \leq 1\). If we multiply both sides of (2.18) by \((1 - \beta)\) and of (2.13) by \(\beta\)

\[
\text{Re } J(\beta, s(z)) \geq \frac{-r^3|\sin \lambda|\{2(1-\alpha)\cos^2 \lambda - 1\} + r^2[2(1-\alpha)\cos^2 \lambda|\sin \lambda|}{(1-r)^2(1+r)|\sin \lambda|} \\
- (\beta - |\sin \lambda|) \cos \lambda - |\sin \lambda| - r\{2(1-\alpha)\cos \lambda(\beta + |\sin \lambda|)} \\
\frac{+(|\sin \lambda| + |\sin \lambda|)}{(1-r)^2(1+r)|\sin \lambda|}.
\]

Since \(T(0) < 0\) and \(T(1) > 0\), there exist a real root of \(T(r) = 0\) in \((0,1)\). Let \(R(\lambda, \alpha, \beta)\) be the smallest positive root \(T(r) = 0\) in \((0,1)\). Then \(s(z)\) is \(\beta\)-convex in \(|z| < R(\lambda, \alpha, \beta)\). We obtain sharp for the extremal function is given by (2.14).

**Corollary 2.7.** If \(\beta = 1\), then we obtain the radius of convexity for the class of \(\lambda\)-spirallike of order \(\alpha\) functions which is given in Theorem 2.5.

**Corollary 2.8.** If \(\beta = 0\), then

\[
r = \frac{(1-\alpha)\cos \lambda - \sqrt{1 - (1-\alpha^2) \cos^2 \lambda}}{2(1-\alpha) \cos^2 \lambda - 1}.
\]
Remark 3. If $\alpha = 0$ in Corollary 2.8, then $r = \frac{1}{|\sin \lambda| + \cos \lambda}$.

It is the radius of starlikeness for $\lambda$-spirallike functions, which was obtained by M. S. Robertson [10] and R. J. Libera [2].

3. The radius of $\beta$-convexity for $p$-fold $\lambda$-spirallike functions

Theorem 3.1. If $f(z) \in S_{\lambda p}^{*}(\alpha) (\lambda \neq 0)$, then $f(z)$ is $\beta$-convex in $|z| < R(\lambda, \alpha, \beta, p)$, where $R(\lambda, \alpha, \beta, p)$ is the smallest positive root of the equation

\[ T(r) = r^{3p}|\sin \lambda||2(1 - \alpha) \cos^2 \lambda - 1| - r^{2p}[2(1 - \alpha)|\cos^2 \lambda|\sin \lambda| - (\beta p - |\sin \lambda|) \cos \lambda] + |\sin \lambda| - |\sin \lambda| + r^p\{2(1 - \alpha) \cos \lambda(\beta p + |\sin \lambda|) + |\sin \lambda|\} - |\sin \lambda|. \]

Proof. From the relation (1.8) we obtain

\[ 1 + z^p \frac{s''(z^p)}{s'(z^p)} = \frac{1}{p} \left( 1 + z \frac{f'(z)}{f(z)} \right) + \left( 1 - \frac{1}{p} \right) z \frac{f'(z)}{f(z)}. \]

From a simple calculation of (1.8), (2.13) and (2.16) we obtain

\[ \text{Re} \ J \left( \frac{1}{p}, f(z) \right) \geq \frac{-r^{3p}|\sin \lambda||2(1 - \alpha) \cos^2 \lambda - 1|}{(1 - r^p)^2(1 + r^p)|\sin \lambda|} + \frac{r^{2p}[2(1 - \alpha)|\cos^2 \lambda|\sin \lambda| - (1 - |\sin \lambda|) \cos \lambda] - |\sin \lambda|}{(1 - r^p)^2(1 + r^p)|\sin \lambda|} - \frac{r^p\{2(1 - \alpha) \cos \lambda(\beta p + |\sin \lambda|) + |\sin \lambda|}{(1 - r^p)^2(1 + r^p)|\sin \lambda|}. \]
\[ \text{(3.3)} \quad \Re z \frac{f'(z)}{f(z)} \geq \frac{2(1 - \alpha) \cos^2 \lambda - 1}{1 - r^{2p}} - 2(1 - \alpha) r^p \cos \lambda + 1. \]

If we multiply both sides of (3.2) by \( \gamma \) and (3.3) by \( 1 - \gamma \), the add the corresponding members, we obtain

\[
\Re J \left( \frac{\gamma}{p}, f(z) \right) 
\geq -r^{3p} |\sin \lambda| \frac{(1 - \beta p - |\sin \lambda|) \cos \lambda}{(1 - r^p)^2 (1 + r^p)} \sin \lambda 
\]

\[
+ r^{2p} \frac{2(1 - \alpha) \left\{ \cos^2 \lambda |\sin \lambda| - (\gamma - |\sin \lambda|) \cos \lambda \right\} - |\sin \lambda|}{(1 - r^p)^2 (1 + r^p)} \sin \lambda 
\]

\[
- r^p \frac{2(1 - \alpha) \left\{ \cos \lambda (\gamma + |\sin \lambda|) + |\sin \lambda| \right\} + |\sin \lambda|}{(1 - r^p)^2 (1 + r^p)} \sin \lambda 
\]

where \( 0 \leq \gamma \leq 1 \). If we take \( \frac{\gamma}{p} = \beta \) the inequality (3.2) can be written in the form

\[
\Re J (\beta, f(z)) 
\geq -r^{3p} |\sin \lambda| \frac{(1 - \beta p - |\sin \lambda|) \cos \lambda}{(1 - r^p)^2 (1 + r^p)} \sin \lambda 
\]

\[
+ r^{2p} \frac{2(1 - \alpha) \left\{ \cos^2 \lambda |\sin \lambda| - (\beta p - |\sin \lambda|) \cos \lambda \right\} - |\sin \lambda|}{(1 - r^p)^2 (1 + r^p)} \sin \lambda 
\]

\[
- r^p \frac{2(1 - \alpha) \left\{ \cos \lambda (\beta p + |\sin \lambda|) + |\sin \lambda| \right\} + |\sin \lambda|}{(1 - r^p)^2 (1 + r^p)} \sin \lambda 
\]

where \( 0 \leq \beta \leq 1 \).

Since \( T(0) < 0 \) and \( T(1) > 0 \), there exist a real root of \( T(r) = 0 \) in \( (0, 1) \). Let \( R(\lambda, \alpha, \beta, p) \) be the smallest positive root \( T(r) = 0 \) in \( (0, 1) \). Then \( f(z) \) is \( \beta \)-convex in \( |z| < R(\lambda, \alpha, \beta, p) \). We obtain sharp because the extremal \( f(z) = z/(1 - z^p)^{2(1-\alpha) \cos \lambda \exp(-i\lambda)/p} \). This shows that the theorem is true.
Corollary 3.2. If $p = 1$, then we obtain the radius of $\beta$-convexity for the class of $\lambda$-spirallike functions which is given in Theorem 2.5.

Corollary 3.3. If $\alpha = 0$, then we obtain the radius of $\beta$-convexity for the class of $\lambda$-spirallike functions.

Corollary 3.4. For $\beta = 0$ we obtain
$$ r = \sqrt{\frac{(1 - \alpha) \cos \lambda - \sqrt{1 - (1 - \alpha) \cos^2 \lambda}}{2(1 - \alpha) \cos^2 \lambda - 1}}. $$
This is the radius of starlikeness for the $p$-fold $\lambda$-spirallike function. If we take $p = 1$, $\alpha = 0$ and $\beta = 0$, we obtain $r = (|\sin \lambda| + \cos \lambda)^{-1}$, which was obtained by M. S. Robertson [8] and R. J. Libera [2].

Corollary 3.5. In the case $\lambda = 0$, we obtain the radius of $\beta$-convexity for the class of $p$-fold starlike functions. If we take $\alpha = 0$ we obtain the radius of $\beta$-convexity for the class of $p$-fold starlike functions.

Corollary 3.6. For $p = 1$, $\beta = 0$, $\lambda = 0$ and $\alpha = 0$, we obtain $r = 2 - \sqrt{3}$, the radius obtained by R. J. Libera [2].

REFERENCES


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