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THE RADIUS OF $\beta$-CONVEXITY FOR THE CLASSES OF $\lambda$-SPIRALLIKE ORDER $\alpha$ FUNCTIONS

OH SANG, KWON AND SHIGEYOSHI, OWA

ABSTRACT. We get sharp bounds for the radius of $\beta$-convexity for the classes of $\lambda$-spirallike of order $\alpha$ and $p$-fold $\lambda$-spirallike of order $\alpha$ functions.

1. Introduction

Let $A$ denote the class of functions of the form

\begin{equation}
    s(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

which are analytic in unit disk $D = \{z : |z| < 1\}$. And let $S$ denote the subclass of $A$ consisting of analytic and univalent function $s(z)$ in unit disk $D$.

A function $s(z)$ in $S$ is said to be starlike if

\begin{equation}
    \text{Re} \left\{ \frac{zs'(z)}{s(z)} \right\} > 0 \quad (z \in D).
\end{equation}

We denote by $S^*$ the class of all starlike functions. A function $s(z)$ in $S$ is said to be convex if

\begin{equation}
    \text{Re} \left\{ 1 + \frac{zs''(z)}{s'(z)} \right\} > 0 \quad (z \in D).
\end{equation}

And we denote by $K$ the class of all convex functions. These classes $S^*$

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Definition 1.1. A function $s(z)$ in $S$ is said to be $\lambda$-spirallike if

$$\Re \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > 0 \quad (z \in D)$$

for some real $\lambda \left( |\lambda| < \frac{\pi}{2} \right)$. The class of these functions is denoted by $S_{\lambda}^*$.

Definition 1.2. A function $s(z)$ in $S$ is said to be $\lambda$-spirallike of order $\alpha$ if

$$\Re \left\{ e^{i\lambda} z \frac{s'(z)}{s(z)} \right\} > \alpha \cos \lambda \quad (z \in D)$$

for some real $\lambda \left( |\lambda| < \frac{\pi}{2} \right)$ and $\alpha \ (0 \leq \alpha < 1)$. The above classes were introduced by Spacek ([12]). For $\lambda = 0$ in (1.4) the class is a starlike function (1.2).

Definition 1.3. Let $F$ denote a non-empty collection of functions $s(z)$ each of which is univalent in $D$, and let $\beta$ be given $0 \leq \beta \leq 1$. Then the real number

$$R_{\alpha}(F) = \sup \{ R | \Re J(\beta, s(z)) > 0, |z| < R, s(z) \in F \}$$

is called the radius of $\beta$-convexity of $F$, where $J(\beta, s(z))$ is defined by the relation,

$$J(\beta, s(z)) = (1 - \beta) z \frac{s'(z)}{s(z)} + \beta \left( 1 + z \frac{s''(z)}{s'(z)} \right).$$

The radius of $\beta$-convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). For $\beta = 0$ and $\beta = 1$ in (1.7), we define a starlike function (1.2) and a convex function (1.3), respectively.
**Definition 1.4.** Consider a function \( s(z) = z + a_2z^2 + a_3z^3 + \cdots \) which is univalent in \( U \). Then the function defined by the relation

\[
f(z) = (s(z^p))^\frac{1}{p} = z + \sum_{n=1}^{\infty} a_{np+1}z^{np+1}
\]

is also univalent in \( U \), and \( f(z) \) is called \( p \)-fold univalent function. If the function \( f(z) \) defined by the relation (1.8) satisfies the collection

\[
Re \left\{ e^{i\lambda}z \frac{f'(z)}{f(z)} \right\} > 0 \quad (z \in D),
\]

then the function \( f(z) \) is called a \( p \)-fold \( \lambda \)-spirallike function in \( U \), for some real \( \lambda \left( |\lambda| < \frac{\pi}{2} \right) \) ([1]), and the class of these functions is denoted by \( S_{\lambda p}^* \). And also we can define a \( p \)-fold \( \lambda \)-spirallike function of order \( \alpha \) in \( U \), denoted by \( S_{\lambda p}^*(\alpha) \).

The radius of \( \beta \)-convexity was introduced by S. S. Miller, P. T. Mocanu and M. O. Reade ([4]). There are many open problems about the radius of starlikeness, convexity and \( \beta \)-convexity for the classes \( S \) ([1]). So, we get sharp bounds for the radius of \( \beta \)-convexity for the classes of \( \lambda \)-spirallike of order \( \alpha \) and \( p \)-fold \( \lambda \)-spirallike of order \( \alpha \) functions.

### 2. The radius of \( \beta \)-convexity

**Lemma 2.1** ([5]). If \( s(z) \in S_{\lambda}^*(\alpha) \), then

\[
\left| z \frac{s'(z)}{s(z)} - \frac{1 + \{2(1 - \alpha)\cos \lambda e^{-i\lambda} - 1\}r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r \cos \lambda}{1 - r^2}.
\]
Lemma 2.2 ([10]). If \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) is analytic in \( D \), and satisfies the conditions \( \text{Re} \, p(z) > 0, \, p(0) = 1 \). Then

\[
|zp'(z)/p(z)| \leq \frac{2r}{1-r^2}. \tag{2.2}
\]

Lemma 2.3 ([7]). If \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) is analytic in \( D \), and satisfies the conditions \( \text{Re} \, p(z) > 0 \), then

(i) \[ |p_n| \leq 2 \quad \text{for} \quad n \geq 1, \]

(ii) \[ |p(z)| \leq \frac{1+|z|}{1-|z|}, \]

\[ \text{Re} \, p(z) \geq \frac{1-|z|}{1+|z|}. \]

Lemma 2.4. If \( s(z) \in S^*_\lambda(\alpha) \), then

\[
(2.3)
\]

(i) for \( \lambda \neq 0 \),

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + \{2(1-\alpha)\cos\lambda e^{-i\lambda} - 1\}r^2}{1-r^2} \right|
\leq \frac{2(1-\alpha)r\{1+r+(1-r)|\sin\lambda|\} \cos \lambda}{(1-r^2)(1+r)|\sin \lambda|}
\]

and

(ii) for \( \lambda = 0 \),

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + (1-2\alpha)r^2}{1-r^2} \right|
\leq \frac{4r(1-\alpha)(1+(1-\alpha)r)}{(1-r^2)((1-\alpha)(1+r)+\alpha(1-r))}.\]

Proof. (i) for \( \lambda \neq 0 \), since \( s(z) \in S^*_\lambda(\alpha) \), then

\[
eq p(z),
\]

(2.4) \[ e^{i\lambda} \frac{zs'(z)}{s(z)} - \alpha \cos \lambda - i \sin \lambda
\]

\[ \frac{(1-\alpha) \cos \lambda}{(1-\alpha) \cos \lambda} = p(z), \]
where $p(z)$ is analytic in $D$, and satisfies the conditions $Re \ p(z) > 0$, $p(0) = 1$. Logarithmic differentiation yields

\[(2.5) \quad 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} = \frac{z(1 - \alpha) \cos \lambda p'(z)}{(1 - \alpha) \cos \lambda p(z) + \alpha \cos \lambda + i \sin \lambda}.
\]

By Lemma 2.2 and putting $\frac{1}{p(z)} = U + iV$, we have

\[\left|1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)}\right| = \frac{\left|\frac{zp'(z)}{p(z)}\right|}{\left|1 + \frac{1}{1 - \alpha p(z)} + i \frac{1}{1 - \alpha \tan \lambda} \frac{1}{p(z)}\right|} \leq (1 - \alpha) \frac{2r}{(1 - r^2) \left|\tan \lambda\right|}.
\]

Using Lemma 2.3 and (2.6), we have the following results.

\[(2.7) \quad \left|1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)}\right| \leq \frac{2(1 - \alpha) r}{(1 - r^2) |\tan \lambda|}.
\]

And by Lemma 2.3 and (2.7), we get

\[(2.8) \quad \left|1 + z \frac{s''(z)}{s'(z)} - \frac{1 + \{2(1 - \alpha) \cos \lambda e^{i\lambda} - 1\} r^2}{1 - r^2}\right| \leq \frac{2(1 - \alpha) r \{1 + r + (1 - r)|\sin \lambda|\} \cos \lambda}{(1 - r^2)(1 + r)|\sin \lambda|}.
\]
(i) for $\lambda = 0$, from (2.4) we get

\[
\frac{zs'(z)}{s(z)} - \alpha = (1 - \alpha)p(z).
\]  

Using Lemma 2.2 and (2.9), by similar method as $\lambda \neq 0$,

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - z \frac{s'(z)}{s(z)} \right| \leq \frac{2r(1 - \alpha)}{\{(1 - \alpha)(1 + r) + \alpha(1 - r)\}(1 - r)}.
\]  

From Lemma 2.1 ($\lambda = 0$), we get

\[
\left| 1 + z \frac{s''(z)}{s'(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{4r(1 - \alpha)\{1 + (1 - \alpha)r\}}{(1 - r^2)\{(1 - \alpha)(1 + r) + \alpha(1 - r)\}}.
\]  

**Theorem 2.5.** If $s(z) \in S_\lambda^*(\alpha)$ ($\lambda \neq 0$), then $s(z)$ is convex in $|z| < R(\lambda, \alpha)$, where $R(\lambda, \alpha)$ is the smallest positive root of the equation

\[
T(r) = r^3|\sin \lambda|\{2(1 - \alpha)\cos^2 \lambda - 1\} - r^2[2(1 - \alpha)\{\cos^2 \lambda|\sin \lambda| - (1 - |\sin \lambda|)\cos \lambda\} - |\sin \lambda|]
\]  

\[
- (1 - |\sin \lambda|)\cos \lambda - |\sin \lambda| + r\{|\sin \lambda| + 2(1 - \alpha)(1 + |\sin \lambda|)\cos \lambda\} - |\sin \lambda|,
\]

the result is sharp.

**Proof.** From Lemma 2.4, we obtain

\[
Re\left(1 + z \frac{s''(z)}{s'(z)}\right) \geq \frac{-r^3|\sin \lambda|\{2(1 - \alpha)\cos^2 \lambda - 1\}}{(1 - r)^2(1 + r)|\sin \lambda|} + \frac{r^2[2(1 - \alpha)\{\cos^2 \lambda|\sin \lambda| - (1 - |\sin \lambda|)\cos \lambda\} - |\sin \lambda|]}{(1 - r)^2(1 + r)|\sin \lambda|} - \frac{r\{|\sin \lambda| + 2(1 - \alpha)(1 + |\sin \lambda|)\cos \lambda\} + |\sin \lambda|}{(1 - r)^2(1 + r)|\sin \lambda|}.
\]  

Since \( T(0) < 0 \) and \( T(1) > 1 \), there exists a real root of \( T(r) = 0 \) in \((0, 1)\).

Let \( R(\lambda, \alpha) \) be the smallest positive root of \( T(r) = 0 \) in \((0, 1)\). Then \( s(z) \) is convex in \(|z| < R(\lambda, \alpha)\). Sharpness is attained for the function,

\[
s(z) = \frac{z}{(1 - z)^{2(1 - \alpha)\cos \lambda \exp(-i\lambda)}}.
\]

\begin{equation}
(2.14)
\end{equation}

**Remark 1.** In the case \( \lambda = 0 \), from Lemma 2.4(i) we get

\[
Re \left( 1 + z \frac{s''(z)}{s'(z)} \right) \geq \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} - \frac{4r(1 - \alpha)(1 + r - \alpha r)}{(1 + r)(1 - \alpha) + \alpha(1 - r)}
\]

\[
= \frac{(1 - 2\alpha)^2 r^3 - (4\alpha^2 - 6\alpha + 3)r^2 + (2\alpha - 3)r + 1}{(1 - r^2)(1 - \alpha)(1 + r) + \alpha(1 - r)}
\]

We have \( s(z) \) is convex in \(|z| < R(\alpha)\), where \( R(\alpha) \) is the smallest positive root of the equation

\[
T(r) = (1 - 2\alpha)^2 r^3 - (4\alpha^2 - 6\alpha + 3)r^2 + (2\alpha - 3)r + 1.
\]

**Remark 2.** If \( \alpha = 0 \) in (2.16), we get \( r = 2 - \sqrt{3} \). This result is obtained by R. J. Libera [2].

**Theorem 2.6.** If \( s(z) \in S_\lambda^*(\alpha) \) (\( \lambda \neq 0 \)), then \( s(z) \) is \( \beta \)-convex in \(|z| < R(\lambda, \alpha, \beta)\), where \( R(\lambda, \alpha, \beta) \) is the smallest positive root of the equation

\[
T(r) = r^3|\sin \lambda|\{2(1 - \alpha)\cos^2 \lambda - 1\} - r^2[2(1 - \alpha)\cos^2 \lambda|\sin \lambda|
\]

\[
- (\beta - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|]
\]

\[
+ r\{2(1 - \alpha)\cos \lambda(\beta + |\sin \lambda|) + |\sin \lambda|\} - |\sin \lambda|,
\]

\begin{equation}
(2.17)
\end{equation}
the result is sharp.

**Proof.** From inequatlity (2.1) we have

\[(2.18) \quad Re \frac{z s'(z)}{s(z)} \geq \frac{\{2(1 - \alpha) \cos^2 \lambda - 1\} r^2 - 2(1 - \alpha) r \cdot \cos \lambda + 1}{1 - r^2}.
\]

Let \(0 \leq \beta \leq 1\). If we multiply both sides of (2.18) by \((1 - \beta)\) and of (2.13) by \(\beta\)

\[
Re J(\beta, s(z)) \geq \frac{-r^3 |\sin \lambda| \{2(1 - \alpha) \cos^2 \lambda - 1\} + r^2 [2(1 - \alpha) \cos^2 \lambda |\sin \lambda|}{(1 - r)^2 (1 + r) |\sin \lambda|} \\
(2.19) \quad (\beta - |\sin \lambda|) \cos \lambda \} - |\sin \lambda| - r \{2(1 - \alpha) \cos \lambda (\beta + |\sin \lambda|) + |\sin \lambda|}{(1 - r)^2 (1 + r) |\sin \lambda|}.
\]

Since \(T(0) < 0\) and \(T(1) > 0\), there exist a real root of \(T(r) = 0\) in \((0, 1)\). Let \(R(\lambda, \alpha, \beta)\) be the smallest positive root \(T(r) = 0\) in \((0, 1)\). Then \(s(z)\) is \(\beta\)-convex in \(|z| < R(\lambda, \alpha, \beta)\). We obtain sharp for the extremal function is given by (2.14).

**Corollary 2.7.** If \(\beta = 1\), then we obtain the radius of convexity for the class of \(\lambda\)-spirallike of order \(\alpha\) functions which is given in Theorem 2.5.

**Corollary 2.8.** If \(\beta = 0\), then

\[
r = \frac{(1 - \alpha) \cos \lambda - \sqrt{1 - (1 - \alpha^2) \cos^2 \lambda}}{2(1 - \alpha) \cos^2 \lambda - 1}.
\]
Remark 3. If $\alpha = 0$ in Corollary 2.8, then $r = \frac{1}{|\sin \lambda| + \cos \lambda}$. It is the radius of starlikeness for $\lambda$-spirallike functions, which was obtained by M. S. Robertson [10] and R. J. Libera [2].

3. The radius of $\beta$-convexity for $p$-fold $\lambda$-spirallike functions

Theorem 3.1. If $f(z) \in S_{\lambda p}^{\ast}(\alpha)$ ($\lambda \neq 0$), then $f(z)$ is $\beta$-convex in $|z| < R(\lambda, \alpha, \beta, p)$, where $R(\lambda, \alpha, \beta, p)$ is the smallest positive root of the equation

\begin{equation}
T(r) = r^{3p}|\sin \lambda|\{2(1-\alpha)\cos^2 \lambda - 1\} - r^{2p}\{2(1-\alpha)\{\cos^2 \lambda|\sin \lambda| - (\beta p - |\sin \lambda|)\cos \lambda\} - |\sin \lambda|\} + r^p\{2(1-\alpha)\cos \lambda(\beta p + |\sin \lambda|) + |\sin \lambda|\} - |\sin \lambda|.
\end{equation}

Proof. From the relation (1.8) we obtain

\begin{equation}
1 + z^p \frac{s''(z^p)}{s'(z^p)} = \frac{1}{p} \left(1 + z \frac{f'(z)}{f(z)}\right) + \left(1 - \frac{1}{p}\right)z \frac{f'(z)}{f(z)}.
\end{equation}

From a simple calculation of (1.8), (2.13) and (2.16) we obtain

\begin{equation}
Re \ J \left(\frac{1}{p}, f(z)\right) \geq \frac{-r^{3p}|\sin \lambda|\{2(1-\alpha)\cos^2 \lambda - 1\}}{(1-r^p)^2(1+r^p)|\sin \lambda|} + \frac{r^{2p}\{2(1-\alpha)\{\cos^2 \lambda|\sin \lambda| - (1 - |\sin \lambda|)\cos \lambda\} - |\sin \lambda|\}}{(1-r^p)^2(1+r^p)|\sin \lambda|} - \frac{r^p\{|\sin \lambda| + 2(1-\alpha)(1 + |\sin \lambda|)\cos \lambda\} + |\sin \lambda|}{(1-r^p)^2(1+r^p)|\sin \lambda|}.
\end{equation}
If we multiply both sides of (3.2) by $\gamma$ and (3.3) by $1 - \gamma$, the add the corresponding members, we obtain

\[
Re J\left(\frac{\gamma}{p}, f(z)\right) \geq \frac{-r^{3p} \sin \lambda \{2(1 - \alpha) \cos^2 \lambda - 1\}}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|} + \frac{r^{2p} \{2(1 - \alpha) \cos^2 \lambda |\sin \lambda| - (\gamma - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|} - \frac{r^p \{2(1 - \alpha) \cos \lambda (\gamma + |\sin \lambda|) + |\sin \lambda|\} + |\sin \lambda|}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|}
\]

where $0 \leq \gamma \leq 1$. If we take $\frac{\gamma}{p} = \beta$ the inequality (3.2) can be written in the form

\[
Re J(\beta, f(z)) \geq \frac{-r^{3p} \sin \lambda \{2(1 - \alpha) \cos^2 \lambda - 1\}}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|} + \frac{r^{2p} \{2(1 - \alpha) \cos^2 \lambda |\sin \lambda| - (\beta p - |\sin \lambda|) \cos \lambda\} - |\sin \lambda|}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|} - \frac{r^p \{2(1 - \alpha) \cos \lambda (\beta p + |\sin \lambda|) + |\sin \lambda|\} + |\sin \lambda|}{(1 - r^p)^2 (1 + r^p) |\sin \lambda|}
\]

where $0 \leq \beta \leq 1$.

Since $T(0) < 0$ and $T(1) > 0$, there exist a real root of $T(r) = 0$ in $(0,1)$. Let $R(\lambda, \alpha, \beta, p)$ be the smallest positive root $T(r) = 0$ in $(0,1)$. Then $f(z)$ is $\beta$-convex in $|z| < R(\lambda, \alpha, \beta, p)$. We obtain sharp because the extremal $f(z) = z/(1 - z^p)^{2(1 - \alpha) \cos \lambda \exp(-i\lambda)/p}$. This shows that the theorem is true.
Corollary 3.2. If $p = 1$, then we obtain the radius of $\beta$-convexity for the class of $\lambda$-spirallike of order $\alpha$ functions which is given in Theorem 2.5.

Corollary 3.3. If $\alpha = 0$, then we obtain the radius of $\beta$-convexity for the class of $\lambda$-spirallike functions.

Corollary 3.4. For $\beta = 0$ we obtain
\[
r = \sqrt{\frac{(1 - \alpha) \cos \lambda - \sqrt{1 - (1 - \alpha) \cos^2 \lambda}}{2(1 - \alpha) \cos^2 \lambda - 1}}.
\]
This is the radius of starlikeness for the $p$-fold $\lambda$-spirallike function. If we take $p = 1$, $\alpha = 0$ and $\beta = 0$, we obtain $r = (|\sin \lambda| + \cos \lambda)^{-1}$, which was obtained by M. S. Roberston [8] and R. J. Libera [2].

Corollary 3.5. In the case $\lambda = 0$, we obtain the radius of $\beta$-convexity for the class of $p$-fold starlike of order $\alpha$ functions. If we take $\alpha = 0$ we obtain the radius of $\beta$-convexity for the class of $p$-fold starlike functions.

Corollary 3.6. For $p = 1$, $\beta = 0$, $\lambda = 0$ and $\alpha = 0$, we obtain $r = 2 - \sqrt{3}$, the radius obtained by R. J. Libera [2].

References


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