ON THE FEKETE-SZEGŐ AND ARGUMENT INEQUALITIES FOR STRONGLY CLOSE-TO-STAR FUNCTIONS

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ABSTRACT. Let $CS(\beta)$ be the class of normalized strongly close-to-star functions of order $\beta$ in the open unit disk. We obtain sharp Fekete-Szegő inequalities for functions belonging to the class $CS(\beta)$. Some sufficient conditions for close-to-star functions also are investigated in a sector. Furthermore, we consider the integral preserving properties for functions in $CS(\beta)$.

1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $S$ be the subclass of $A$ consisting of all univalent functions. We also denote by $S^*$, $K$ and $C$ the subclasses of $A$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{U}$ (see, e.g., Srivastava and Owa [18]).

For analytic functions $g$ and $h$ with $g(0) = h(0)$, $g$ is said to be subordinate to $h$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathcal{U}$), and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

Let

$$S^*[A, B] = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \ (z \in \mathcal{U} ; \ -1 \leq B < A \leq 1) \right\}$$

and

$$K[A, B] = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \ (z \in \mathcal{U} ; \ -1 \leq B < A \leq 1) \right\}.$$

The class $S^*[A, B]$ was studied by Janowski [5] and (more recently) by Silverman and Silvia [17]. Applying the Briot-Bouquet differential
subordination \[10, \text{p. 81}\], we can easily see that \( \mathcal{K}[A, B] \subset S^*[A, B] \). We also note that \( S^*[1, -1] = S^* \) and \( \mathcal{K}[1, -1] = \mathcal{K} \). Furthermore, Silverman and Silvia \[17\] proved that a function \( f \) is in \( S^*[A, B] \) if and only if
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \quad (z \in \mathcal{U} ; B \neq -1) \tag{1.2}
\]
and
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1-A}{2} \quad (z \in \mathcal{U} ; B = -1). \tag{1.3}
\]

A classical result of Fekete and Szegö \[4\] determines the maximum value of \( |a_3 - \mu a_2^2| \), as a function of the real parameter \( \mu \), for functions belonging to \( S \). There are now several results of this type in the literature, each of them dealing with \( |a_3 - \mu a_2^2| \) for various classes of functions (see, e.g., \[2,6-8,14\]).

Denote by \( CS(\beta) \) the class of strongly close-to-star functions of order \( \beta (\beta \geq 0) \). Thus \( f \in CS(\beta) \) if and only if there exists \( g \in S^* \) such that for \( z \in \mathcal{U} \),
\[
\left| \arg \left\{ \frac{f(z)}{g(z)} \right\} \right| \leq \frac{\pi}{2} \beta.
\]
For the case \( \beta = 1 \), \( CS(\beta) \) is the class of close-to-star functions introduced by Reade \[16\]. The close-to-star and similar other functions have been extensively studied by Ahuja and Mogra \[1\], Padmanabhan and Parvatham \[12\], Paravatham and Srinivasan \[13\], Sudharsan et. al. \[19\] and others.

In the present paper, we prove sharp Fekete-Szegö inequalities for functions belonging to the class \( CS(\beta) \). Argument properties also are investigated, which give conditions for close-to-star functions. Furthermore, we consider the integral preserving properties for functions in the class \( CS(\beta) \).

2. Results

To prove our main results, we need the following lemmas.

**Lemma 2.1** \[3,15\]. Let \( p \) be analytic in \( \mathcal{U} \) and satisfy \( \text{Re} \{p(z)\} > 0 \) for \( z \in \mathcal{U} \), with \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \). Then
\[
|p_n| \leq 2 \quad (n \geq 1)
\]
and
\[
\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.
\]
Lemma 2.2 [11]. Let $p$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\left| \arg \{p(z)\} \right| < \frac{\pi}{2} \eta \quad \text{for} \quad |z| < |z_0|$$

and

$$\left| \arg \{p(z_0)\} \right| = \frac{\pi}{2} \eta(0 < \eta \leq 1).$$

Then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{p(z_0)\} = \frac{\pi}{2} \eta,$$

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{p(z_0)\} = -\frac{\pi}{2} \eta,$$

and

$$\{p(z_0)\}^\frac{1}{\eta} = \pm ia \quad (a > 0).$$

Lemma 2.3 [9]. Let $h$ be convex (univalent) function in $\mathcal{U}$ and $\omega$ be an analytic function in $\mathcal{U}$ with $\Re \{\omega(z)\} \geq 0$. If $p$ is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$

With the help of Lemma 2.1, we now derive

Theorem 2.1. Let $f \in CS(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
1 + 2(1 + \beta)^2(1 - 2\mu) & \text{if} \quad \mu \leq \frac{\beta}{2(1+\beta)}, \\
1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)} & \text{if} \quad \frac{\beta}{2(1+\beta)} \leq \mu \leq \frac{1}{2}, \\
1 + 2\beta & \text{if} \quad \frac{1}{2} \leq \mu \leq \frac{2+\beta}{2(1+\beta)}, \\
-1 + 2(1 + \beta)^2(2\mu - 1) & \text{if} \quad \mu \geq \frac{2+\beta}{2(1+\beta)}. 
\end{cases}$$

For each $\mu$, there is a function in $CS(\beta)$ such that equality holds in all cases.
Proof. Let \( f \in \text{CS}(\beta) \). Then it follows from the definition that we may write

\[
\frac{f(z)}{g(z)} = p^\beta(z),
\]

where \( g \) is starlike and \( p \) has positive real part. Let \( g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \), and let \( p \) be given as in Lemma 2.1. Then by equating coefficients, we obtain

\[
a_2 = b_2 + \beta p_1
\]

and

\[
a_3 = b_3 + \beta p_1 b_2 + \frac{\beta(\beta - 1)}{2} p_1^2 + \beta p_2.
\]

So, with \( x = 1 - 2\mu \), we have

\[
(a_3 - \mu a_2^2) = b_3 + \frac{1}{2} (x - 1) b_2^2 + \beta \left( p_2 + \frac{1}{2} (\beta x - 1) p_1^2 \right) + \beta x p_1 b_2.
\] (2.7)

Since rotations of \( f \) also belong to \( \text{CS}(\beta) \), we may assume, without loss of generality, that \( a_3 - \mu a_2^2 \) is positive. Thus we now estimate \( \text{Re}(a_3 - \mu a_2^2) \).

For some functions \( h(z) = 1 + k_1 z + k_2 z^2 + \cdots (z \in \mathcal{U}) \) with positive real part, we have \( zg'(z) = g(z) h(z) \). Hence, by equating coefficients, \( b_2 = k_1 \) and \( b_3 = (k_2 + k_1^2)/2 \). So by Lemma 2.1,

\[
\text{Re}\left(b_3 + \frac{1}{2} (x - 1) b_2^2\right) = \frac{1}{2} \text{Re}\left(k_2 - \frac{1}{2} k_1^2\right) + \frac{1 + 2x}{4} \text{Re} k_1^2
\]

\[
\leq 1 - \rho^2 + (1 + 2x) \rho^2 \cos 2\phi,
\] (2.8)

where \( b_2 = k_1 = 2\rho e^{i\theta \phi} \) for some \( \rho \) in \([0,1]\). We also have

\[
\text{Re}\left(p_2 + \frac{1}{2} (\beta x - 1) p_1^2\right) = \text{Re}\left(p_2 - \frac{1}{2} p_1^2\right) + \frac{1}{2} \beta x \text{Re} p_1^2
\]

\[
\leq 2(1 - r^2) + 2 \beta x r^2 \cos 2\theta,
\] (2.9)

where \( p_1 = 2r e^{i\theta} \) for some \( r \) in \([0,1]\). From (2.7-9), we obtain

\[
\text{Re}(a_3 - \mu a_2^2) \leq 1 - \rho^2 + (1 + 2x) \rho^2 \cos 2\phi + 2 \beta ((1 - r^2)
\]

\[
+ \beta x r^2 \cos 2\theta + 2 x r \cos(\theta + \phi)),
\] (2.10)

and we now proceed to maximize the right-hand side of (2.10). This function will be denote \( \psi \) whenever all parameters except \( x \) are held constant.

Assume that \( \beta/(2(1 + \beta)) \leq \mu \leq 1/2 \), so that \( 0 \leq x \leq 1/(1 + \beta) \). Since the expression \(-t^2 + t^2 \beta x \cos 2\theta + 2xt \) is the largest when \( t = x/(1 - \beta x \cos 2\theta) \), we have
Thus

\[-t^2 + t^2 \beta x \cos 2\theta + 2xt \leq \frac{x^2}{1 - \beta x \cos 2\theta} \leq \frac{x^2}{1 - \beta x}.
\]

Thus

\[
\psi(x) \leq 1 + 2x + 2\beta \left(1 + \frac{x^2}{1 - \beta x}\right) = 1 + 2\beta + \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)}
\]

and with (2.10) this establishes the second inequality in the theorem. Equality occurs only if

\[
p_1 = \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)}, \quad p_2 = b_2 = 2, \quad b_3 = 3,
\]

and the corresponding function \(f\) is defined by

\[
f(z) = \frac{z}{(1 - z)^2} \left(\frac{1+z}{1-z} + (1-\lambda)\frac{1-z}{1+z}\right)^\beta, \quad f(0) = 0,
\]

where

\[
\lambda = \frac{1 + (1 - 2\beta)(1 - 2\mu)}{2(1 - \beta(1 - 2\mu))}.
\]

We now prove the first inequality. Let \(\mu \leq \beta/(2(1 + \beta))\), so that \(x \geq 1/(1 + \beta)\). With \(x_0 = 1/(1 + \beta)\), we have

\[
\psi(x) = \psi(x_0) + 2(x - x_0)(\rho^2 \cos 2\phi + \beta^2 \rho^2 \cos 2\theta + 2\rho \beta \cos(\theta + \phi))
\]
\[
\leq \psi(x_0) + 2(x - x_0)(1 + \beta)^2
\]
\[
\leq 1 + 2(1 + \beta)^2(1 - 2\mu),
\]

as required. Equality occurs only if \(p_1 = p_2 = b_2 = 2, \quad b_3 = 3\), and the corresponding function \(f\) is defined by

\[
f(z) = \frac{z}{(1 - z)^2} \left(\frac{1+z}{1-z}\right)^\beta, \quad f(0) = 0.
\]

Let \(x_1 = -1/(1 + \beta)\). We shall find that \(\psi(x_1) = 1 + 2\beta\), and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

\[
\psi(x) \leq \psi(x_1) + 2|x - x_1|(1 + \beta)^2
\]
\[
\leq -1 + 2(1 + \beta)^2(2\mu - 1),
\]

if \(x \leq x_1\), that is, \(\mu \geq (2 + \beta)/(2(1 + \beta))\). Equality occurs only if \(p_1 = 2i, \quad p_2 = -2, \quad b_2 = 2i, \quad b_3 = -3\), and the corresponding function \(f\) is defined by

\[
f(z) = \frac{z}{(1 - iz)^2} \left(\frac{1+iz}{1-iz}\right)^\beta, \quad f(0) = 0.
\]
Also, for $0 \leq \lambda \leq 1$,

$$\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0) \leq \lambda(1 + 2\beta) + (1 - \lambda)(1 + 2\beta) = 1 + 2\beta,$$

so, we obtain $\psi(x) \leq 1 + 2\beta$ for $x_1 \leq x \leq 0$, i.e., $1/2 \leq \mu \leq (2 + \beta)/(1 + \beta)$. Equality occurs only if $p_1 = b_2 = 0$, $p_2 = 2$, $b_3 = 1$, and the corresponding function $f$ is defined by

$$f(z) = \frac{z(1 + z^2)^\beta}{(1 - z^2)^{1 + \beta}}, \quad f(0) = 0.$$

We now show that $\psi(x_1) \leq 1 + 2\beta$. We have

$$-t^2 + t^2 \beta x \cos 2\theta + 2xt \rho \cos(\theta + \phi) \leq \frac{x^2 \rho^2 \cos^2(\theta + \phi)}{1 - \beta x \cos 2\theta}$$

for real $t$, and so

$$\psi(x) - 1 - 2\beta \leq \rho^2 \left(-1 + (1 + 2x) \cos 2\phi + \frac{\beta x^2 (1 + \cos 2(\theta + \phi))}{1 - \beta x \cos 2\theta}\right).$$

Thus we consider the inequality

$$\beta x^2(1 + \cos 2(\theta + \phi)) + (1 - \beta x \cos 2\theta)(-1 + (1 + 2x) \cos 2\phi) \leq 0$$

with $x = x_1$. After some simplifications, this becomes

$$2\beta^2 \sin^2 \phi \cos^2 \phi + 2\beta \cos \theta \sin \theta \sin \phi + \cos^2 \phi \geq 0. \quad (2.11)$$

Now, for all real $t$, we note that

$$2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,$$

so, by taking $t = \beta \sin \phi \cos \theta$, we obtain (2.11). Therefore we complete the proof of Theorem 2.1.

Next, we prove

**Theorem 2.2.** Let $f \in A$. If

$$\left| \arg \left\{ \left( \frac{f'(z)}{g'(z)} \right)^\alpha (\frac{f(z)}{g(z)})^\beta \right\} \right| < \frac{\pi}{2} \delta \quad (\alpha > 0; \beta \in \mathbb{R}; 0 < \delta \leq 1)$$

for some $g \in \mathcal{K}[A, B]$, then

$$\left| \arg \left( \frac{f(z)}{g(z)} \right) \right| < \frac{\pi}{2} \eta,$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equation:
$$\delta = \begin{cases} 
(\alpha + \beta)\eta + \frac{2}{\pi} \alpha \tan^{-1} \left( \frac{\eta \sin \left[ \frac{\pi}{2} \{1 - t(A, B)\} \right]}{1 + A \pm B + \eta \cos \left[ \frac{\pi}{2} \{1 - t(A, B)\} \right]} \right) & (B \neq -1) \\
(\alpha + \beta)\eta & (B = -1) 
\end{cases} \quad (2.12)$$

and

$$t(A, B) = \frac{2}{\pi} \sin^{-1} \left( \frac{A - B}{1 - AB} \right). \quad (2.13)$$

**Proof.** Let

$$p(z) = \frac{f(z)}{g(z)} \quad \text{and} \quad q(z) = \frac{zg'(z)}{g(z)}.$$ 

Then, by a simple calculation, we have

$$\left( \frac{f'(z)}{g'(z)} \right)^\alpha \left( \frac{f(z)}{g(z)} \right)^\beta = (p(z))^{\alpha + \beta} \left( 1 + \frac{1}{q(z)} \frac{zp'(z)}{p(z)} \right)^\alpha.$$

Since $g \in \mathcal{K}[A, B], g \in \mathcal{S}^*[A, B]$. If we let

$$q(z) = \rho e^{i\frac{\pi}{2}\phi} \quad (z \in \mathcal{U}),$$

then it follows from (1.2) and (1.3) that

$$\begin{cases} 
\frac{1-A}{1+B} < \rho < \frac{1+A}{1+B} \\
-t(A, B) < \phi < t(A, B) & (B \neq -1) 
\end{cases}$$

and

$$\begin{cases} 
\frac{1-A}{2} < \rho < \infty \\
-1 < \phi < 1 & (B = -1), 
\end{cases}$$

where $t(A, B)$ is defined by (2.13).

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.2) we obtain (2.3) under the restrictions (2.4-6).

At first, we suppose that

$$\{p(z_0)\}^{\frac{1}{\eta}} = ia \quad (a > 0).$$

For the case $B \neq -1$, we then obtain
\[
\arg \left\{ \left( \frac{f'(z_0)}{g'(z_0)} \right)^\alpha \left( \frac{f(z_0)}{g(z_0)} \right)^\beta \right\} = \arg \left\{ (p(z_0))^{\alpha+\beta} \left( 1 + \frac{1}{q(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right)^\alpha \right\}
\]
\[
= \arg \left\{ (p(z_0))^{\alpha+\beta} \right\} + \arg \left\{ \left( 1 + i\eta k(\rho e^{i\frac{\pi}{2}\phi})^{-1} \right)^\alpha \right\}
\]
\[
\geq (\alpha + \beta) \frac{\pi}{2} \eta + \alpha \tan^{-1} \left( \frac{\eta k \sin \left[ \frac{\pi}{2}(1 - \phi) \right]}{\rho + \eta k \cos \left[ \frac{\pi}{2}(1 - \phi) \right]} \right)
\]
\[
= \frac{\pi}{2} \delta,
\]
where \( \delta \) and \( t(A, B) \) are given by (2.12) and (2.13), respectively. Similarly, for the case \( B = -1 \), we have

\[
\arg \left\{ \left( \frac{f'(z_0)}{g'(z_0)} \right)^\alpha \left( \frac{f(z_0)}{g(z_0)} \right)^\beta \right\} \geq (\alpha + \beta) \frac{\pi}{2} \eta = \frac{\pi}{2} \delta.
\]
These evidently contradict the assumption of the theorem.

Next, in the case \( p(z_0)^{\frac{1}{\eta}} = -ia \) \((a > 0)\), applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

By setting \( \alpha = 1, \beta = 0, \delta = 1, A = 1 \) and \( B = -1 \) in Theorem 2.2, we have

**Corollary 2.1.** Every close-to-convex function is close-to-star in \( \mathcal{U} \).

If we put \( g(z) = z \) in Theorem 2.2, then, by letting \( B \to A \) \((A < 1)\), we obtain

**Corollary 2.2.** If \( f \in \mathcal{A} \) and

\[
\left| \arg \left\{ \left( f'(z) \right)^\alpha \left( \frac{f(z)}{z} \right)^\beta \right\} \right| < \frac{\pi}{2} \delta \quad (\alpha > 0; \beta \in \mathbb{R}; 0 < \delta \leq 1),
\]
then

\[
\left| \arg \{ f'(z) \} \right| < \frac{\pi}{2} \eta,
\]
where \( \eta \) \((0 < \eta \leq 1)\) is the solution of the equation:

\[
\delta = (\alpha + \beta) \eta + \frac{2}{\pi} \alpha \tan^{-1} \eta.
\]
For a function \( f \) belonging to the class \( A \), we define the integral operator \( F_c \) as follows:

\[
F_c(f) := F_c(f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1}g(t)dt \quad (c \geq 0 ; \ z \in \mathcal{U}). 
\] (2.14)

For various interesting developments involving the operator (2.14), the reader may be referred (for example) to the recent works of Miller and Mocanu [10] and Srivastava and Owa [18].

Finally, we prove

**Theorem 2.3.** Let \( f \in A \). If

\[
\left| \arg\left( \frac{f(z)}{g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta \quad (0 < \gamma \leq 1 ; \ 0 < \delta \leq 1)
\]

for some \( g \in S^*[A, B] \), then

\[
\left| \arg\left( \frac{F_c(f)}{F_c(g)} - \gamma \right) \right| < \frac{\pi}{2} \eta,
\]

where the operator \( F_c \) is given by (2.14) and \( \eta(0 < \eta \leq 1) \) is the solution of the equation

\[
\delta = \begin{cases} 
\eta + \frac{2}{\pi} \tan^{-1} \left( \frac{3 \sin \frac{\pi}{2} (1-t(A,B,c))}{(1+\delta)+\eta \cos \frac{\pi}{2} (1-t(A,B,c))} \right) & \text{for } B \neq -1, \\
\eta & \text{for } B = -1,
\end{cases}
\]

when

\[
t(A, B, c) = \frac{2}{\pi} \sin^{-1} \left( \frac{A-B}{1-AB+c(1-B^2)} \right) 
\] (2.15)

**Proof.** Let

\[
p(z) = \frac{1}{1-\gamma} \left( \frac{F_c(f)}{F_c(g)} - \gamma \right) \quad \text{and} \quad q(z) = \frac{zF_c'(g)}{F_c(g)}.
\]

From the assumption for \( g \) and an application of Briot-Bouquet differential equation [10, p. 81], we see that \( F_c(g) \in S^*[A, B] \). Using the equation

\[
zF_c'(f)(z) + cF_c(f)(z) = (1+c)f(z)
\]

and simplifying, we obtain

\[
\frac{1}{1-\gamma} \left( \frac{f(z)}{g(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{q(z) + c}.
\]

Then, by applying (1.2) and (1.3), we have

\[
q(z) + c = \rho e^{i\frac{\pi}{2} \phi},
\]
where
\[
\begin{cases}
\frac{1-A}{1-B} + c < \rho < \frac{1+A}{1+B} + c \\
-t(A, B, c) < \phi < t(A, B, c)
\end{cases}
\]
for \(B \neq -1\),
when \(t(A, B, c)\) is given by (2.16), and
\[
\begin{cases}
\frac{1-A}{2} + c < \rho < \infty \\
-1 < \phi < 1
\end{cases}
\]
for \(B = -1\).

Here, we note that \(p\) is analytic in \(U\) with \(p(0) = 1\) and \(\text{Re } p(z) > 0\) in \(U\) by applying the assumption and Lemma 2.3 with \(\omega(z) = 1/(q(z)+c)\). Hence \(p(z) \neq 0\) in \(U\). The remaining part of the proof of Theorem 2.3 is similar to that of Theorem 2.2, and so we omit it.

**Remark.** From Theorem 2.3, we see easily that every function in \(CS(\delta) (0 < \delta \leq 1)\) preserves the angles under the integral operator defined by (2.14).

By letting \(A = 1 - 2\beta (0 \leq \beta \leq 1)\), \(B = -1\), \(\delta = 1\) in Theorem 2.3, we obtain

**Corollary 2.3.** If \(f \in A\) and
\[
\text{Re } \left\{ \frac{f(z)}{g(z)} \right\} > \gamma \quad (0 \leq \gamma < 1; \; z \in \mathcal{U}),
\]
for some \(g\) such that
\[
\text{Re } \left\{ \frac{zg'(z)}{g(z)} \right\} > \beta \quad (0 \leq \beta < 1; \; z \in \mathcal{U}),
\]
then
\[
\text{Re } \left\{ \frac{F_c(f)}{F_c(g)} \right\} > \gamma \quad (0 \leq \gamma < 1; \; z \in \mathcal{U}),
\]
where \(F_c\) is given by (2.14).

If we take \(g(z) = z\) in Theorem 2.3, then, by letting \(B \to A \quad (A < 1)\), we have

**Corollary 2.4.** If \(f \in A\) and
\[
\left| \arg \left( \frac{f(z)}{z} - \gamma \right) \right| < \frac{\pi}{2\delta} \quad (0 \leq \gamma < 1; \; 0 < \delta \leq 1),
\]
then
\[
\left| \arg \left( \frac{F_c(f)}{z} - \gamma \right) \right| < \frac{\pi}{2\eta},
\]
where $F_c$ is given by (2.14) and $\eta(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1}\left(\frac{\eta}{1+c}\right).$$

References


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