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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2001(1192): 1-12</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64778">http://hdl.handle.net/2433/64778</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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ON THE FEKETE-SZEGÖ AND ARGUMENT INEQUALITIES FOR STRONGLY CLOSE-TO-STAR FUNCTIONS

NAK EUN CHO AND SHIGEYOSHI OWA

ABSTRACT. Let $CS(\beta)$ be the class of normalized strongly close-to-star functions of order $\beta$ in the open unit disk. We obtain sharp Fekete-Szegő inequalities for functions belonging to the class $CS(\beta)$. Some sufficient conditions for close-to-star functions also are investigated in a sector. Furthermore, we consider the integral preserving properties for functions in $CS(\beta)$.

1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $S$ be the subclass of $A$ consisting of all univalent functions. We also denote by $S^*$, $K$ and $C$ the subclasses of $A$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{U}$ (see, e.g., Srivastava and Owa [18]).

For analytic functions $g$ and $h$ with $g(0) = h(0)$, $g$ is said to be subordinate to $h$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathcal{U}$), and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

Let

$$S^*[A, B] = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \ (z \in \mathcal{U} ; \ -1 \leq B < A \leq 1) \right\}$$

and

$$K[A, B] = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \ (z \in \mathcal{U} ; \ -1 \leq B < A \leq 1) \right\}.$$
subordination [10, p. 81], we can easily see that $\mathcal{K}[A, B] \subset S^*[A, B]$. We also note that $S^*[1, -1] = S^*$ and $\mathcal{K}[1, -1] = \mathcal{K}$. Furthermore, Silverman and Silvia [17] proved that a function $f$ is in $S^*[A, B]$ if and only if
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (z \in \mathcal{U} ; B \neq -1)
\] (1.2)
and
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1 - A}{2} \quad (z \in \mathcal{U} ; B = -1).
\] (1.3)

A classical result of Fekete and Szegő [4] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter $\mu$, for functions belonging to $S$. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [2, 6-8, 14]).

Denote by $CS(\beta)$ the class of strongly close-to-star functions of order $\beta (\beta \geq 0)$. Thus $f \in CS(\beta)$ if and only if there exists $g \in S^*$ such that for $z \in \mathcal{U}$,
\[
\left| \arg \left\{ \frac{f(z)}{g(z)} \right\} \right| \leq \frac{\pi}{2} \beta.
\]
For the case $\beta = 1$, $CS(\beta)$ is the class of close-to-star functions introduced by Reade [16]. The close-to-star and similar other functions have been extensively studied by Ahuja and Mogra [1], Padmanabhan and Parvatham [12], Parvatham and Srinivasan [13], Sudharsan et al. [19] and others.

In the present paper, we prove sharp Fekete-Szegő inequalities for functions belonging to the class $CS(\beta)$. Argument properties also are investigated, which give conditions for close-to-star functions. Furthermore, we consider the integral preserving properties for functions in the class $CS(\beta)$.

2. Results

To prove our main results, we need the following lemmas.

Lemma 2.1 [3, 15]. Let $p$ be analytic in $\mathcal{U}$ and satisfy $\text{Re} \{p(z)\} > 0$ for $z \in \mathcal{U}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then
\[
|p_n| \leq 2 \quad (n \geq 1)
\]
and
\[
\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.
\]
Lemma 2.2 [11]. Let \( p \) be analytic in \( \mathcal{U} \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) in \( \mathcal{U} \). Suppose that there exists a point \( z_0 \in \mathcal{U} \) such that

\[
\left| \arg \{p(z)\} \right| < \frac{\pi}{2} \eta \quad \text{for} \quad |z| < |z_0| \tag{2.1}
\]

and

\[
\left| \arg \{p(z_0)\} \right| = \frac{\pi}{2} \eta \quad (0 < \eta \leq 1).
\tag{2.2}
\]

Then

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,
\tag{2.3}
\]

where

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{p(z_0)\} = \frac{\pi}{2} \eta, \tag{2.4}
\]

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg \{p(z_0)\} = -\frac{\pi}{2} \eta, \tag{2.5}
\]

and

\[
\{p(z_0)\}^{\frac{1}{n}} = \pm ia \quad (a > 0). \tag{2.6}
\]

Lemma 2.3 [9]. Let \( h \) be convex(univalent) function in \( \mathcal{U} \) and \( \omega \) be an analytic function in \( \mathcal{U} \) with \( \text{Re} \{\omega(z)\} \geq 0 \). If \( p \) is analytic in \( \mathcal{U} \) and \( p(0) = h(0) \), then

\[
p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathcal{U})
\]

implies

\[
p(z) \prec h(z) \quad (z \in \mathcal{U}).
\]

With the help of Lemma 2.1, we now derive

Theorem 2.1. Let \( f \in \text{CS}(\beta) \) and be given by (1.1). Then for \( \beta \geq 0 \), we have

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
1 + 2(1 + \beta)^2(1 - 2\mu) & \text{if} \quad \mu \leq \frac{\beta}{2(1+\beta)}, \\
1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)} & \text{if} \quad \frac{\beta}{2(1+\beta)} \leq \mu \leq \frac{1}{2}, \\
1 + 2\beta & \text{if} \quad \frac{1}{2} \leq \mu \leq \frac{2+\beta}{2(1+\beta)}, \\
-1 + 2(1 + \beta)^2(2\mu - 1) & \text{if} \quad \mu \geq \frac{2+\beta}{2(1+\beta)}.
\end{cases}
\]

For each \( \mu \), there is a function in \( \text{CS}(\beta) \) such that equality holds in all cases.
Proof. Let $f \in CS(\beta)$. Then it follows from the definition that we may write

$$\frac{f(z)}{g(z)} = p^\beta(z),$$

where $g$ is starlike and $p$ has positive real part. Let $g(z) = z + b_2z^2 + b_3z^3 + \cdots$, and let $p$ be given as in Lemma 2.1. Then by equating coefficients, we obtain

$$a_2 = b_2 + \beta p_1$$

and

$$a_3 = b_3 + \beta p_1 b_2 + \frac{\beta(\beta - 1)}{2} p_1^2 + \beta p_2.$$

So, with $x = 1 - 2\mu$, we have

$$(a_3 - \mu a_2^2) = b_3 + \frac{1}{2}(x - 1)b_2^2 + \beta \left(p_2 + \frac{1}{2}(\beta x - 1)p_1^2\right) + \beta xp_1 b_2. \tag{2.7}$$

Since rotations of $f$ also belong to $CS(\beta)$, we may assume, without loss of generality, that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\text{Re}(a_3 - \mu a_2^2)$.

For some functions $h(z) = 1 + k_1z + k_2z^2 + \cdots \ (z \in \mathcal{U})$ with positive real part, we have $zg'(z) = g(z)h(z)$. Hence, by equating coefficients, $b_2 = k_1$ and $b_3 = (k_2 + k_1^2)/2$. So by Lemma 2.1,

$$\text{Re}\left(b_3 + \frac{1}{2}(x - 1)b_2^2\right) = \frac{1}{2}\text{Re}\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1 + 2x}{4}\text{Re}k_1^2 \leq 1 - \rho^2 + (1 + 2x)\rho^2 \cos 2\phi, \tag{2.8}$$

where $b_2 = k_1 = 2\rho e^{i\theta}$ for some $\rho$ in $[0,1]$. We also have

$$\text{Re}\left(p_2 + \frac{1}{2}(\beta x - 1)p_1^2\right) = \text{Re}\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{2}\beta x\text{Re}p_1^2 \leq 2(1 - r^2) + 2\beta xr^2 \cos 2\theta, \tag{2.9}$$

where $p_1 = 2re^{i\theta}$ for some $r$ in $[0,1]$. From (2.7-9), we obtain

$$\text{Re}(a_3 - \mu a_2^2) \leq 1 - \rho^2 + (1 + 2x)\rho^2 \cos 2\phi + 2\beta((1 - r^2)$$

$$+ \beta xr^2 \cos 2\theta + 2xr \rho \cos(\theta + \phi)), \tag{2.10}$$

and we now proceed to maximize the right-hand side of (2.10). This function will be denote $\psi$ whenever all parameters except $x$ are held constant.

Assume that $\beta/(2(1 + \beta)) \leq \mu \leq 1/2$, so that $0 \leq x \leq 1/(1 + \beta)$. Since the expression $-t^2 + t^2\beta x \cos 2\theta + 2xt$ is the largest when $t = x/(1 - \beta x \cos 2\theta)$, we have
\[-t^2 + t^2 \beta x \cos 2\theta + 2xt \leq \frac{x^2}{1 - \beta x \cos 2\theta} \leq \frac{x^2}{1 - \beta x}.

Thus

\[\psi(x) \leq 1 + 2x + 2\beta \left(1 + \frac{x^2}{1 - \beta x}\right) = 1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)}\]

and with (2.10) this establishes the second inequality in the theorem. Equality occurs only if

\[p_1 = \frac{2(1-2\mu)}{1-\beta(1-2\mu)}, \ p_2 = b_2 = 2, \ b_3 = 3,\]

and the corresponding function \(f\) is defined by

\[f(z) = \frac{z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^\beta, \ f(0) = 0,\]

where

\[\lambda = \frac{1+(1-2\beta)(1-2\mu)}{2(1-\beta(1-2\mu))}.

We now prove the first inequality. Let \(\mu \leq \beta/(2(1+\beta))\), so that \(x \geq 1/(1+\beta)\). With \(x_0 = 1/(1+\beta)\), we have

\[\psi(x) = \psi(x_0) + 2(x - x_0)(\rho^2 \cos 2\phi + \beta^2 \rho^2 \cos 2\theta + 2\rho \beta \rho \cos(\theta + \phi)
\leq \psi(x_0) + 2(x - x_0)(1 + \beta)^2
\leq 1 + 2(1+\beta)^2(1-2\mu),\]

as required. Equality occurs only if \(p_1 = p_2 = b_2 = 2, \ b_3 = 3,\) and the corresponding function \(f\) is defined by

\[f(z) = \frac{z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^\beta, \ f(0) = 0.\]

Let \(x_1 = -1/(1+\beta)\). We shall find that \(\psi(x_1) = 1 + 2\beta\), and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

\[\psi(x) \leq \psi(x_1) + 2|x - x_1|(1+\beta)^2
\leq -1 + 2(1+\beta)^2(2\mu - 1),\]

if \(x \leq x_1\), that is, \(\mu \geq (2 + \beta)/(2(1+\beta))\). Equality occurs only if \(p_1 = 2i, \ p_2 = -2, \ b_2 = 2i, \ b_3 = -3,\) and the corresponding function \(f\) is defined by

\[f(z) = \frac{z}{(1-iz)^2} \left(\frac{1+iz}{1-iz}\right)^\beta, \ f(0) = 0.\]
Also, for $0 \leq \lambda \leq 1$,

$$\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0) \leq \lambda(1 + 2\beta) + (1 - \lambda)(1 + 2\beta) = 1 + 2\beta,$$

so, we obtain $\psi(x) \leq 1 + 2\beta$ for $x_1 \leq x \leq 0$, i.e., $1/2 \leq \mu \leq (2 + \beta)/(1 + \beta)$. Equality occurs only if $p_1 = b_2 = 0$, $p_2 = 2$, $b_3 = 1$, and the corresponding function $f$ is defined by

$$f(z) = \frac{z(1 + z^2)^{\beta}}{(1 - z^2)^{1+\beta}}, \quad f(0) = 0.$$

We now show that $\psi(x_1) \leq 1 + 2\beta$. We have

$$-t^2 + t^2 \beta x \cos 2\theta + 2xt \rho \cos(\theta + \phi) \leq \frac{x^2 \rho^2 \cos^2(\theta + \phi)}{1 - \beta x \cos 2\theta}$$

for real $t$, and so

$$\psi(x) - 1 - 2\beta \leq \rho^2 \left(-1 + (1 + 2x) \cos 2\phi + \frac{\beta x^2(1 + \cos 2(\theta + \phi))}{1 - \beta x \cos 2\theta}\right).$$

Thus we consider the inequality

$$\beta x^2(1 + \cos 2(\theta + \phi)) + (1 - \beta x \cos 2\theta)(-1 + (1 + 2x) \cos 2\phi) \leq 0$$

with $x = x_1$. After some simplifications, this becomes

$$2\beta^2 \sin^2 \phi \cos^2 \phi + 2\beta \cos \theta \sin \theta \sin \phi + \cos^2 \phi \geq 0. \quad (2.11)$$

Now, for all real $t$, we note that

$$2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,$$

so, by taking $t = \beta \sin \phi \cos \theta$, we obtain (2.11). Therefore we complete the proof of Theorem 2.1.

Next, we prove

**Theorem 2.2.** Let $f \in A$. If

$$\left| \arg \left( \frac{f'(z)}{g'(z)} \right)^{\alpha} \left( \frac{f(z)}{g(z)} \right)^{\beta} \right| < \frac{\pi}{2} \delta \quad (\alpha > 0; \ \beta \in \mathbb{R}; \ 0 < \delta \leq 1)$$

for some $g \in \mathcal{K}[A, B]$, then

$$\left| \arg \left( \frac{f(z)}{g(z)} \right) \right| < \frac{\pi}{2} \eta,$$

where $\eta \ (0 < \eta \leq 1)$ is the solution of the equation:
\[
\delta = \begin{cases} 
(\alpha + \beta)\eta + \frac{2}{\pi} \frac{\eta \sin \left[ \frac{\pi}{2} \{1 - t(A, B)\} \right]}{1 + \frac{A}{1 + B} + \eta \cos \left[ \frac{\pi}{2} \{1 - t(A, B)\} \right]} & (B \neq -1) \\
(\alpha + \beta)\eta & (B = -1)
\end{cases}
\] 

and

\[
t(A, B) = \frac{2}{\pi} \sin^{-1} \left( \frac{A - B}{1 - AB} \right).
\]

**Proof.** Let

\[
p(z) = \frac{f(z)}{g(z)} \quad \text{and} \quad q(z) = \frac{zg'(z)}{g(z)}.
\]

Then, by a simple calculation, we have

\[
\left( \frac{f'(z)}{g'(z)} \right)^{\alpha} \left( \frac{f(z)}{g(z)} \right)^{\beta} = (p(z))^{\alpha + \beta} \left( 1 + \frac{1}{q(z)} \frac{zp'(z)}{p(z)} \right)^{\alpha}.
\]

Since \( g \in \mathcal{K}[A, B] \), \( g \in \mathcal{S}^*[A, B] \). If we let

\[
q(z) = \rho e^{i\frac{\pi}{2}\phi} \quad (z \in \mathcal{U}),
\]

then it follows from (1.2) and (1.3) that

\[
\begin{cases}
\frac{1 - A}{1 - B} < \rho < \frac{1 + A}{1 + B} \\
-t(A, B) < \phi < t(A, B)
\end{cases} \quad (B \neq -1)
\]

and

\[
\begin{cases}
\frac{1 - A}{2} < \rho < \infty \\
-1 < \phi < 1
\end{cases} \quad (B = -1),
\]

where \( t(A, B) \) is defined by (2.13).

If there exists a point \( z_0 \in \mathcal{U} \) such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.2) we obtain (2.3) under the restrictions (2.4-6).

At first, we suppose that

\[
\{p(z_0)\}^{\frac{1}{n}} = ia \quad (a > 0).
\]

For the case \( B \neq -1 \), we then obtain
\[
\arg \left\{ \left( \frac{f'(z_0)}{g'(z_0)} \right)^\alpha \left( \frac{f(z_0)}{g(z_0)} \right)^\beta \right\} = \arg \left\{ (p(z_0))^{\alpha+\beta} \left( 1 + \frac{1}{q(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right)^\alpha \right\} = \arg \left\{ (p(z_0))^{\alpha+\beta} \cdot \left( 1 + \frac{1}{q(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right)^\alpha \right\}
\]

\[
= (\alpha + \beta) \frac{\pi}{2} \eta + \alpha \tan^{-1} \left( \frac{\eta k \sin[\frac{\pi}{2}(1 - \phi)]}{\rho + \eta k \cos[\frac{\pi}{2}(1 - \phi)]} \right) \geq (\alpha + \beta) \frac{\pi}{2} \eta + \alpha \tan^{-1} \left( \frac{\eta \sin[\frac{\pi}{2}(1 - t(A, B))]}{\frac{1+A}{1+B} + \eta \cos[\frac{\pi}{2}(1 - t(A, B))]} \right) = \frac{\pi}{2} \delta,
\]

where \( \delta \) and \( t(A, B) \) are given by (2.12) and (2.13), respectively. Similarly, for the case \( B = -1 \), we have

\[
\arg \left\{ \left( \frac{f'(z_0)}{g'(z_0)} \right)^\alpha \left( \frac{f(z_0)}{g(z_0)} \right)^\beta \right\} \geq (\alpha + \beta) \frac{\pi}{2} \eta = \frac{\pi}{2} \delta.
\]

These evidently contradict the assumption of the theorem.

Next, in the case \( p(z_0)^{\frac{1}{\eta}} = -ia \) (\( a > 0 \)), applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

By setting \( \alpha = 1, \beta = 0, \delta = 1, A = 1 \) and \( B = -1 \) in Theorem 2.2, we have

**Corollary 2.1.** Every close-to-convex function is close-to-star in \( \mathcal{U} \).

If we put \( g(z) = z \) in Theorem 2.2, then, by letting \( B \to A (A < 1) \), we obtain

**Corollary 2.2.** If \( f \in \mathcal{A} \) and

\[
\left| \arg \left\{ \left( \frac{f'(z)}{z} \right)^\alpha \left( \frac{f(z)}{z} \right)^\beta \right\} \right| < \frac{\pi}{2} \delta \quad (\alpha > 0; \beta \in \mathbb{R}; 0 < \delta \leq 1),
\]

then

\[
|\arg \{f'(z)\}| < \frac{\pi}{2} \eta,
\]

where \( \eta (0 < \eta \leq 1) \) is the solution of the equation:

\[
\delta = (\alpha + \beta) \eta + \frac{2}{\pi} \alpha \tan^{-1}(\eta).
\]
For a function $f$ belonging to the class $A$, we define the integral operator $F_c$ as follows:

\[
F_c(f) := F_c(f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} g(t) \, dt \quad (c \geq 0; \; z \in \mathcal{U}).
\] (2.14)

For various interesting developments involving the operator (2.14), the reader may be referred (for example) to the recent works of Miller and Mocanu [10] and Srivastava and Owa [18].

Finally, we prove

**Theorem 2.3.** Let $f \in A$. If

\[
\left| \arg \left( \frac{f(z)}{g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta \quad (0 < \gamma \leq 1; \; 0 < \delta \leq 1)
\]

for some $g \in S^*[A, B]$, then

\[
\left| \arg \left( \frac{F_c(f)}{F_c(g)} - \gamma \right) \right| < \frac{\pi}{2} \eta,
\]

where the operator $F_c$ is given by (2.14) and $\eta(0 < \eta \leq 1)$ is the solution of the equation

\[
\delta = \begin{cases} 
\frac{\eta}{\left(1+B^2\right)} + \frac{3\sin \frac{\pi}{2}(1-t(A,B,c))}{\left(1+B^2\right) + \eta \cos \frac{\pi}{2}(1-t(A,B,c))} & \text{for } B \neq -1, \\
\eta & \text{for } B = -1,
\end{cases}
\]

when

\[
t(A, B, c) = \frac{2}{\pi} \sin^{-1} \left( \frac{A-B}{1-AB+c(1-B^2)} \right)
\] (2.15)

**Proof.** Let

\[
p(z) = \frac{1}{1-\gamma} \left( \frac{F_c(f)}{F_c(g)} - \gamma \right) \quad \text{and} \quad q(z) = \frac{zf_c'(g)}{F_c(g)}.
\]

From the assumption for $g$ and an application of Briot-Bouquet differential equation [10, p. 81], we see that $F_c(g) \in S^*[A, B]$. Using the equation

\[
zF_c'(f)(z) + cF_c(f)(z) = (1+c)f(z)
\]

and simplying, we obtain

\[
\frac{1}{1-\gamma} \left( \frac{f(z)}{g(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{q(z)+c}.
\]

Then, by applying (1.2) and (1.3), we have

\[q(z) + c = \rho e^{i\frac{\pi}{2}}\phi,\]
where

\[
\begin{cases}
\frac{1-A}{1-B} + c < \rho < \frac{1+A}{1+B} + c \\
-t(A, B, c) < \phi < t(A, B, c)
\end{cases}
\]

for \( B \neq -1 \),

when \( t(A, B, c) \) is given by (2.16), and

\[
\begin{cases}
\frac{1-A}{2} + c < \rho < \infty \\
-1 < \phi < 1
\end{cases}
\]

for \( B = -1 \).

Here, we note that \( p \) is analytic in \( U \) with \( p(0) = 1 \) and \( \text{Re} \, p(z) > 0 \) in \( U \) by applying the assumption and Lemma 2.3 with \( \omega(z) = 1/(q(z)+c) \). Hence \( p(z) \neq 0 \) in \( U \). The remaining part of the proof of Theorem 2.3 is similar to that of Theorem 2.2, and so we omit it.

**Remark.** From Theorem 2.3, we see easily that every function in \( \mathcal{S}(\delta) \) \((0 < \delta \leq 1)\) preserves the angles under the integral operator defined by (2.14).

By letting \( A = 1 - 2\beta(0 \leq \beta \leq 1), \, B = -1, \, \delta = 1 \) in Theorem 2.3, we obtain

**Corollary 2.3.** If \( f \in \mathcal{A} \) and

\[
\text{Re} \left\{ \frac{f(z)}{g(z)} \right\} > \gamma \quad (0 \leq \gamma < 1; \, z \in \mathcal{U}),
\]

for some \( g \) such that

\[
\text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} > \beta \quad (0 \leq \beta < 1; \, z \in \mathcal{U}),
\]

then

\[
\text{Re} \left\{ \frac{F_c(f)}{F_c(g)} \right\} > \gamma \quad (0 \leq \gamma < 1; \, z \in \mathcal{U}),
\]

where \( F_c \) is given by (2.14).

If we take \( g(z) = z \) in Theorem 2.3, then, by letting \( B \to A \) \((A < 1)\), we have

**Corollary 2.4.** If \( f \in \mathcal{A} \) and

\[
| \arg\left( \frac{f(z)}{z} - \gamma \right) | < \frac{\pi}{2} \delta \quad (0 \leq \gamma < 1; \, 0 < \delta \leq 1),
\]

then

\[
| \arg\left( \frac{F_c(f)}{z} - \gamma \right) | < \frac{\pi}{2} \eta,
\]
where $F_c$ is given by (2.14) and $\eta(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1}\left(\frac{\eta}{1+c}\right).$$

References


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