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The Lévy Laplacian and the Number Operator

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Abstract

In this paper, we introduce a homeomorphism to connect the Lévy Laplacian to the Number operator. Moreover we also give a relationship between a stochastic process generated by some function of the Lévy Laplacian and the semi-group generated by the Number operator.

1. Introduction

An infinite dimensional Laplacian, the Lévy Laplacian, was introduced by P. Lévy [17]. This Laplacian was introduced into the framework of white noise analysis initiated by T. Hida [4]. L. Accardi et al. [1] obtained an important relationship between this Laplacian and the Yang-Mills equations. It has been studied by many authors (see [1, 2, 3, 5, 7, 8, 13, 15, 16, 18, 21, 22, 23, 24 etc]).

In the previous papers [25,26] we obtained stochastic processes generated by the powers of an extended Lévy Laplacian and also in [29] we obtained stochastic processes generated by some functions of the Laplacian.

The purpose of this paper is to present recent developments on stochastic processes generated by functions of the Lévy Laplacian acting on white noise distributions based on the idea in [29] and to give a stochastic expression of an equi-continuous semigroup of class (C_0) generated by the Laplacian related to an infinite dimensional Ornstein-Uhlenbeck process following [27].

The paper is organized as follows. In Section 2 we summarize some basic definitions and results in white noise analysis. In Section 3 we introduce a Hilbert space as a domain of the extended Lévy Laplacian which is self-adjoint on the domain following our previous paper [27], and we give an equi-continuous semigroup of class (C_0) generated by some functions of the extended Lévy Laplacian. In the last section we give a stochastic expression of a semigroup generated by some function of the Number operator.

2. Preliminaries

In this section we assemble some basic notations of white noise analysis following [7, 12, 15, 19].

Let $L^2(\mathbf{R})$ be the Hilbert space of real-valued square-integrable functions on $\mathbf{R}$. We take the space $E^* \equiv \mathcal{S}'(\mathbf{R})$ of tempered distributions with a probability measure μ which satisfies

$$\int_{E^*} \exp\{i\langle x,\xi\rangle\}\, d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E \equiv \mathcal{S}(\mathbf{R}),$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^* \times E$ and $|\cdot|_0$ is the $L^2(\mathbf{R})$-norm.

The differential operator $A = -(d/du)^2 + u^2 + 1$ is a densely defined self-adjoint operator on $L^2(\mathbf{R})$. There exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2(\mathbf{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. Let E_p be the completion of E with respect to the norm $|\cdot|_p$ defined by $|f|_p = |A^p f|_0$ for $f \in E$ and $p \in \mathbf{R}$. Then E_p is a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' of E_p is the same as E_{-p} (see Ref. 10). With the projective limit space E of $\{E_p; p \geq 0\}$ and the inductive limit space E^* of $\{E_{-p}; p \geq 0\}$, we have a chain of Hilbert spaces: for $0 \leq p \leq q$,

$$E \subset E_q \subset E_p \subset L^2(\mathbf{R}) \subset E_{-p} \subset E_{-q} \subset E^*.$$

We denote the complexifications of $L^2(\mathbf{R})$, E and E_p by $L^2_{\mathbf{C}}(\mathbf{R})$, $E_{\mathbf{C}}$ and $E_{\mathbf{C},p}$, respectively.

Let $(L^2) = L^2(E^*,\mu)$ be the Hilbert space of complex-valued square-integrable functionals defined on E^* with norm denoted by $\|\cdot\|_0$. By the Wiener-Itô theorem every φ in (L^2) can be represented uniquely by

$$\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n), \quad f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n},$$

where $\mathbf{I}_n$ denotes the multiple Wiener integrals of order $n \in \mathbf{N}$. Let $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbf{C}}(\mathbf{R})$. Moreover, for the (L^2)-norm $\|\varphi\|_0$ of φ we have

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2},$$

where $|\cdot|_0$ is the norm of $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p\varphi\|_0$, where $\Gamma(A)$ is the second quantization operator densely defined on (L^2) by

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(A^{\otimes n} f_n), \quad \varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n).$$

If $p \geq 0$, let $(E)_p = \{\varphi \in (L^2); \|\varphi\|_p < \infty\}$. If $p < 0$, let $(E)_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(E)_p$, $p \in \mathbf{R}$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(E)_p^*$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in E_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $E_{\mathbf{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $E_{\mathbf{C}}$. In fact we have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in E_{\mathbf{C},p}^{\hat{\otimes} n}\}$ for any

$p \in \mathbf{R}$, where $E_{\mathbf{C},p}^{\hat{\otimes}n}$ is also the n-fold symmetric tensor product of $E_{\mathbf{C},p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n!|f_n|_p^2\right)^{1/2}, \qquad f_n \in E_{\mathbf{C},p}^{\hat{\otimes}n},$$

where the norm of $E_{\mathbf{C},p}^{\hat{\otimes}n}$ is denoted also by $|\cdot|_p$.

The projective limit space (E) of spaces $(E)_p$, $p \in \mathbf{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)_p, p \in \mathbf{R}$ is nothing but the dual space of (E). The space $(E)^*$ is called the space of *generalized white noise functionals.* The canonical bilinear form on $(E)^* \times (E)$ is denoted by $\ll \cdot, \cdot \gg$. Then we have

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} n!\langle F_n, f_n\rangle \quad \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (E)^*, \quad \varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E),$$

where the canonical bilinear form on $(E_{\mathbf{C}}^{\otimes n})^* \times E_{\mathbf{C}}^{\otimes n}$ is denoted also by $\langle \cdot, \cdot \rangle$. The Schwarz inequality takes the form:

$$| \ll \Phi, \varphi \gg | \leq \|\Phi\|_{-p}\|\varphi\|_p, \quad p \in \mathbf{R}.$$

Since $\phi_\xi(\cdot) = \exp\left(\langle \cdot, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right) \in (E)$, the *S-transform* is defined on $(E)^*$ by

$$S[\Phi](\xi) = \ll \Phi, \phi_\xi \gg, \qquad \xi \in E_{\mathbf{C}}.$$

3. An equi-continuous semigroup of class (C_0) generated by a function of the Lévy Laplacian

Let Φ be in $(E)^*$. Then the S-transform $S[\Phi]$ of Φ is *Fréchet differentiable*, i.e.

$$S[\Phi](\xi + \eta) = S[\Phi](\xi) + S[\Phi]'(\xi)(\eta) + o(\eta),$$

where $o(\eta)$ means that there exists $p \geq 0$ depending on ξ such that $o(\eta)/|\eta|_p \to 0$ as $|\eta|_p \to 0$.

Fix a finite interval T in $\mathbf{R}$. Take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E$ for $L^2(T)$ satisfying the equally dense and uniform boundedness property (see [7,15,16,18,24, etc]). Let $\mathcal{D}_L$ denote the set of all $\Phi \in (E)^*$ such that the limit

$$\widetilde{\Delta}_L S[\Phi](\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(\zeta_n, \zeta_n)$$

exists for any $\xi \in E_{\mathbf{C}}$ and is in $S[(E)^*]$. The Lévy Laplacian Δ_L is defined by

$$\Delta_L \Phi = S^{-1}\widetilde{\Delta}_L S\Phi$$

for $\Phi \in \mathcal{D}_L$. We denote the set of all functionals $\Phi \in \mathcal{D}_L$ such that $S[\Phi](\eta) = 0$ for all $\eta \in E$ with $\mathrm{supp}(\eta) \subset T^c$ by $\mathcal{D}_L^T$.

Take a generalized white noise functional

$$\Phi = \int_{T^n} f(u_1, \ldots, u_n) : e^{ia_1 x(u_1)} \ldots e^{ia_n x(u_n)} : d\mathbf{u} \in \mathcal{D}_L^T, \tag{3.1}$$

$$f \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}, a_k \in \mathbf{R}, k = 1, 2, \ldots, n,$$

where $: \cdot :$ means the Wick ordering. Its S-transform is given by

$$S[\Phi](\xi) = \int_{T^n} f(\mathbf{u}) e^{ia_1 \xi(u_1)} \ldots e^{ia_n \xi(u_n)} d\mathbf{u}. \tag{3.2}$$

Theorem 1.[27] *A generalized white noise functional Φ as in (3.1) satisfies the equation*

$$\Delta_L \Phi = -\frac{1}{|T|} \sum_{k=1}^{n} a_k^2 \Phi. \tag{3.3}$$

We set

$$\mathbf{D}_n = \left\{ \int_{T^n} f(\mathbf{u}) : \prod_{\nu=1}^{n} e^{ix(u_\nu)} : d\mathbf{u} \in \mathcal{D}_L^T; f \in E_{\mathbf{C}}^{\hat{\otimes} n} \right\}$$

for each $n \in \mathbf{N}$ and set $\mathbf{D}_0 = \mathbf{C}$. Then $\mathbf{D}_n$ is a linear subspace of $(E)_{-p}$ for any $p \geq 1$. We define a space $\overline{\mathbf{D}}_n$ by the completion of $\mathbf{D}_n$ in $(E)_{-p}$ with respect to $\|\cdot\|_{-p}$. Then for each $n \in \mathbf{N} \cup \{0\}$, $\overline{\mathbf{D}}_n$ becomes a Hilbert space with the inner product of $(E)_{-p}$. For each $n \in \mathbf{N} \cup \{0\}$, the operator Δ_L becomes a continuous linear operator from $\overline{\mathbf{D}}_n$ into itself satisfying

$$\|\Delta_L \Phi\|_{-p} = \frac{n}{|T|} \|\Phi\|_{-p} \text{ for any } \Phi \in \overline{\mathbf{D}}_n.$$

The operator $\overline{\Delta_L}$ is a self-adjoint operator on $\overline{\mathbf{D}}_n$ for each $n \in \mathbf{N} \cup \{0\}$.

Proposition 2. [27] *Let $\Phi = \sum_{n=0}^{\infty} \Phi_n, \Psi = \sum_{n=0}^{\infty} \Psi_n$ be generalized white noise functionals such that Φ_n and Ψ_n are in $\overline{\mathbf{D}}_n$ for each $n \in \mathbf{N} \cup \{0\}$. If $\Phi = \Psi$ in $(E)^*$, then $\Phi_n = \Psi_n$ in $(E)^*$ for each $n \in \mathbf{N} \cup \{0\}$.*

Let $\alpha_N(n) = \sum_{\ell=0}^{N} \left(\frac{n}{|T|}\right)^{2\ell}$. Proposition 2 says that $\sum_{n=0}^{\infty} \Phi_n, \Phi_n \in \overline{\mathbf{D}}_n$, is uniquely determined as an element of $(E)^*$. Therefore, for any $N \in \mathbf{N} \cup \{0\}$, we can define a space $\mathbf{E}_{-p,N}$ by

$$\mathbf{E}_{-p,N} = \left\{ \sum_{n=0}^{\infty} \Phi_n \in (E)^*; \sum_{n=0}^{\infty} \alpha_N(n) \|\Phi_n\|_{-p}^2 < \infty, \Phi_n \in \overline{\mathbf{D}}_n, n = 0, 1, 2, \ldots \right\}$$

with the norm $|||\cdot|||_{-p,N}$ given by

$$|||\Phi|||_{-p,N} = \left(\sum_{n=0}^{\infty} \alpha_N(n) \|\Phi_n\|_{-p}^2 \right)^{1/2}, \quad \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,N}$$

for each $N \in \mathbf{N}\cup\{0\}$ and $p \geq 1$. For any $N \in \mathbf{N}\cup\{0\}$ and $p \geq 1$, $\mathbf{E}_{-p,N}$ is a Hilbert space with the norm $|||\cdot|||_{-p,N}$.

Put $\mathbf{E}_{-p,\infty} = \bigcap_{N\geq 1}\mathbf{E}_{-p,N}$ with the projective limit topology. Define a Hilbert space $\mathbf{E}_{-p,-N}$ by the completion of $\mathbf{E}_{-p,\infty}$ with respect to the norm $|||\cdot|||_{-p,-N}$ given by

$$|||\Phi|||_{-p,-N} = \left(\sum_{n=0}^{\infty}\alpha_N(n)^{-1}\|\Phi_n\|_{-p}^2\right)^{1/2}, \quad \Phi = \sum_{n=0}^{\infty}\Phi_n \in \mathbf{E}_{-p,\infty}$$

for each $N \geq 0$. With the inductive limit space $\mathbf{E}_{-p,-\infty} \equiv \cup_{N\geq 1}\mathbf{E}_{-p,-N}$, for any $N \geq 0$, we have the following inclusion relations:

$$\mathbf{E}_{-p,\infty} \subset \mathbf{E}_{-p,N+1} \subset \mathbf{E}_{-p,N} \subset \mathbf{E}_{-p,1} \subset \mathbf{E}_{-p,0} \subset \mathbf{E}_{-p,-1} \subset \mathbf{E}_{-p,-N} \subset \mathbf{E}_{-p,-N-1} \subset \mathbf{E}_{-p,-\infty}.$$

The space $\mathbf{E}_{-p,\infty}$ includes $\overline{\mathbf{D}_n}$ for any $n \in \mathbf{N}\cup\{0\}$. The operator $\overline{\Delta_L}$ can be extended to a continuous linear operator defined on $\mathbf{E}_{-p,-\infty}$, denoted by the same notation $\overline{\Delta_L}$, satisfying $|||\overline{\Delta_L}\Phi|||_{-p,N} \leq |||\Phi|||_{-p,N+1}$, $\Phi \in \mathbf{E}_{-p,N+1}$, for each $N \in \mathbf{Z}$. Any restriction of $\overline{\Delta_L}$ is also denoted by the same notation $\overline{\Delta_L}$. Using the simillar method of Theorem 2 in [27], we have the following:

Theorem 3. *The operator $\overline{\Delta_L}$ restricted on $\mathbf{E}_{-p,N+1}$ is a self-adjoint operator densely defined on $\mathbf{E}_{-p,N}$ for each $N \in \mathbf{Z}$ and $p \geq 1$.*

Let $c(t,z)$ be a complex-valued bounded function on $\mathbf{R}^2$ which is differentiable in t and continuous in z. There exists a constant $M > 0$ such that $\sup_{t,z}|c(t,z)| \leq M$. For each $t \geq 0$ we consider an operator G_t on $\mathbf{E}_{-p,-\infty}$ defined by

$$G_t\Phi = \sum_{n=0}^{\infty} c\left(t, \frac{n}{|T|}\right)\Phi_n$$

for $\Phi = \sum_{n=0}^{\infty}\Phi_n \in \mathbf{E}_{-p,-\infty}$. For any $\Phi = \sum_{n=0}^{\infty}\Phi_n$ in $\mathbf{E}_{-p,-\infty}$, there exists $N \in \mathbf{Z}$ such that $\Phi \in \mathbf{E}_{-p,N}$. Then, for any $t \geq 0, p \geq 1$, the norm $|||G_t\Phi|||_{-p,N}$ is estimated as follows:

$$\begin{aligned} |||G_t\Phi|||_{-p,N}^2 &= \sum_{n=0}^{\infty}\alpha_N(n)\left\|c\left(t,\frac{n}{|T|}\right)\Phi_n\right\|_{-p}^2 \\ &\leq M^2\sum_{n=0}^{\infty}\alpha_N(n)\|\Phi_n\|_{-p}^2 = M^2|||\Phi|||_{-p,N}^2, \end{aligned}$$

where $\alpha_N(n)$ means $\alpha_{-N}(n)^{-1}$ for $N \leq 0$.

Thus the operator G_t is a continuous linear operator from $\mathbf{E}_{-p,-\infty}$ into itself.

For any $p \geq 1$ and complex-valued continuous function $h(z), z \in \mathbf{R}$ satisfying the condition:

(P) there exists a polynomial $r(z)$ of $z \in \mathbf{R}$ such that $|h(z)| \leq r(|z|)$ for all $z \in \mathbf{R}$, the operator $h(-\overline{\Delta_L})$ on $\mathbf{E}_{-p,-\infty}$ is given by

$$h(-\overline{\Delta_L})\Phi = \textstyle\sum_{n=0}^{\infty} h\left(\frac{n}{|T|}\right)\Phi_n, \text{ for } \Phi = \sum_{n=0}^{\infty}\Phi_n \in \mathbf{E}_{-p,-\infty}.$$

Theorem 4. *Let $h(z)$ be a complex-valued bounded function $h(z)$ of $z \in \mathbf{R}$ such that $h(z)$ is continuous, $h(0) = 0$ and $c(t,z) = e^{h(z)t}$ for all $t \geq 0$. If $h(z)$ satisfies the condition* (**P**), *then the family $\{G_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by $h(-\overline{\Delta_L})$.*

Proof. If there exists a complex-valued continuous function $h(z)$ of $z \in \mathbf{R}$ such that $c(t,z) = e^{h(z)t}$, then it is easily checked that $G_0 = I$, $G_tG_s = G_{t+s}$ for each $t, s \geq 0$. Moreover we can estimate that

$$\begin{aligned} |||G_t\Phi - G_{t_0}\Phi|||_{-p,N}^2 &= \sum_{n=0}^{\infty} \alpha_N(n)\left|c\left(t,\frac{n}{|T|}\right) - c\left(t_0,\frac{n}{|T|}\right)\right|^2 ||\Phi_n||_{-p}^2 \\ &\leq 4M^2\sum_{n=0}^{\infty}\alpha_N(n)||\Phi_n||_{-p}^2 = 4M^2|||\Phi|||_{-p,N}^2 < \infty \end{aligned}$$

for each $t, t_0 \geq 0, N \in \mathbf{Z}^*$ and $\Phi = \sum_{n=0}^{\infty}\Phi_n \in \mathbf{E}_{-p,N}$. Therefore, by the Lebesgue convergence theorem, we get that

$$\lim_{t\to t_0} G_t\Phi = G_{t_0}\Phi \quad \text{in} \quad \mathbf{E}_{-p,\infty}$$

for each $t_0 \geq 0$ and $\Phi \in \mathbf{E}_{-p,-\infty}$. Thus the family $\{G_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0). Let $p \geq 1$ and let $\Phi = \sum_{n=0}^{\infty}\Phi_n \in \mathbf{E}_{-p,-\infty}$. Then, there exists $N \in \mathbf{Z}$ such that $\Phi \in \mathbf{E}_{-p,N}$. Let d_r be the degree of the polynomial r in the condition (**P**). Then we note that

$$\left|\left|\left|\frac{G_t\Phi - \Phi}{t} - h(-\overline{\Delta_L})\Phi\right|\right|\right|_{-p,N-d_r}^2 = \sum_{n=0}^{\infty}\alpha_{N-d_r}(n)\left|\left|\left(\frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t} - h\left(\frac{n}{|T|}\right)\right)\Phi_n\right|\right|_{-p}^2. \tag{3.4}$$

By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0,1)$ such that

$$\left|\frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t}\right| = \left|h\left(\frac{n}{|T|}\right)\right| e^{Re\left[h\left(\frac{n}{|T|}\right)\right]t\theta} \leq Mr\left(\frac{n}{|T|}\right).$$

Therefore we get that

$$\begin{aligned} \left|\left|\frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t}\Phi_n - h\left(\frac{n}{|T|}\right)\Phi_n\right|\right|_{-p}^2 &= \left|\frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t} - h\left(\frac{n}{|T|}\right)\right|^2 ||\Phi_n||_{-p}^2 \\ &\leq 2(M+1)r\left(\frac{n}{|T|}\right)^2 ||\Phi_n||_{-p}^2. \end{aligned}$$

Since there exists a positive constant C_r depending on r such that $\alpha_{N-d_r}(n)r\left(\frac{n}{|T|}\right)^2 \leq C_r\alpha_N(n)$, we have

$$\sum_{n=0}^{\infty} \alpha_{N-d_r}(n)r\left(\frac{n}{|T|}\right)^2 ||\Phi_n||_{-p}^2 < \infty. \tag{3.5}$$

By (3.4), (3.5) and

$$\lim_{t\to 0}\left|\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t} - h\left(\frac{n}{|T|}\right)\right| = 0,$$

the Lebesgue convergence theorem admits

$$\lim_{t\to 0}\left|\left|\left|\frac{G_t\Phi-\Phi}{t} - h(-\overline{\Delta_L})\Phi\right|\right|\right|^2_{-p,N-d_r} = 0.$$

Thus the proof is completed. □

4. A relationship to the Number operator

Let $\beta(p) = \sum_{k=0}^{\infty}(2k+2)^{-2p}|e_k|_0^2$ and set

$$[E]_{p,N} = \left\{\varphi = \sum_{n=0}^{\infty}\mathbf{I}_n(f_n) \in (L^2); \sum_{n=0}^{\infty}\alpha_N(n)e^{n^2\beta(p)}|f_n|_0^2 < \infty,\ \operatorname{supp}(f_n)\subset T, n=0,1,2,\dots\right\}$$

for $p \geq 0$ and $N \geq 0$. The space $[E]_{p,N}$ is a Hilbert space with norm $||\cdot||_{[E]_{p,N}}$ given by

$$||\varphi||_{[E]_{p,N}} = \left(\sum_{n=0}^{\infty}\alpha_N(n)e^{n^2\beta(p)}|f_n|_0^2\right)^{1/2}$$

for $\varphi = \sum_{n=0}^{\infty}\mathbf{I}_n(f_n) \in (L^2)$.

Define an operator K on $[E]_{p,N}$ by

$$K[\Phi] = S^{-1}[S[\Phi](e^{i\xi})].$$

Then we have the following:

Proposition 4. *Let $p \geq 1$ and let $N \geq 1$. Then the operator K is a continuous linear operator from $[E]_{p,N}$ into $\mathbf{E}_{-p,N}$.*

Proof. Let $p \geq 1$ and let $N \geq 1$. Then for each $\ell \geq 1$ we can calculate the norm $|||K[\varphi]|||^2_{-p,N}$ of $K[\varphi]$ for $\varphi = \sum_{n=0}^{\infty}\mathbf{I}_n(f_n) \in [E]_{p,N}$ as follows:

$$\begin{aligned}
|||K[\varphi]|||^2_{-p,N} &= \sum_{n=0}^{\infty}\alpha_N(n)||\langle : (e^{ix})^{\otimes n} :, f_n\rangle||^2_{-p} \\
&\leq \sum_{n=0}^{\infty}\alpha_N(n)\sum_{\ell=0}^{\infty}\ell! \sum_{k_1,\dots,k_\ell=0}^{\infty}\prod_{j=1}^{\ell}(2k_j+2)^{-2p}\left|\sum_{|\nu|=\ell}\frac{1}{\nu!}\langle F_\nu, e_{k_1}\otimes\cdots\otimes e_{k_\ell}\rangle\right|^2,
\end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_n) \in \mathbf{N}\cup\{0\}$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\nu! = \nu_1! \dots \nu_n!$ and $F_\nu = \int_{\mathbf{R}^n} f(\mathbf{u}) \hat{\otimes}_{j=1}^{n} \delta_{u_j}^{\hat{\otimes}\nu_j} d\mathbf{u}$. Since there exists $q \geq 0$ such that

$$\begin{aligned} &\sum_{k_1,\dots,k_\ell=0}^{\infty} \prod_{j=1}^{\ell} (2k_j+2)^{-2p} \left| \sum_{|\nu|=\ell} \frac{1}{\nu!} \langle F_\nu, e_{k_1} \otimes \cdots \otimes e_{k_\ell} \rangle \right|^2 \\ \leq\ & |f_n|_0^2 \left(\sum_{|\nu|=\ell} \frac{1}{\nu!} \right)^2 \left(\sum_{k=0}^{\infty} (2k+2)^{-2p} |e_k|_0^2 \right)^{\ell}, \end{aligned}$$

we get that

$$\begin{aligned} |||K[\varphi]|||_{-p,N}^2 &\leq \sum_{n=0}^{\infty} \alpha_N(n) e^{n^2\beta(p)} |f_n|_0^2 \\ &= ||\varphi||_{[E]_{p,N}}. \end{aligned}$$

Thus the proof is completed. □

Remark: Regarding K as an operator from $[E]_{p,N}$ onto $K[[E]_{p,N}]$, it is a bijection. Define a norm $[\![\cdot]\!]_{-p,N}$ on $K[[E]_{p,N}]$ by

$$[\![\Phi]\!]_{-p,N} = ||K^{-1}\Phi||_{p,N}$$

for $\Phi \in K[[E]_{p,N}]$. Let $\mathcal{K}_{-p,N} = \{\Phi \in K[[E]_{p,N}]; [\![\Phi]\!]_{-p,N} < \infty\}$. Then this becomes a Hilbert space with the norm $[\![\cdot]\!]_{-p,N}$ and the operator K is a homeomorpism from $[E]_{p,N}$ onto $\mathcal{K}_{-p,N}$.

Let $[E]_{p,\infty}$ be the projective limit space of spaces $[E]_{p,N}$, $N \in \mathbf{N}$. Then the operator K is a continuous linear operator from $[E]_{p,\infty}$ into $\mathbf{E}_{-p,\infty}$. Let $\{X_t; t \geq 0\}$ be a stochastic process with the characteristic function of X_t given by

$$E[e^{izX_t}] = e^{th(z)}$$

and let $\{G_t; t \geq 0\}$ be an equi-continuous semigroup of class (C_0) generated by $h(-\overline{\Delta_L})$ in Theorem 4. Take a smooth function $\eta_T \in E$ with $\eta_T = \frac{1}{|T|}$ on T. Define an operator $\widetilde{G_t}$ acting on $S[\mathbf{E}_{-p,\infty}]$ by

$$\widetilde{G_t} = SG_tS^{-1}.$$

Here the space $S[\mathbf{E}_{-p,\infty}]$ is endowed with the topology induced from $\mathbf{E}_{-p,\infty}$ by the S-transform. Then by Theorem 3.5, $\{\widetilde{G_t};\ t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the operator $h(-\widetilde{\Delta_L})$, where $\widetilde{\Delta_L} = S\overline{\Delta_L}S^{-1}$. Then we have the following theorem.

Theorem 5.[cf. 26, 29] *For all $F \in S[\mathbf{E}_{-p,\infty}]$, the following equality holds*

$$\widetilde{G_t}F(\xi) = \mathrm{E}[F(\xi + X_t^\gamma \eta_T)].$$

The operator K implies a relationship between $\overline{\Delta_L}$ and the number operator $\mathcal{N}$ on $(E)^*$ given by

$$\mathcal{N}\Phi = \sum_{n=0}^{\infty} n\mathbf{I}_n(f_n) \quad \text{for} \quad \Phi = \textstyle\sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)^*.$$

Proposition 6. *For any $\Phi \in [E]_{p,N}$ we have*

$$\overline{\Delta_L} K[\varphi] = -\frac{1}{|T|} K[\mathcal{N}[\varphi]].$$

Theorem 7. *For any $\varphi \in [E]_{p,N} \cap (E)$ we have*

$$\widetilde{K}[e^{th\left(-\frac{1}{|T|}\mathcal{N}\right)}\varphi](\xi) = E[\widetilde{K}[\varphi](\xi + X_t^{\gamma}\eta_T)],$$

where $\widetilde{K}[\varphi](\xi) = S[\varphi](e^{i\xi})$, $\xi \in E_{\mathbf{C}}$.

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