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ASYMPTOTIC BEHAVIOR OF CONTINUOUS-TIME MARKOV CHAINS OF THE EHRENFEST TYPE WITH APPLICATIONS*

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Abstract

Kac [6] considers the discrete-time Markov chain for the original Ehrenfest urn model and gives the expression of the n-step transition probabilities in terms of the Krawtchouk polynomials; moreover, Kac's formulas for transition laws show that the model yields Newton's law of cooling at the level of the evolution of the averages. Here we first give a version of Newton's law of cooling for Dette's [4] Ehrenfest urn model. We next discuss the continuous-time formulation of Markov chain for Krafft and Schaefer's [9] Ehrenfest urn model and calculate probability law of transitions and first-passage times in both the reduced description with states counting the number of balls in the specified urn and the full description with states enumerating the vertices of the N-cube. Our results correspond to a generalization of the work of Bingham [3]. We also treat several applications of urn models to queueing network and reliability theory.

1. Introduction

Let s and t be two parameters such that $0 < s \leqq 1$ and $0 < t \leqq 1$, and consider the homogeneous Markov chain with state $\{0, 1, \ldots, N\}$ and transition probabilities

$$p_{ij} = \begin{cases} \left(1 - \frac{i}{N}\right) s, & \text{if } j = i + 1, \\ \frac{i}{N} t, & \text{if } j = i - 1, \\ 1 - \left(1 - \frac{i}{N}\right) s - \frac{i}{N} t, & \text{if } j = i, \quad 0 \leqq j \leqq N, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

which is proposed by Krafft and Schaefer [9]. In case $s = t = 1$, the Markov chain is the same as P. and T. Ehrenfest [5] use for illustration of the *process of heat exchange* between two bodies that are in contact and insulated from the outside. The temperatures are assumed to change in steps of one unit and are represented by the numbers of balls in two urns. The two urns are marked I and II and there are N balls labeled $1, 2, \ldots, N$ and distributed in two urns. The chain is in state i when there are i balls in I. At each point of time n one ball is chosen at random (i.e. with equal probabilities $1/N$ among ball

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numbers $1, 2, \ldots, N$) and moved from its urn to other urn by the following way. If it is in urn I, it is placed in II with probability t and returned to I with probability $1-t$. If it is in II, it is placed in I with probability s and returned to II with probability $1-s$. The case where $s+t=1$ is the *one-parameter* Ehrenfest urn model as investigated by Karlin and McGregor [7]. The case where $s=t$ is the *thinning* of the Ehrenfest urn model as proposed by Vincze [14] and Takács [13].

Remark 1.1. The process of the Markov chain in the model (1.1) corresponds to a 'reduced description' with $N+1$ states, $0, 1, \cdots, N$ counting the number of balls in urn I. If we use a 'full description' with 2^N states, representing the possible configurations by N-tuples $i = (i_1, \ldots, i_k, \ldots, i_N)$, where $i_k = 1,\ 0$ if ball k is in urn I, II, respectively, one may identify the states with the vertices of the N-cube.

Let a and b denote real numbers such that $a, b > -1$ or $a, b < -N$, and consider the following homogeneous Morkov chain with state space $\{0, 1, \ldots, N\}$ and transition probabilities

$$p_{ij} = \begin{cases} \left(1 - \frac{i}{N}\right)\left(\frac{a+1+i}{N+a+b+2}\right)\nu, & \text{if } j = i+1, \\ \frac{i}{N}\left(\frac{N+b+1-i}{N+a+b+2}\right)\nu, & \text{if } j = i-1, \\ 1 - \left(1 - \frac{i}{N}\right)\left(\frac{a+1+i}{N+a+b+2}\right)\nu - \frac{i}{N}\left(\frac{N+b+1-i}{N+a+b+2}\right)\nu, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

where $\nu > 0$ is arbitrary such that $0 \leqq p_{ij} \leqq 1$ for all $i, j = 0, 1, \ldots, N$.

Remark 1.2. The *two-parameter* Ehrenfest urn model (1.1) is obtained by putting $a = su/(s+t)$, $b = tu/(s+t)$ and $\nu = s+t$ and taking the limit as $u \to \infty$.

For the model (1.2), Dette [4] obtains the more general integral representations for the n-step transition probabilities and the (unique) stationary probability distribution with respect to the spectral measure of random walk and shows the following remark.

Remark 1.3.

(i) The n-step transition probabilities $p_{ik}^{(n)}$ in the model (1.1) are given by

$$p_{ik}^{(n)} = \binom{N}{k}\left(\frac{p}{q}\right)^k \sum_{x=0}^{N} \binom{N}{x} p^x q^{N-x} K_i(x,p,N) K_k(x,p,N) \left(1 - \frac{s+t}{N}x\right)^n,$$

where $p = s/(s+t)$ and $q = t/(s+t)$. Here $\{K_n(x,p,N) : n = 0, 1, \ldots, N\}$ is the family of Krawtchouk polynomials, which will appear in Remark 1.4 below. This result is also found by Krafft and Schaefer [9].

(ii) The (unique) stationary probability distribution $\pi = (\pi_0, \ldots, \pi_N)$ in the model (1.2) is given by

$$\pi_j = \binom{N}{j} \frac{\beta(a+1+j, N+b+1-j)}{\beta(a+1, b+1)},$$

where $\beta(x, y)$ denotes the beta function, that is,

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 u^{x-1}(1-u)^{y-1}\,du \quad \text{for } x, y > 0.$$

For the model (1.1) with $s = t = 1$, Kac [6] finds the same form with (i) of Remark 1.3 and shows *Newton's law of cooling* that concerns the estimate of the excess of the numbers of balls in urn I over the mean of the stationary probability distribution.

Our purpose is the following:

• For the model (1.2), we will show the same result with Newton's law of cooling [Theorem 2.1].

• For the model (1.1), we will give the continuous-time formulation and obtain the law of transition probabilities and first-passage times in both the 'reduced' and the 'full' descriptions, using the method of Bingham [3] [Theorems 2.2-2.3].

Remark 1.4. With $0 < p = 1 - q < 1$ and integer $N \geqq 0$ the *Krawtchouk polynomials* are defined by the hypergeometric series:

$$\begin{aligned} K_n(x,p,N) &= {}_2F_1(-n,-x;-N;1/p) \\ &= \sum_{\nu=0}^{n} (-1)^\nu \frac{\binom{n}{\nu}\binom{x}{\nu}}{\binom{N}{\nu}} \left(\frac{1}{p}\right)^\nu, \qquad n = 0,1,\dots,N. \end{aligned}$$

When the meaning is clear, we write simply $K_n(x)$ instead of $K_n(x,p,N)$. From Karlin and McGregor [7] we cite the properties for $K_n(x)$, which will be used in after sections.

$$\text{(i)} \qquad \sum_{n=0}^{N} \binom{N}{n} K_n(x) z^n = (1+z)^{N-x} \left(1 - \frac{q}{p} z\right)^x, \qquad x = 0,1,\dots,N.$$

$$\text{(ii)} \qquad K_n(0) = 1, \quad K_0(x) = 1.$$

$$\text{(iii)} \qquad K_n(x,p,N) = K_x(n,p,N), \quad n,x = 0,1,\dots,N.$$

$$\text{(iv)} \qquad K_n(N-x,p,N) = (-1)^n \left(\frac{q}{p}\right)^n K_n(x,q,N).$$

2. Theorems

We first concern ourselves with evaluation of the excess of the number of balls in urn I over the value averaged by the stationary probability distribution for the model (1.2).

Theorem 2.1. *Let $\{X_n : n = 0,1,\dots\}$ be the Markov chain with transition probabilities* (1.2) *with $a > -1$ and $b > -1$. Let $\pi = (\pi_0,\dots,\pi_N)$ be the stationary probabilitity distribution given by* (ii) *of Remark* 1.3, *and denote by $< \pi >$ the mean of π, that is, $< \pi > = \sum_{j=0}^{N} j\pi_j$. Set $Y_n = X_n - < \pi >$ and put $e_n = E_i(Y_n)$, that is, the expected value of Y_n given $X_0 = i$. Then we have the following:*

$$\text{(i)} \qquad < \pi > = N\left(\frac{a+1}{a+b+2}\right).$$

(ii) $$e_n = (i - <\pi>)\left[1 - \nu\left(\frac{a+b+2}{N(N+a+b+2)}\right)\right]^n.$$

Remark 2.1. Suppose that the frequency of transitions is τ per second. Then in time T there are $n = T\tau$ transitions. Write

$$c = -\log\left[1 - \nu\left(\frac{a+b+2}{N(N+a+b+2)}\right)\right]^\tau.$$

Then (ii) of the above theorem yields

$$e_n = (i - <\pi>)\exp[-cT]. \tag{2.1}$$

Let a, b and ν be taken as in Remark 1.2. Then

$$<\pi> \longrightarrow N\left(\frac{s}{s+t}\right) \quad \text{and} \quad 1 - \nu\left(\frac{a+b+2}{N(N+a+b+2)}\right) \longrightarrow 1 - \frac{s+t}{N}$$

in the limit as $u \to \infty$, which shows

$$e_n = \left\{i - N\left(\frac{s}{s+t}\right)\right\}\exp[-cT] \quad \text{with} \quad c = -\log\left[1 - \frac{s+t}{N}\right]^\tau. \tag{2.2}$$

Newton's law of cooling in Kac [6] is the special case of (2.2) where $s = t = 1$.
Remark 2.2. Consider the Markov chain $\{X_n : n = 0, 1, \dots\}$ for the model (1.2) with $a = -N - 1 - l$ and $b = -N - 1 - k$, where k and l are nonnegative integers. Then, as follows from Dette [4], the stationary probability distribution is the hypergeometric distribution:

$$\pi_j = \frac{\binom{N+l}{j}\binom{N+k}{N-j}}{\binom{2N+l+k}{N}}, \qquad j = 0, 1, \dots, N,$$

having its mean $<\pi> = \sum_{j=0}^{N} j\,\pi_j = N\left(\frac{N+l}{2N+l+k}\right)$. For X_n, define e_n by the same way as that in Theorem 2.1. Then a straightforward calculation as taken in the proofs of (i) of Theorem 2.1 and Remark 2.1 leads us to the following result:

$$e_n = E_i[X_n - <\pi>] = (i - <\pi>)\left[1 - \nu\left(\frac{2N+l+k}{N(N+l+k)}\right)\right]^n.$$

$$e_n = (i - <\pi>)\exp[-cT] \quad \text{with} \quad c = -\log\left[1 - \nu\left(\frac{2N+l+k}{N(N+l+k)}\right)\right]^\tau.$$

Namely, Theorem 2.1 and Remark 2.1 hold for the model (1.2) with the choice

$$a = -N - 1 - l \quad \text{and} \quad b = -N - 1 - k.$$

In the following, let us consider the two-parameter Ehrenfest urn model (1.1). Then we will give the continuous-time formulation for this model.

We first consider the reduced description in continuous-time. Let $X(u)$ be the number of balls in urn I at time u. We shall sometimes speak of this number as the state of the system. Then $X(u)$ is a random function which can take integer values from 0 to N. Associate with the ith ball a random function $X_i(u)$ defined as follows:

$$\begin{aligned} X_i(u) &= 1 \quad \text{if the } i\text{th ball is in urn I at the time } u, \\ X_i(u) &= 0 \quad \text{otherwise.} \end{aligned} \tag{2.3}$$

Assumption 2.1. The family $\{X_i(u) : i = 1, 2, \ldots, N\}$ of random functions are independent. Each $X_i(u)$ is a Markov process with continuous-time parameter $u \geqq 0$ on the state space $\{0, 1\}$ and governed by the following Q-matrix (infinitesimal generator):

$$Q = \begin{pmatrix} -s/N & s/N \\ t/N & -t/N \end{pmatrix}.$$

Remark 2.3. Under Assumption 2.1, the transition probability matrix $\exp[uQ]$ is given by

$$\begin{pmatrix} p_{00}(u) & p_{01}(u) \\ p_{10}(u) & p_{11}(u) \end{pmatrix},$$

where

$$\begin{aligned} p_{00}(u) &= q + p\exp[-(s+t)u/N], & p_{01}(u) &= p - p\exp[-(s+t)u/N], \\ p_{10}(u) &= q - q\exp[-(s+t)u/N], & p_{11}(u) &= p + q\exp[-(s+t)u/N], \end{aligned}$$

with parameters $0 < s, t \leqq 1$, $p = s/(s+t)$ and $q = t/(s+t)$.

Since $X(u)$ is the number of balls in urn I, we note that $X(u) = \sum_{i=1}^{N} X_i(u)$.

Suppose that initially there are $X(0)$ balls in urn I and the trials are independent. Then $\{X(u) : u \geqq 0\}$ is a homogeneous Markov chain with state space $\{0, 1, \ldots, N\}$ and transition probabilities

$$P_{ik}(u) = P(\, X(u) = k \,|\, X(0) = i \,), \quad i, k = 0, 1, \ldots, N. \tag{2.4}$$

Secondly we consider the continuous-time Markov chain in the full description. For this process write

$$\widetilde{X}(u) = (\, X_1(u), X_2(u), \ldots, X_N(u) \,),$$

where $X_i(u)$ is given by (2.3), and put

$$\widetilde{P}_{ij}(u) = P(\, \widetilde{X}(u) = j \,|\, \widetilde{X}(0) = i \,), \quad i, j \in V, \tag{2.5}$$

where V is the vertex-set. Then we obtain the next theorem.

Theorem 2.2. *Under Assumption* 2.1, *we have the following*:
(i) *In the full description, the transition probabilities* (2.5) *are given by*

$$\widetilde{P}_{ij}(u) = \frac{1}{2^N}[\, p_{00}(u) + p_{11}(u) \,]^{N-m}[\, p_{01}(u) + p_{10}(u) \,]^m, \quad m = |i - j|,$$

where $p_{00}(u)$, $p_{01}(u)$, $p_{10}(u)$ *and* $p_{11}(u)$ *are the functions as given in Remark* 2.3, *and for* N*-tuples* $i = (i_1, \dots, i_k, \dots, i_N)$ *and* $j = (j_1, \dots, j_k, \dots, j_N)$, *the distance* $|i-j|$ *of* j *from* i *is defined by* $|i-j| = \sum_{k=1}^{N} |i_k - j_k|$.

(ii) *In the reduced description, the transition probabilities* (2.4) *are given by*

$$P_{ik}(u) = \binom{N}{k}\left(\frac{p}{q}\right)^k \sum_{x=0}^{N} \binom{N}{x} p^x\, q^{N-x}\, K_i(x,p,N) K_k(x,p,N) \exp[\,-x(s+t)u/N\,],$$

where $K_n(x,p,N)$ *are the Krawtchouk polynomials as given in Remark* 1.4.

Remark 2.4. Bingham [3] shows Theorem 2.2 for the particular case where $s = t = 1$ (i.e. $p = q = \frac{1}{2}$). The form (i) of Remark 1.3 in the discrete-time parameter is a corollary of Theorem 2.2. Indeed, we condition on the total number of transitions in $[0, u]$, which is Poisson with parameter u:

$$P_{ik}(u) = \sum_{n=0}^{\infty} \widetilde{p}_{ik}(n) e^{-u} \frac{u^n}{n!},$$

where $\widetilde{p}_{ik}(n)$ stand for the n-step transition probabilities in the discrete-time chain. Then, by (ii) of Theorem 2.2 we see

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widetilde{p}_{ik}(n) \frac{u^n}{n!} = P_{ik}(u) e^u \\
&= \binom{N}{k}\left(\frac{p}{q}\right)^k \sum_{x=0}^{N} \binom{N}{x} p^x\, q^{N-x}\, K_i(x,p,N) K_k(x,p,N) \exp[\,\{1 - x(s+t)/N\}u\,].
\end{aligned}$$

Equating the coefficients of $u^n/n!$, we get the form of $\widetilde{p}_{ik}(n)$ which is just the same expression with (i) of Remark 1.3.

Lastly, consider the Markov chain with transition probabilities (1.1) in the reduced description of the random walk on the N-cube in discrete-time, starting at the origin (state 0). By T_{0N} denote the *first-passage time* to the opposite vertex (state N), and let $F_{0N}(z)$ and μ_{0N} be the probability generating function and the mean of T_{0N}, respectively:

$$F_{0N}(z) = E[\,z^{T_{0N}}\,] \quad \text{and} \quad \mu_{0N} = E[\,T_{0N}\,].$$

Let $\widetilde{T}_{0N}$ be the continuous-time analogue of T_{0N}. By $\widetilde{F}_{0N}(z)$ and $\widetilde{\mu}_{0N}$ denote the Laplace-Stieltjes transform and mean of $\widetilde{T}_{0N}$, repectively:

$$\widetilde{F}_{0N}(z) = E[\exp(\,-z\,\widetilde{T}_{0N}\,)\,] \quad \text{and} \quad \widetilde{\mu}_{0N} = E[\,\widetilde{T}_{0N}\,].$$

Then we will investigate the asymptotics of the first passage to the opposite vertex as N increases, extending the theorem of Bingham [3].

Theorem 2.3. *Consider the model* (1.1). *Put*

$$p = \frac{s}{s+t}, \quad q = \frac{t}{s+t} \quad \text{and} \quad \lambda = \frac{s+t}{N}.$$

In (iv), (v) *and* (vi) *below, suppose that* $s \leqq t$. *Then we have the following:*

(i) $$F_{0N}(z) = \left[\sum_{x=0}^{N} (-1)^x \binom{N}{x} \left\{ \frac{1}{1 - z(1-\lambda x)} \right\} \right] \times \left[\sum_{x=0}^{N} \left(\frac{q}{p}\right)^x \binom{N}{x} \left\{ \frac{1}{1 - z(1-\lambda x)} \right\} \right]^{-1}.$$

(ii) $$\widetilde{F}_{0N}(z) = F_{0N}\left(\frac{1}{1+z}\right) = \left[\sum_{x=0}^{N} (-1)^x \binom{N}{x} \left\{ \frac{1}{z+\lambda x} \right\} \right] \times \left[\sum_{x=0}^{N} \left(\frac{q}{p}\right)^x \binom{N}{x} \left\{ \frac{1}{z+\lambda x} \right\} \right]^{-1}.$$

(iii) $\mu_{0N} = \widetilde{\mu}_{0N}$.

(iv) $$\mu_{0N} = \frac{1}{\lambda} \sum_{x=1}^{N} \left[\left(\frac{q}{p}\right)^x - (-1)^x \right] \binom{N}{x} \frac{1}{x}.$$

(v) For large N, $\mu_{0N} \sim \dfrac{1}{t\, p^N}$.

(vi) $(t\, p^N) T_{0N}$ converges in distribution to the exponential distribution with parameter 1 as $N \to \infty$.

3. Proof of Theorem 2.1

(i) Write $\Pi(z)$ for the probability generating function of the stationary probability distribution $\pi = (\pi_0, \dots, \pi_N)$ that is given in (ii) of Remark 1.3:

$$\Pi(z) = \sum_{j=0}^{N} \pi_j\, z^j = \frac{1}{\beta(a+1, b+1)} \int_0^1 u^a (1-u)^b \{ z\, u + (1-u) \}^N \, du$$

with constants $a, b > -1$, where $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

From this we calculate the differential coefficient $\Pi'(1)$ and obtain the result.
(ii) First of all we rewrite (1.2) as follows:

$$p_{i,i+1} = \frac{\nu}{N(N+a+b+2)} \{ N(a+1) + (N-a-1)i - i^2 \},$$

$$p_{i,i-1} = \frac{\nu}{N(N+a+b+2)} \{ (N+b+1)i - i^2 \},$$

$$p_{i,i} = 1 - \frac{\nu}{N(N+a+b+2)} \{ N(a+1) + (2N-a+b)i - 2\, i^2 \}.$$

So, a straightforward calculation yields

$$\begin{aligned}&(i+1)\cdot p_{i,i+1}+(i-1)\cdot p_{i,i-1}+i\cdot p_{i,i}\\&\quad=i+\frac{\nu}{N(N+a+b+2)}\{N(a+1)-(a+b+2)i\}.\end{aligned}\tag{3.1}$$

For the Markov chain with transition probabilities (1.2), let $E(X_n\,|\,X_{n-1})$ be the conditional expectation of X_n given X_{n-1}. Then (3.1) implies

$$E(X_n\,|\,X_{n-1})=X_{n-1}+\frac{\nu}{N(N+a+b+2)}\{N(a+1)-(a+b+2)X_{n-1}\},$$

and so

$$E(X_n\,|\,X_{n-1})-X_{n-1}=-\nu\left(\frac{a+b+2}{N(N+a+b+2)}\right)\{X_{n-1}-<\pi>\}\tag{3.2}$$

with $<\pi>=N(\frac{a+1}{a+b+2})$. For $Y_n=X_n-<\pi>$, set $e_n=E_i(Y_n)$, that is, the expected value of Y_n given $X_0=i$. Then, taking note of that $E_i(X_n-X_{n-1})=E_i[E(X_n\,|\,X_{n-1})-X_{n-1}]$, by (3.2) we can calculate as follows:

$$\begin{aligned}e_n&=E_i(X_n-<\pi>)=E_i[(X_{n-1}-<\pi>)+(X_n-X_{n-1})]\\&=E_i(X_{n-1}-<\pi>)+E_i(X_n-X_{n-1})\\&=e_{n-1}-\nu\left(\frac{a+b+2}{N(N+a+b+2)}\right)E_i(X_{n-1}-<\pi>)\\&=\left[1-\nu\left(\frac{a+b+2}{N(N+a+b+2)}\right)\right]e_{n-1}.\end{aligned}$$

Thus, by successive applications of the above relation, we get

$$e_n=\left[1-\nu\left(\frac{a+b+2}{N(N+a+b+2)}\right)\right]^n e_0,$$

where $e_0=E_i(X_0-<\pi>)=i-<\pi>$.
Hence the proof is complete.

4. Proof of Theorem 2.2

(i) Consider the matrix $\widetilde{P}(u)=(\widetilde{P}_{ij}(u))$ with $\widetilde{P}_{ij}(u)$ given by (2.5). Then we must verify that $\widetilde{P}(u)$ is a stochastic matrix, satisfying $\widetilde{P}(u)\to I$ as $u\to 0$, where I is the identity matrix. By (2.3) and Remark 2.3 it is easy to see that the transition probabilities $\widetilde{P}_{ij}(u)$ can be expressed by the form of (i). Since for each i there are $\binom{N}{m}$ states j with $m=|i-j|$, we have

$$\begin{aligned}\sum_j\widetilde{P}_{ij}(u)&=\frac{1}{2^N}\sum_{m=0}^{N}\binom{N}{m}[p_{00}(u)+p_{11}(u)]^{N-m}[p_{01}(u)+p_{10}(u)]^m\\&=\frac{1}{2^N}[\{p_{00}(u)+p_{11}(u)\}+\{p_{01}(u)+p_{10}(u)\}]^N\\&=\frac{1}{2^N}[\{p_{00}(u)+p_{01}(u)\}+\{p_{10}(u)+p_{11}(u)\}]^N\\&=\frac{1}{2^N}[\{q+p\}+\{q+p\}]^N=\frac{1}{2^N}[1+1]^N=1.\end{aligned}$$

Namely, $\widetilde{P}(u)$ is stochastic. On the other hand, by Remark 2.3, since

$$p_{00}(0) + p_{11}(0) = (q+p) + (p+q) = 1 + 1 = 2$$

and since

$$p_{01}(0) + p_{10}(0) = (p-p) + (q-q) = 0,$$

we have that $\widetilde{P}(u) \to I$ as $u \to 0$. Further we note that each $\widetilde{P}_{ij}(u)$ has the right-hand derivative at $u = 0$, associating with the Q-matrix $Q = (\ \widetilde{P}'_{ij}(0)\ )$ such that

$$\widetilde{P}'_{ij}(0) = -\frac{s+t}{2} \quad \text{if} \quad m = 0, \qquad \frac{s+t}{2N} \quad \text{if} \quad m = 1, \qquad 0 \quad \text{if} \quad m > 1.$$

(ii) Consider the coordinate process $X_i(u)$ given by (2.3). Then we may define the probability generating function of $X_i(u)$ by

$$P(\ X_i(u) = 0\ ) + z\, P(\ X_i(u) = 1\ ),$$

which is from Remark 2.3,

$$p_{00}(u) + z\, p_{01}(u) = (\, q + p\, z\,) \left[1 + \left(\frac{1-z}{q+p\,z} \right) p \exp[\, -(s+t)u/N\,] \right],$$

or

$$p_{10}(u) + z\, p_{11}(u) = (\, q + p\, z\,) \left[1 - \left(\frac{1-z}{q+p\,z} \right) q \exp[\, -(s+t)u/N\,] \right],$$

according as $X_i(0)$ is 0 or 1. Remember that $X(u) = \sum_{i=1}^N X_i(u)$ and $P_{ik}(u) = P(\ X(u) = k \,|\, X(0) = i\ )$. Now, suppose that there are initially i balls in urn I and $N - i$ in urn II. Then, since the quantities $X_i(u)$, where $i = 1, 2, \ldots, N$, are independent, the probability generating function of $X(u)$ is calculated as follows:

$$\begin{aligned}
\sum_{k=0}^{N} P_{ik}(u)\, z^k &= E_i \left[\, z^{X(u)}\, \right] = \Pi_{j=1}^N E_i \left[\, z^{X_j(u)}\, \right] \qquad\qquad (\, E_i(\,\cdot\,) = E(\,\cdot\, | X(0) = i)\,) \\
&= [\, p_{10}(u) + z\, p_{11}(u)\,]^i\ [\, p_{00}(u) + z\, p_{01}(u)\,]^{N-i} \\
&= (q + p\,z)^N \left[1 - \frac{q}{p}\, \xi \right]^i [\, 1 + \xi\,]^{N-i} \\
&= (q + p\,z)^N \sum_{x=0}^{N} \binom{N}{x} K_x(i)\, \xi^x \qquad \left(\xi = \left(\frac{1-z}{q+p\,z} \right) p \exp[\, -(s+t)u/N\,] \right) \\
&= \sum_{x=0}^{N} \binom{N}{x} K_x(i)\, p^x\, q^{N-x} \left\{ [\, 1 + \eta\,]^{N-x} \left[1 - \frac{q}{p}\, \eta \right]^x \right\} \exp[\, -x(s+t)u/N\,] \\
&\qquad\qquad \left(\eta = \frac{p}{q}\, z \right) \\
&= \sum_{x=0}^{N} \binom{N}{x} K_x(i)\, p^x\, q^{N-x} \left\{ \sum_{k=0}^{N} \binom{N}{k} K_k(x)\, \eta^k \right\} \exp[\, -x(s+t)u/N\,] \\
&= \sum_{k=0}^{N} \binom{N}{k} \left(\frac{p}{q} \right)^k \left[\sum_{x=0}^{N} \binom{N}{x} p^x\, q^{N-x}\, K_x(i)\, K_k(x) \exp[\, -x(s+t)u/N\,] \right] z^k.
\end{aligned}$$

Here $K_n(x)$ with $n, x = 0, 1, \dots, N$ are the Krawtchouk polynomials appearing in Remark 1.4, and in the above equalities the property of the generating function of $K_n(x)$ is used at twice. Note the symmetry relation such that $K_x(i) = K_i(x)$. Then, equating the coefficients of z^k, we obtain the expression of $P_{ik}(u)$. Hence the proof is complete.

5. Proof of Theorem 2.3

(i) Consider the Markov chain with transition probabilities (1.1). Denote by $p_{ij}^{(n)}$ and T_{ij} the n-step transition probabilities and the first-passage times to the state j for this chain starting at the state i, respectively. Set

$$P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n \quad \text{and} \quad F_{ij}(z) = \sum_{n=0}^{\infty} P(T_{ij} = n) z^n.$$

Then, since $P_{ij}(z) = P_{jj}(z)F_{ij}(z)$ for $i \neq j$, we see that $F_{0N}(z) = P_{0N}(z)/P_{NN}(z)$. By Remark 1.3, $p_{ij}^{(n)}$ is expressed in terms of the Krawtchouk polynomials, which implies

$$\begin{aligned} P_{0N}(z) &= \sum_{n=0}^{\infty} p_{0N}^{(n)} z^n \\ &= \sum_{n=0}^{\infty} \binom{N}{N} \left(\frac{p}{q}\right)^N \sum_{x=0}^{N} \binom{N}{x} p^x q^{N-x} K_0(x) K_N(x) [(1-\lambda x) z]^n \\ &= \left(\frac{p}{q}\right)^N \sum_{x=0}^{N} \binom{N}{x} p^x q^{N-x} K_0(x) K_N(x) \left[\frac{1}{1 - z(1-\lambda x)}\right]. \end{aligned}$$

By Remark 1.4, since $K_0(x) = 1$ and $K_N(x) = K_x(N) = (-1)^x \left(\frac{q}{p}\right)^x$, we get

$$P_{0N}(z) = p^N \sum_{x=0}^{N} (-1)^x \binom{N}{x} \left[\frac{1}{1 - z(1-\lambda x)}\right].$$

By the same way, we get

$$P_{NN}(z) = \sum_{n=0}^{\infty} p_{NN}^{(n)} z^n = p^N \sum_{x=0}^{N} \left(\frac{q}{p}\right)^x \binom{N}{x} \left[\frac{1}{1 - z(1-\lambda x)}\right],$$

from which follows (i).
(ii) The time reversibility shows that $\widetilde{F}_{0N}(z) = E(\exp[-z\widetilde{T}_{0N}]) = F_{0N}(\frac{1}{1+z})$. So, the above formula (i) with z replaced by $1/(1+z)$ yields (ii).
(iii) Evidently,

$$\widetilde{\mu}_{0N} = E(\widetilde{T}_{0N}) = -\widetilde{F}'_{0N}(0) = -\left[\frac{d}{dz} F_{0N}\left(\frac{1}{1+z}\right)\right]_{z=0} = F'_{0N}(1) = E(T_{0N}) = \mu_{0N}.$$

(iv) The above result (ii) can be rewritten by the form

$$\widetilde{F}_{0N}(z) = \left(\frac{1}{z} + f_1(z) \right) \left(\frac{1}{z} + f_2(z) \right)^{-1},$$

where $f_1(z) = \sum_{x=1}^{N} (-1)^x \binom{N}{x} \left[\frac{1}{z+\lambda x} \right]$ and $f_2(z) = \sum_{x=1}^{N} \left(\frac{q}{p}\right)^x \binom{N}{x} \left[\frac{1}{z+\lambda x} \right]$. Since $f_1'(z)$ and $f_2'(z)$ exist at $z = 0$, and since

$$-\widetilde{F}_{0N}'(z) = -\left[\frac{g(z) - h(z)}{(1 + z\, f_2(z))^2} \right]$$

with functions

$$g(z) = -f_2(z) + z\, f_1'(z) + z^2\, f_1'(z)\, f_2(z) \quad \text{and} \quad h(z) = -f_1(z) + z\, f_2'(z) + z^2\, f_1(z)\, f_2'(z),$$

we have that $\mu_{0N} = \widetilde{\mu}_{0N} = -\widetilde{F}_{0N}'(0) = f_2(0) - f_1(0) = \sum_{x=1}^{N} \left[\left(\frac{q}{p}\right)^x - (-1)^x \right] \binom{N}{x} \frac{1}{\lambda x}.$

Here, the assumption on s and t such that $s \leqq t$ implies

$$\left(\frac{q}{p}\right)^x - (-1)^x = \left(\frac{t}{s}\right)^x - (-1)^x \geqq 0 \quad \text{for} \quad x = 1, 2, \ldots, N,$$

which guarantees that $\mu_{0N} \geqq 0$, giving (iv).
(v) Put

$$a_N = \frac{1}{\lambda} \sum_{x=1}^{N} \left(\frac{q}{p}\right)^x \binom{N}{x} \frac{1}{x} \quad \text{and} \quad b_N = \frac{1}{\lambda} \sum_{x=1}^{N} (-1)^x \binom{N}{x} \frac{1}{x}. \tag{5.1}$$

Then we evaluate $a_N - b_N$ for large N. Set

$$p(x) = \left(\frac{p^N}{1 - p^N}\right) \left(\frac{q}{p}\right)^x \binom{N}{x}, \quad x = 1, 2, \ldots, N,$$

so that

$$\sum_{x=1}^{N} p(x) = \left(\frac{p^N}{1 - p^N}\right) \left[\left(1 + \frac{q}{p}\right)^N - 1 \right] = 1.$$

Namely, $\{p(x) : x = 1, 2, \ldots, N\}$ is a probability distribution, having the probability generating function

$$\left(\frac{p^N}{1 - p^N}\right) \left[\left(1 + \frac{q}{p} z\right)^N - 1 \right]$$

and mean $Nq/(1 - p^N)$. For large N, the binomial distribution can be approximated by the standard normal distribution, and so the mass of $p(x)$ concentrates on the intervals of the form $(Nq/(1-p^N)-C\sqrt{N},\quad Nq/(1-p^N)+C\sqrt{N})$ with a constant $C>0$. Namely, on such intervals, $1/x \sim (1-p^N)/Nq$. Hence

$$\sum_{x=1}^{N} p(x)\left(\frac{Nq}{1-p^N}\right)\frac{1}{x} \sim 1 \qquad (N\to\infty),$$

which implies

$$\begin{aligned}\frac{1}{\lambda}\sum_{x=1}^{N}\left(\frac{q}{p}\right)^x\binom{N}{x}\frac{1}{x} &\sim \frac{1}{\lambda}\left(\frac{1-p^N}{Nq}\right)\left(\frac{1-p^N}{p^N}\right)\\ &= \left(\frac{N}{s+t}\right)\left(\frac{s+t}{Nt}\right)\frac{(1-p^N)^2}{p^N} \qquad (N\to\infty).\end{aligned}$$

Here, $0< \ p=s/(s+t)\ <1$, and so $(1-p^N)^2\sim 1 \quad (N\to\infty)$. Thus

$$a_N \sim \frac{1}{t\,p^N} \qquad (N\to\infty). \tag{5.2}$$

On the other hand, as Bingham [3, p.604] shows, we have

$$\frac{N}{2}\sum_{x=1}^{N}(-1)^x\binom{N}{x}\frac{1}{x} = o(2^N) \qquad (N\to\infty).$$

Notice that

$$\begin{aligned}(t\,p^N)\,b_N &= (t\,p^N)\left(\frac{2}{s+t}\right)\left\{\frac{N}{2}\sum_{x=1}^{N}(-1)^x\binom{N}{x}\frac{1}{x}\right\}\\ &= (2p)^N\left(\frac{2t}{s+t}\right)\frac{1}{2^N}\left\{\frac{N}{2}\sum_{x=1}^{N}(-1)^x\binom{N}{x}\frac{1}{x}\right\}\end{aligned}$$

and consider that for $s\leqq t$, $0<(2p)^N=[\,2s/(s+t)\,]^N\leqq 1$. Then

$$b_N = o\left(\frac{1}{t\,p^N}\right) \qquad (N\to\infty). \tag{5.3}$$

Therefore, (5.2) and (5.3) yield

$$a_N - b_N \sim \frac{1}{t\,p^N} \qquad (N\to\infty),$$

giving (v).
(vi) The above result (ii) shows

$$E\left[\exp\{-(t\,p^N)\widetilde{T}_{0N}\,z\}\right] = \widetilde{F}_{0N}\left(\,(t\,p^N)\,z\,\right) = \frac{B_N}{A_N}, \tag{5.4}$$

where

$$A_N = \frac{1}{(t\,p^N)z} + \sum_{x=1}^{N} \left(\frac{q}{p}\right)^x \binom{N}{x} \left[\frac{1}{(t\,p^N)z + \lambda x}\right] \sim \frac{1}{(t\,p^N)z} + a_N \qquad (N \to \infty)$$

and

$$B_N = \frac{1}{(t\,p^N)z} + \sum_{x=1}^{N} (-1)^x \binom{N}{x} \left[\frac{1}{(t\,p^N)z + \lambda x}\right] \sim \frac{1}{(t\,p^N)z} + b_N \qquad (N \to \infty)$$

with a_N and b_N as defined by (5.1). Accordingly, combining (5.4) with (5.3) and (5.2), we have

$$\begin{aligned} E\left[\exp\left\{-(t\,p^N)\widetilde{T}_{0N}\,z\right\}\right] &= \frac{\frac{1}{(t\,p^N)z} + o\left(\frac{1}{t\,p^N}\right)}{\frac{1}{(t\,p^N)z} + \frac{1}{t\,p^N} + o\left(\frac{1}{t\,p^N}\right)} \\ &\longrightarrow \frac{\frac{1}{z}}{\frac{1}{z}+1} = \frac{1}{1+z} \qquad (N \to \infty). \end{aligned}$$

Since a random variable X having the exponential distribution with parameter 1 satisfies that $E[\,\exp\{-zX\}\,] = 1/(1+z)$, the above relation implies that $(t\,p^N)\widetilde{T}_{0N}$ converges in distribution to the exponential distribution with parameter 1 as $N \to \infty$, and (vi) follows similarly or by (ii). Hence the proof is complete.

5. Complements

In Theorem 2.2 there is a limit distribution in each case:
Since $\lim_{u\to\infty} p_{00}(u) = \lim_{u\to\infty} p_{10}(u) = q = t/(s+t)$ and $\lim_{u\to\infty} p_{01}(u) = \lim_{u\to\infty} p_{11}(u) = p = s/(s+t)$, we have

$$\lim_{u\to\infty} \widetilde{P}_{ij}(u) = \frac{1}{2^N}\,[q+p]^{N-m}\,[p+q]^m = \frac{1}{2^N},$$

so that the limit law is uniform for the 'full description'.
On the other hand, by Remark 1.4, since $K_i(0) = K_k(0) = 1$, we have

$$\lim_{u\to\infty} P_{ik}(u) = \binom{N}{k}\left(\frac{p}{q}\right)^k p^0\, q^{N-0} = \binom{N}{k} p^k\, q^{N-k}, \quad \text{where} \quad i,k = 0,1,\ldots,N.$$

Namely, the limit law is binomial for the 'reduced description'.

By contrast, in Kac's theorem [6] for the n-step transition probability of the discrete-time Markov chain that is the Ehrenfest model (1.1) with $s = t = 1$, $\pi_k = 2^{-N}\binom{N}{k}$ gives an invariant (stationary) probability distribution, but there is no limit distribution: Indeed, let $\widetilde{\mathsf{p}} = (\widetilde{p}_{ik})$ be the transition probability matrix for the model (1.1) with $s = t = 1$. Then it is easy to see that

$$\lim_{n\to\infty} \widetilde{\mathsf{p}}^{2n} \neq \lim_{n\to\infty} \widetilde{\mathsf{p}}^{2n+1}.$$

If we start at any vertex that we relabel as the 'origin' $(0,\dots,0)$, the vertex it takes longest to hit is $(1,\dots,1)$. Now if we apply a unit voltage between these two vertices so that the voltage at $(0,\dots,0)$ is 1 and the voltage at $(1,\dots,1)$ is 0, then all vertices having the same number of 1's share the same voltage and can be shorted. In general what we obtain is a new graph with $N+1$ vertices, where the kth new vertex consists of the shorting of all vertices in the unit cube with k1's. Thus, as follows from Palacios [10] and [11], the N-cube yields the platonic graphs in electric networks.

The well-known *Ornstein-Uhlenbeck process* is obtained in the diffusion limit of the Ehrenfest model (1.1) with $s=t=1$, where we centre at $N/2$, scale space and time and let $N\to\infty$.

The Ehrenfest urn models are applicable to the field of neuron firing and have a complementary aspect which is the study of the time taken to approach stationarity —measured in terms of variation distance— and the 'cut-off phenomenon' typically exhibited.

Our work for a generalization is inspired by the above point of view.

6. Discrete-time Ehrenfest urn models with applications

The transition matrix $P=(\,p_{ij}\,)$ as defined in (1.1) or (1.2) is a special cae of tridiagonal matrix

$$P=\begin{pmatrix} \nu_0 & \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & \nu_1 & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_2 & \nu_2 & \lambda_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \mu_{N-1} & \nu_{N-1} & \lambda_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu_N & \nu_N \end{pmatrix}, \tag{6.1}$$

satisfying

$$\lambda_i=p_{i,i+1},\quad \mu_i=p_{i,i-1},\quad \nu_i=p_{i,i}\quad (i=0,1,\dots,N),\quad \lambda_N=0,\quad \mu_0=0.$$

This chain is a random walk on $\{0,1,\dots,N\}$. The model (1.2) is a generalization of the model (1.1) such that the transition probabilities $p_{i,i+1}$ and $p_{i,i-1}$ are allowed to be quadratic functions of the current state i. In this section, for simplicity of the notation and the calculation, we take the discrete-time Ehrenfest model (1.1), that is, the model (6.1) satisfying

$$\lambda_i=\left(1-\frac{i}{N}\right)s,\quad \mu_i=\frac{i}{N}t,\quad \nu_i=1-(\,\lambda_i+\mu_i\,), \tag{6.2}$$

where $0<s,\,t\leqq 1$ and $i=0,1,\dots,N$.

If $s=t=1$, then $(\nu_0,\lambda_0)=(0,1)$ and $(\mu_N,\nu_N)=(1,0)$, and such a model is a *birth-death chain* with *reflecting boundaries* at 0 and N. For the case $s+t=1$, which we call a *one-parameter* Ehrenfest model, we interpret s and $t(=1-s)$ as probabilities of choosing urns I and II, respectively. If $s\neq 1$ and $t\neq 1$, then the chain is a *randam walk* with *sticky boundaries* at 0 and N.

6.1 Steady state and time reversibility

The Ehrenfest model (6.1) with rates (6.2) satisfies the *global balance equations*

$$\pi_i = \sum_{j=0}^{N} \pi_j \, p_{ji} \quad \text{for} \quad i = 0, 1, \ldots, N$$

and also satisfies the *detailed balance equations*

$$\pi_i \, p_{ij} = \pi_j \, p_{ji} \quad \text{for} \quad i, j = 0, 1, \ldots, N.$$

Hence the chain is *time reversible*, having (unique) stationary probability distribution

$$\pi_j = \binom{N}{j} \left(\frac{s}{s+t}\right)^j \left(\frac{t}{s+t}\right)^{N-j}, \quad j = 0, 1, \ldots, N, \quad \text{where} \quad 0 < s, t \leqq 1.$$

6.2 Convergence vs. recurrence

Here we discusss the Ehrenfest model (6.1) satisfying (6.2) with $s = t = 1$. Then we can show that recurrence is *not observable* for states far from the steady state $M = N/2$, assuming that N is even. For instance, the average time to reach 0 from state M, where M is the mean for the binomial distribution (stationary probability distribution), is

$$\frac{1}{2M} 2^{2M} \left(1 + O\left(\frac{1}{M}\right)\right)$$

whereas the average time to reach state M from state 0 is less than

$$M + M \log M + O(1).$$

With $M = 10^4$ balls and rate of transition one ball per second, the return time to equilibrium from state 0 is on the order of 102103 seconds, which is less than 29 hours (only about a day), whereas it would take on the order of 10^{6000} years to go from M to 0, which is an astronomical time (see Bhattacharya and Waymire [2, pp.250-251]).

Newton's law of cooling (Theorem 2.1) is another manifestation of thermodynamic irreversibility in the Ehrenfest model. On the other hand, in much more general settings, several authors investigate 'relaxation time', measuring the time taken for the random walk to approximate the limit law in variation norm (see Aldous [1] and Kijima [8]).

Example 6.1(*Finite Buffer*). A discrete-time communication channel with a finite buffer of size N behaves as follows: During the ith slot, either a new message arrives (if there is room for it), with positive probability, or one of the messages in the buffer (if any) is transmitted, with positive probability, or there is no change. This system can be modelled by a Markov chain which is in state i when there are i messages in the buffer ($i = 0, 1, \ldots, N$). From state i, there are transitions to states $i - 1$ (except when $i = 0$), i and $i + 1$ (except when $i = N$), with probabilities

$$\mu_i = p_{i,i-1}, \quad \nu_i = p_{i,i} = 1 - (\lambda_i + \mu_i) \quad \text{and} \quad \lambda_i = p_{i,i+1},$$

respectively. The chain with rates of the form (6.2) is an example of such a transmission.
Example 6.2(*Portfolio*). There will be one risky investment opportunity (such as a common stock) whose market price changes with time in probabilistic way. This risky asset will form a Markov chain called a random walk with sticky boundaries. Prices 0 and N are (certainly or uncertainly) reflecting states. If the price now is 0, then at the next time instant the price will be 1 with certainty or uncertainty. Similarly, if the price is N now, the price at the next instant of time will be $N-1$ or N. Otherwise, for states between 0 and N, the price will increase by 1 with probability $\lambda_i = p_{i,i+1}$, or decrease by 1 with probability $\mu_i = p_{i,i-1}$, or remain with probability $\nu_i = p_{i,i} = 1-(\lambda_i+\mu_i)$. The chain with rates of the form (6.2) is an example of such a market. We follow the progress of an investor who owns shares in this asset, and who also owns shares in a risk-free asset (such as a government bond) whose value is multiplied by a constant $\gamma > 1$, at each discrete instant of time. The investor can allocate his wealth between the risky and the risk-free asset at each instant, and also, if he chooses, consume some of his wealth. The investor is to decide how to change his porfolio and how to consume money as time progresses, in order to maximize his expected total consumption. In a finite time horizon investment-consumption problem, all wealth is consumed at the terminal time. In an infinite-horizon discounted problem, the invester will maximize the expected total discounted consumption. The optimal-value function for the control problem contains two parameters s and t such that $0 < s, t \leqq 1$. We may be concerned with asymptotic analysis as $s \uparrow 1$ and $t \uparrow 1$.

7. Continuou-time Ehrenfest urn models with applications

Here we treat the Markov process which is the continuous-time formulation of the Markov chain with transition matrix (6.1) satisfying (6.2). First of all, we note that the Krawtchouk polynomials $K_n(x,p,N)$ as described in Remark 1.4, which play an essential role in the continuous-time formulation, have the following equivalent formula:

$$K_n(x,p,N) = C^{-1}\sum_{\nu=0}^{n}(-1)^{\nu}\binom{x}{\nu}\binom{N-x}{n-\nu}\left(\frac{q}{p}\right)^{\nu}, \qquad n = 0,1,\ldots,N,$$

where

$$C = \binom{N}{n} \quad \text{and} \quad 0 < p = 1-q < 1.$$

The Krawtchouk polynomials are orthogonal with respect to the binomial distribution with masses $\binom{N}{x}p^x q^{N-x}$ at the $N+1$ points $x = 0,1,\ldots,N$.

For $i = 0,1,\ldots,N$, define k_{ij} by

$$k_{ij} = K_i(j,p,N).$$

Then, as follows from Karlin and McGregor [7], the polynomials k_{ij} satisfy the following recurrence relation:

$$-j\,k_{ij} = i\,q\,k_{i-1,j} - [i\,q + (N-i)p]\,k_{ij} + (N-i)p\,k_{i+1,j}.$$

7.1 Expression of the Krawtchouk polynomials

The above-mentioned formula and Remark 1.4 yield the following explicit expression of the polynomials k_{ij}:

$$k_{0j} = 1, \qquad k_{1j} = \frac{1}{pN}(Np - j),$$

$$k_{Nj} = (-1)^j \left(\frac{q}{p}\right)^j, \qquad k_{N-1,j} = (-1)^j \frac{q^{j-1}}{p^j N}(Nq - j),$$

$$k_{ij} = k_{ji}, \qquad i, j = 0, 1, \ldots, N.$$

Let s and t be the parameters such that $0 < s, t \leqq 1$, and put

$$p = \frac{s}{s+t}, \qquad q = \frac{t}{s+t},$$

$$K = (\, p\, k_{ij}\,)_{0 \leqq i,j \leqq N}, \quad \lambda_j = 1 - \frac{j}{N}(s+t), \quad 0 \leqq j \leqq N, \quad \Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N).$$

Let P be the transition matrix as given by (6.1) and (6.2). Then, it follows from Krafft and Schaefer [9] that $P\,K = K\,\Lambda$. Namely, the Krawtchouk polynomials provide right eigenvectors of P corresponding to the eigenvalues λ_j.

7.2 Q-matrix

Let $X(u)$ and $\widetilde{X}(u)$ be the Markov processes with continuos parameter $u \geqq 0$, which are the continuou-time formulation of the Ehrenfest urn model (1.1) in the 'reduced description' and the 'full description' as described in Remark 1.1, respectively. Then, the transition probabilities of $X(u)$ and $\widetilde{X}(u)$ are denoted by

$$P_{ik}(u), \text{ where } i, k = 0, 1, \ldots, N$$

and

$$\widetilde{P}_{ij}(u), \text{ where } i \text{ and } j \text{ vary in the vertex-set } V,$$

respectively (see (2.4) and (2.5)). By Theorem 2.2 and the expression of the Krawtchouk polynomials k_{ij} in the preceding subsection, we can calculate the right-hand derivatives at $u = 0$ for $\widetilde{P}_{ij}(u)$ and $P_{ik}(u)$, associating the Q-matrices $Q = (\, \widetilde{P}'_{ij}(0)\,)$ and $Q = (\, P'_{ik}(0)\,)$ with the following results:

$$\text{(i)} \quad Q = (\, \widetilde{P}'_{ij}(0)\,), \qquad \widetilde{P}'_{ij}(0) = \begin{cases} -\frac{1}{2}(s+t) & \text{if } m = \mid i - j \mid = 0, \\ \frac{1}{2N}(s+t) & \text{if } m = \mid i - j \mid = 1, \\ 0 & \text{if } m = \mid i - j \mid > 1, \end{cases} \tag{7.1}$$

where $i = (i_1, \ldots, i_k, \ldots, i_N)$ and $j = (j_1, \ldots, j_k, \ldots, j_N)$ are in the vertex-set V, and $m = \mid i - j \mid = \sum_{k=1}^{N} \mid i_k - j_k \mid$, and moreover $0 < s, t \leqq 1$.

$$\text{(ii)} \quad Q = (\, P'_{ik}(0)\,), \qquad P'_{ik}(0) = \begin{cases} \lambda_i & \text{if } k = i + 1, \\ \mu_i & \text{if } k = i - 1, \\ -(\lambda_i + \mu_i) & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases} \tag{7.2}$$

where

$$\lambda_i = \left(1 - \frac{i}{N}\right) s, \qquad \mu_i = \frac{i}{N} t, \qquad 0 < s, t \leqq 1, \qquad i = 0, 1, \dots, N.$$

The Q-matrices (7.1) and (7.2) are called the (formal) *infinitesimal generator* associated with $\widetilde{X}(u)$ and $X(u)$, respectively, which represent the *instantaneous transition rates.*

The Markov proxess $X(u)$ as governed by (7.2) is a finite *birth-death process* with state space $\{0, 1, \dots, N\}$, satisfying the following Kolmogorov's backward equation:

$$\frac{d}{du} P_{i,k}(u) = \left(\frac{i}{N} t\right) P_{i-1,k}(u) - \left(\frac{i}{N} t + \left(1 - \frac{i}{N}\right) s\right) P_{i,k}(u) + \left(\left(1 - \frac{i}{N}\right) s\right) P_{i+1,k}(u), \tag{7.3}$$

$$i, k = 0, 1, \dots, N, \qquad P_{N+1,k}(u) \equiv 0, \qquad P_{-1,k}(u) \equiv 0,$$

and the initial conditions

$$P_{i,k}(0) = 1 \quad \text{if} \quad i = k, \qquad P_{i,k}(0) = 0 \quad \text{if} \quad i \neq k.$$

7.3 Models in 3-cube

(i) *The Markov process $\widetilde{X}(u)$ in the full description.* Define the vertex-set V by

$$V = \{\bar{0},\ \bar{1},\ \bar{2},\ \bar{3},\ \bar{4},\ \bar{5},\ \bar{6},\ \bar{7}\}$$

where

$$\bar{0} = (0,0,0), \qquad \bar{1} = (0,0,1), \qquad \bar{2} = (0,1,0), \qquad \bar{3} = (0,1,1)$$
$$\bar{4} = (1,0,0), \qquad \bar{5} = (1,0,1), \qquad \bar{6} = (1,1,0), \qquad \bar{7} = (1,1,1).$$

Set

$$c = \frac{1}{2}(s + t) \quad \text{and} \quad d = \frac{1}{6}(s + t).$$

Then, by (7.1), the Q-matrix associated with $\widetilde{X}(u)$ on the vertex set V has the following expression:

$$Q = \begin{pmatrix} -c & d & d & 0 & d & 0 & 0 & 0 \\ d & -c & 0 & d & 0 & d & 0 & 0 \\ d & 0 & -c & d & 0 & 0 & d & 0 \\ 0 & d & d & -c & 0 & 0 & 0 & d \\ d & 0 & 0 & 0 & -c & d & d & 0 \\ 0 & d & 0 & 0 & d & -c & 0 & d \\ 0 & 0 & d & 0 & d & 0 & -c & d \\ 0 & 0 & 0 & d & 0 & d & d & -c \end{pmatrix}.$$

(ii) *The Markov process* $X(u)$ *in the reduced description.* By (7.2), the Q-matrix associated with $X(u)$ on $\{0,1,2,3\}$ has the following expression:

$$Q = \begin{pmatrix} -s & s & 0 & 0 \\ \frac{1}{3}t & -\frac{1}{3}(2s+t) & \frac{1}{3}2s & 0 \\ 0 & \frac{1}{3}2t & -\frac{1}{3}(s+2t) & \frac{1}{3}s \\ 0 & 0 & t & -t \end{pmatrix}.$$

Example 7.1(*Mean First-Passage Time*). Let us consider the Markov process $X(u)$ on the 3-cube in the reduced description, starting at the origin (state $0 \Leftrightarrow \overline{0} = (0,0,0)$). Let $\widetilde{T}_{03}$ be the first-passage time of $X(u)$ to the opposite vertex (state $3 \Leftrightarrow \overline{3} = (1,1,1)$), and set $\widetilde{\mu}_{03} = E[\widetilde{T}_{03}]$. Then, by (iii) and (iv) of Theorem 2.3, we obtain

$$\widetilde{\mu}_{03} = \frac{1}{s}\left\{9 + \left(\frac{t}{s} - 1\right)\frac{7}{2} + \left(\frac{t}{s}\right)^2\right\}, \quad \text{and hence} \quad \widetilde{\mu}_{03} = 10 \quad \text{if} \quad s = t = 1.$$

7.4 Reliability theory

Let $X_i(u)$, where $i = 0, 1, \ldots, N$, be the one-particle process as defined in (2.3), satisfying Assumption 2.1. Then we can interpret as follows: Each $X_i(u)$ sojourns in urn I (state 1) and urn II (state 0) with exponentiol holding time of parameter $1/N$, and at the next instant of the end of its sojourn time, $X_i(u)$ remains in urn I and urn II repeatedly with probabilities $1-t$ and $1-s$, respectively, or moves to the opposite urn with probabilities t and s, respectively. The Markov process $X(u)$ in the reduced description is defined by

$$X(u) = \sum_{i=1}^{N} X_i(u),$$

with transition probabilities $P_{ik}(u)$ as given in (2.4). The probability law of $X(u)$ is obtained by the convolution of probability laws of i.i.d. random functions $X_i(u)$. So, the rate of occurrence that $X(u)$ terminates its sojourn is estimated at $1 = N*(1/N)$. Namely, $X(u)$ is a kind of Markov renewal process such that the discrete-time Markov chain with transition probabilities (1.1) moves from state to state with exponential holding time of parameter 1. From this poin of view, the results (ii) and (iii) of Theorem 2.3 are regarded as trivial properties.

Example 7.2(*Parallel System*). Let us consider a machine with Ncomponents, each alternately functioning (up) and defective (down). Suppose that $X_i(u)$ denote the states of the ith component, that is, the process with state space $\{0, 1\}$, where $0 =$ 'up' and $1 =$ 'down', and suppose that Assumption 2.1 holds for the family $\{X_i(u) : i = 1, \ldots, N\}$. Set $X(u) = \sum_{i=1}^{N} X_i(u)$, and consider $X(u)$ on the N-cube. For $X(u)$, starting at the origin (state 0), denote by $\widetilde{T}_{0N}$ the first-passage time of $X(u)$ to the opposite vertex (state N). If the components are in parallel, so that the machine functions if and only if at least one component does, then $\widetilde{T}_{0N}$ is the time to first failure of the machine. The results (iii), (iv) and (v) of Theorem 2.3 show the asymptotc behavior of the mean first-failure time $\widetilde{\mu}_{0N} = E[\widetilde{T}_{0N}]$ as $N \to \infty$: $\widetilde{\mu}_{0N} \sim 2^N$ for large N, if $s = t = 1$.

7.5 Queueing network

Let $X(u)$ be the Markove process in the reduced description with transition probabilities as given in (2.4) in the reduced description. Then $X(u)$ represents a model of a closed queueing network of the Jackson type with two nodes (urn I and urn II), each of which contains N servers whose service times are independent exponentials of parameter $1/N$, satisfying the following:
The total number of customers is N. Each customer, after completion of his service in urn I (resp. urn II), moves to urn II (resp. urn I) with probability t (resp. s) and remains in urn I (resp. urn II) with probability $1-t$ (resp. $1-s$) in order to be served, repeatedly.

The results as obtained in the previous section 2 may be extended to the 'multi-urn' versions of the classical Ehrenfest model in order that the continuous-time formulation can be applied to closed networks with multi-node.

7.6 Infinite system of queue processes

In the finite system of differential equations (7.3), let N tend to infinity. Then we obtain the (formal) infinite system:

$$\begin{aligned} &\frac{d}{du} P^*_{i,k}(u) = -s\, P^*_{i,k}(u) + s\, P^*_{i+1,k}(u), && i,k = 0,1,2,\dots\,, \\ &P^*_{i,k}(0) = 1 \quad \text{if} \quad i = k, && P^*_{i,k}(0) = 0 \quad \text{if} \quad i \neq k, \end{aligned}$$

which are the differential equations associated with the pure birth process with state space $\{0,1,2,\dots\}$.

We can obtain another infinite system under suitable scale change.

Example 7.3(*Infinitely Many Servers*). Fix a positive integer N_0 which is large enough, and choose N so that $N > N_0$. Set $s = a/N$ and $t = 1 - b/N$. Here a and b are positive real numbers, which are chosen so that $0 < s,t \leqq 1$, and fixed. Multiply the equations (7.3) by N_0. Then we see

$$\begin{aligned} N_0 \frac{d}{du} P_{i,k}(u) &= \left(\frac{N_0}{N}\right) i \left(1 - \frac{b}{N}\right) P_{i-1,k}(u) \\ &\quad - \left\{ \left(\frac{N_0}{N}\right) i \left(1 - \frac{b}{N}\right) + \left(\frac{N_0}{N}\right) \left(1 - \frac{i}{N}\right) a \right\} P_{i,k}(u) \\ &\quad + \left(\frac{N_0}{N}\right) \left(1 - \frac{i}{N}\right) a\, P_{i+1,k}(u). \end{aligned}$$

Let N tend to infinity in the avove equations, assuming that $N_0/N \approx 1$ as $N \to \infty$. Then, we obtain the following (formal) infinite system:

$$\begin{aligned} N_0 \frac{d}{du} P^*_{i,k}(u) &= i\, P^*_{i-1,k}(u) - (i+a) P^*_{i,k}(u) + a\, P^*_{i+1,k}(u), \\ i,k &= 0,1,2,\dots\,, \\ P^*_{i,k}(0) &= 1 \quad \text{if} \quad i = k, \qquad P^*_{i,k}(0) = 0 \quad \text{if} \quad i \neq k, \end{aligned}$$

which are the differential equations associated with a birth-death queueing process with infinitely many servers.

7.7 Manufacturing system

We consider the stochastic manufacturing system with the surplus level $x(u) \in R^1$ and the production rate $y(u) \in R^1$ satisfying

$$\frac{d}{du}x(u) = y(u) - z, \qquad x(0) = x,$$

where $z \in R^1$ is the constant demand rate and x is the initial value of $x(u)$.

Let $\mathcal{S} = \{0, 1, 2, \dots, N\}$ denote the set of capacity states, and let $K(\varepsilon, u) \in \mathcal{S}, u \geqq 0$, denote the total production capacity process of the parallel machines in the system, which is given by a finite state Markov process parameterized by a small number $\varepsilon > 0$. Then the production rate $y(u)$ must satisfy $0 \leqq y(u) \leqq K(\varepsilon, u)$ for $u \geqq 0$. We consider the cost function $J^\varepsilon(x, k, y(\cdot))$ with $K(\varepsilon, 0) = k$ defined by

$$J^\varepsilon(x, k, y(\cdot)) = E\left[\int_0^\infty \exp(-\rho u) G(x(u), y(u))\, du\right],$$

where $G(x, y)$ is the running cost of having surplus x and production rate y and $\rho > 0$ is the discount rate. The problem is to find an admissible control $0 \leqq y(u) \leqq K(\varepsilon, u), u \geqq 0$, as a function of the past $K(\epsilon, u)$, that minimizes $J^\varepsilon(x, k, y(\cdot))$.

According to Sethi [12], we make the following assumptions on G and $K(\varepsilon, u)$:
(i) The cost function G satisfies suitable smoothness condition and growth restriction.
(ii) The capacity process $0 \leqq K(\varepsilon, u) \in \mathcal{S}, u \geqq 0$, is a finite Markov process governed by the infinitesimal generator Q^ε of the form

$$Q^\varepsilon = Q^{(1)} + \frac{1}{\varepsilon} Q^{(2)},$$

where $Q^{(l)}$ is a $(N+1) \times (N+1)$ matrix such that $Q^{(l)} = (\, q_{ij}^{(l)} \,)$ with $q_{ij}^{(l)} \geqq 0$ if $i \neq j$ and $q_{ii}^{(l)} = -\sum_{i \neq j} q_{ij}^{(l)}$, for $l = 1, 2$. The matrix $Q^{(2)}$ is weakly irreducible, that is, it has a unique stationary probability distribution.
Under these assumtions, Sethi [12] investigates the limiting control problem as $\varepsilon \to 0$. Sethi [12] also considers $K(\varepsilon, u)$ which is modelled by a birth-death process such that the second Q-matrix $Q^{(2)} = (\, q_{ij}^{(2)} \,)$ in the infinitesimal generator Q^ε has the following expression:

$$q_{ij}^{(2)} = \lambda_i \quad \text{if} \quad j = i+1, \qquad\qquad q_{ij}^{(2)} = \mu_i \quad \text{if} \quad j = i-1,$$
$$q_{ij}^{(2)} = -(\lambda_i + \mu_i), \quad \text{if} \quad j = i, \qquad\qquad q_{ij}^{(2)} = 0 \quad \text{otherwise},$$

with nonnegative constants λ_i and μ_i. In the case where the second Q-matrix $Q^{(2)}$ is given by the Ehrenfest model (7.2), the value function for the above problem contains several parameters, such as $0 < \varepsilon \ll 1$, $0 < s, t \leqq 1$ and $N \geqq 1$. We may be concerned with the asymptotic analysis as $\varepsilon \downarrow 0$, $s \uparrow 1$, $t \uparrow 1$ and $N \uparrow \infty$.

8. Limiting distribution and diffusion approximation

Let $P_{ik}(u)$ be the transition probabilities of the continuous-time Markov chain in the reduced description as given in Theorem 2.2. Then we have the following results for the limiting distribution and diffusion approximation.

8.1 The rate of convergence to the limiting binomial distribution

Theorem 2.2 implies the explicit formula:

$$P_{ik}(u) = \binom{N}{k}\left(\frac{p}{q}\right)^k \sum_{x=0}^{N} \binom{N}{x} p^x q^{N-x} K_i(x) K_k(x) \exp[-\lambda x u],$$

where $K_i(x) = K_i(x, p, N)$ and $\lambda = (s+t)/N$. Using

$$K_n(0) = 1, \quad K_0(x) = 1,$$

which appear in (ii) of Remark 1.4, we have

$$\begin{aligned} P_{ik}(u) &= \binom{N}{k}\left(\frac{p}{q}\right)^k \left\{ q^N K_i(0) K_k(0) + \sum_{x=1}^{N} \binom{N}{x} p^x q^{N-x} K_i(x) K_k(x) \exp[-\lambda x u] \right\} \\ &= \binom{N}{k} p^k q^{N-k} + \exp[-\lambda u]\, \widetilde{\pi}_k \sum_{x=1}^{N} \binom{N}{x} p^x q^{N-x} K_i(x) K_k(x) \exp[-\lambda (x-1) u], \end{aligned}$$

where

$$\widetilde{\pi}_k = \binom{N}{x}\left(\frac{p}{q}\right)^k,$$

that is,

$$P_{ik}(u) = \binom{N}{k} p^k q^{N-k} + O\left(e^{-\lambda u}\right) \quad \text{as} \quad u \to \infty.$$

In other words the rate of convergence to the limiting (binomial) distribution is exponential of order $e^{-\lambda u}$. The constant in the O-symbol is easily expressed in terms of the parameters, in fact

$$\begin{aligned} P_{ik}(u) &= \binom{N}{k} p^k q^{N-k} \left\{ 1 + \sum_{x=1}^{N} \binom{N}{x} p^x q^{-x} K_i(x) K_k(x) \exp[-\lambda x u] \right\} \\ &= \binom{N}{k} p^k q^{N-k} \left\{ 1 + N p q^{-1} \frac{1}{pN}(Np - i) \frac{1}{pN}(Np - k) \exp[-\lambda u] + O\left(\exp[-2\lambda u]\right) \right\} \\ &= \binom{N}{k} p^k q^{N-k} \left\{ 1 + \frac{(Np-i)(Np-k)}{Npq} e^{-\lambda u} + O\left(e^{-2\lambda u}\right) \right\}, \end{aligned}$$

since $K_j(1) = k_{j1} = k_{1j} = (Np - j)/(pN)$ as given in subsection 7.1.

8.2 Convergence to the Ornstein-Uhlenbeck process

The Ehrenfest chain of the model (1.1) is given on the state space $\{0, 1, \dots, N\}$ with transition probabilities :

$$p(i, i-1) = \frac{i}{N} t, \quad p(i, i+1) = \left(1 - \frac{i}{N}\right) s, \quad p(i, i) = 1 - \left(1 - \frac{i}{N}\right) s - \frac{i}{N} t,$$

where $0 < s, t \leqq 1$. Assume that N is an even number. Set $a = N/2$ and let $\{ X_n \}$ be the discrete-time Ehrenfest chain on the state space

$$\{-a, \; -a+1, \; \dots, \; -1, \; 0, \; 1, \; \dots, \; a-1, \; a\}$$

with the following transition probabilities :

$$\begin{aligned} p(k, k-1) &= \left(\frac{k+a}{2a}\right) t = \frac{1}{2}\left(1 + \frac{k}{a}\right) t, \\ p(k, k+1) &= \left(\frac{a-k}{2a}\right) s = \frac{1}{2}\left(1 - \frac{k}{a}\right) s, \\ p(k, k) &= 1 - \frac{1}{2}\left(1 - \frac{k}{a}\right) s - \frac{1}{2}\left(1 + \frac{k}{a}\right) t. \end{aligned}$$

Let $p(n; k, m)$ be the n-step transition probabilities such that

$$p(n; k, m) = P(X_n = m \mid X_0 = k).$$

Then it is clear that

$$\begin{aligned} & p(n+1; k, m) \\ &= p(n; k, m-1) p(m-1, m) + p(n; k, m+1) p(m+1, m) + p(n; k, m) p(m, m) \\ &= \frac{1}{2}\left(1 - \frac{m-1}{a}\right) s \, p(n; k, m-1) + \frac{1}{2}\left(1 + \frac{m+1}{a}\right) t \, p(n; k, m+1) \\ &\quad + \left\{1 - \frac{1}{2}\left(1 - \frac{m}{a}\right) s - \frac{1}{2}\left(1 + \frac{m}{a}\right) t\right\} p(n; k, m). \end{aligned}$$

Let us speed up this nonhomogeneous random walk. Let $a (= N/2) \to \infty$ such that

$$\frac{(\Delta x)^2}{\Delta u} = \sigma^2 \quad \text{and} \quad a(\Delta u) \to \nu^{-1}$$

with constants $\sigma > 0$ and $\nu > 0$.

Let us pass from the n-step transition probability to the transition density

$$f(u, y) = f(x; u, y)$$

by taking $n(\Delta u) = u, \quad m(\Delta x) = x$:

$$\begin{aligned} f(u + \Delta u, y) &= \frac{1}{2}\left(1 - \frac{m-1}{a}\right) s f(u, y - \Delta y) + \frac{1}{2}\left(1 + \frac{m+1}{a}\right) t f(u, y + \Delta y) \\ &\quad + \left\{1 - \frac{1}{2}\left(1 - \frac{m}{a}\right) s - \frac{1}{2}\left(1 + \frac{m}{a}\right) t\right\} f(u, y). \end{aligned}$$

Then

$$
\begin{aligned}
& f(u,y) + (\Delta u)\frac{\partial f}{\partial u} + o(\Delta u) \\
= \; & \frac{1}{2}\left(1 - \frac{m-1}{a}\right) s \left\{ f(u,y) - (\Delta y)\frac{\partial f}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 f}{\partial y^2} \right\} \\
& + \frac{1}{2}\left(1 + \frac{m+1}{a}\right) t \left\{ f(u,y) + (\Delta y)\frac{\partial f}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 f}{\partial y^2} \right\} \\
& + \left\{ 1 - \frac{1}{2}\left(1 - \frac{m}{a}\right) s - \frac{1}{2}\left(1 + \frac{m}{a}\right) t \right\} f(u,y) + o(\,(\Delta y)^2\,) \\
= \; & \left\{ 1 + \frac{t+s}{2a} \right\} f(u,y) + \left\{ \frac{t-s}{2} + \frac{m}{2a}(t+s) + \frac{t-s}{2a} \right\} (\Delta y)\frac{\partial f}{\partial y} \\
& + \left\{ \frac{t+s}{2} + \frac{m}{2a}(t-s) + \frac{t+s}{2a} \right\} \frac{1}{2}\,(\Delta y)^2 \frac{\partial 2 f}{\partial y^2} + o(\,(\Delta y)^2\,).
\end{aligned}
$$

Observe that $m = y/(\Delta y)$, so that

$$
\begin{aligned}
& \frac{\partial f}{\partial u} + \frac{o(\,\Delta u\,)}{\Delta u} \\
= \; & \left(\frac{t+s}{2}\right)\left(\frac{1}{a(\Delta u)}\right) f(u,y) \\
& + \left\{ \left(\frac{t-s}{2}\right)\left(\frac{1}{\Delta u}\right)(\Delta y)\frac{\partial f}{\partial y} + \left(\frac{t+s}{2}\right)\left(\frac{1}{a(\Delta u)}\right) y\,\frac{\partial f}{\partial y} \right. \\
& \qquad \left. + \left(\frac{t-s}{2}\right)\left(\frac{1}{a(\Delta u)}\right)(\Delta y)\,\frac{\partial f}{\partial y} \right\} \\
& + \left\{ \left(\frac{t+s}{2}\right)\frac{1}{2}\left(\frac{(\Delta y)^2}{\Delta u}\right)\frac{\partial^2 f}{\partial y^2} + \left(\frac{t-s}{2}\right)\left(\frac{1}{a(\Delta u)}\right) y\,(\Delta y)\frac{\partial^2 f}{\partial y^2} \right. \\
& \qquad \left. + \left(\frac{t+s}{2}\right)\left(\frac{1}{a(\Delta u)}\right)(\Delta y)^2\,\frac{\partial^2 f}{\partial y^2} \right\} + \frac{o(\,(\Delta y)^2\,)}{\Delta u}.
\end{aligned}
$$

Assume that $t = s$ and choose Δu and Δy so that

$$
\frac{1}{a(\Delta u)} \to \nu \quad (\Delta u \to 0) \quad \text{and} \quad \frac{(\Delta y)^2}{\Delta u} \to \sigma^2 \quad (\Delta u \to 0,\ \Delta y \to 0).
$$

Then, we can obtain the following Kolmogorov's forward equation for the Ornstein-Uhlenbeck process :

$$
\frac{\partial f}{\partial u} = t \left\{ \nu\,\frac{\partial}{\partial y}\,(y\,f) + \frac{1}{2}\,\sigma^2\,\frac{\partial^2 f}{\partial y^2} \right\}.
$$

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