数理解析研究所講究録
1193 巻 2001 年 20-29

Ergodic properties of Fleming-Viot processes with selection and recombination

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1 Introduction

Let E be a locally compact separable metric space and $\mathcal{P}(E)$ be the space of all probability measures on E. For $\mu\in\mathcal{P}(E)$ let us denote $\langle f,\mu\rangle=\int_{E}fd\mu$. For any $f_{1},\cdots,f_{m}\in \mathcal{D}(A)$ and $F\in C^{2}(R^{m})$ let $\varphi(\mu)=F(\langle f_{1},\mu\rangle,\cdots,\langle f_{m},\mu\rangle)$ $=F(\langle\mathbf{f},\mu\rangle)$.

$$\begin{aligned}\mathcal{L}\varphi(\mu) &= \frac{1}{2}\sum_{i,j=1}^{m}(\langle f_{i}f_{j},\mu\rangle-\langle f_{i},\mu\rangle\langle f_{j},\mu\rangle)F_{z_{i}z_{j}}(\langle\mathbf{f},\mu\rangle) \\ &+ \sum_{i=1}^{m}(\langle Af_{i},\mu\rangle+\langle Bf_{i},\mu^{2}\rangle)F_{z_{i}}(\langle\mathbf{f},\mu\rangle) \qquad (1)\\ &+ \sum_{i=1}^{m}\{\langle(f_{i}\otimes 1)\sigma,\mu^{2}\rangle-\langle f_{i},\mu\rangle\langle\sigma,\mu^{2}\rangle\}F_{z_{i}}(\langle\mathbf{f},\mu\rangle).\end{aligned}$$

Here E is the space of genetic types and A is a mutation operator in $\bar{C}(E)(\equiv$ the space of bounded continuous functions on $E)$ which is the generator for a Feller semigroup $\{T(t)\}$ on $\hat{C}(E)(\equiv$ the space of continuous functions vanishing at infinity). Here $\sigma=\sigma(x,y)$ is a bounded symmetric function on $E\times E$ which is selection parameters for types $x,y\in E$ B is a recombination operator defined by

$$Bf(x,y)=\alpha\int_{E}(f(x')-f(x))R((x,y),dx')$$

where $\alpha\geq 0$ and $R((x,y),dx')$ is a one step transition function on $E^{2}\times\mathcal{B}(E)$, and we denote μ^{n} the n-fold product of μ. According to [3], this operator defines a generator corresponding to a Markov process on $\mathcal{P}(E)$ in the sense that the $C_{\mathcal{P}(E)}[0,\infty)$ martingale problem for $\mathcal{L}$ is well posed. This process is called the Fleming-Viot process. The aim of this paper is to consider ergodicity for this process by using the duality in the form

$$E_\mu[\langle f, \mu_t^n \rangle] = \sum_{k=1}^{\infty} \langle f_k(t), \mu^k \rangle$$

for any $t \geq 0$, $n \in \mathbf{N}$ and $f \in \bar{C}(E^n)$ with sup-norm $\|\cdot\|$. Here $f_k(t) \in \bar{C}(E^k)$ and satisfy $\sum_{k=1}^{\infty} \gamma^k \|f_k(t)\| < \infty$ for some $\gamma > 1$ and $f_n(0) = f$ and $f_k(0) = 0$ for $k \neq n$, and we consider a semigroup for this process.

2 Construction of a semigroup

We consider that E is a locally compact separable metric space, and treat the case of the formula (1) and assume $\{T(t)\}$ is a Feller semigroup on $\hat{C}(E)$ with the generator A. Denote the semigroup $T_k(t) = \overbrace{T(t) \otimes \cdots \otimes T(t)}^{k \text{ times}}$ on $\bar{C}(E^k)$ and its generator $A^{(k)}$.

We now consider duality under general condition for the diffusion. In this section we consider the operator of the form

$$(2) \qquad \mathcal{L}\varphi(\mu) = \frac{1}{2} \sum_{i,j=1}^{m} (\langle f_i f_j, \mu \rangle - \langle f_i, \mu \rangle \langle f_j, \mu \rangle) F_{z_i z_j}(\langle \mathbf{f}, \mu \rangle) + \sum_{i=1}^{m} (\langle A f_i, \mu \rangle + \langle \tilde{B} f_i, \mu^\infty \rangle) F_{z_i}(\langle \mathbf{f}, \mu \rangle).$$

Here $\tilde{B}$ is an operator from $\hat{C}(E)$ to $\bar{C}(E^\infty)$ with $\tilde{B}f = \sum_{l=1}^{\infty} B_l f$ and $B_l{:}\hat{C}(E) \to \hat{C}(E^l)$ a bounded operator and $\sum_{l=1}^{\infty} \|B_l\| \gamma^{l-1} < \infty$ for some $\gamma > 1$ and $\langle \tilde{B} f_i, \mu^\infty \rangle = \sum_{k=1}^{\infty} \langle B_k f_i, \mu^k \rangle$. In the formula (1) we consider $\tilde{B} f(x) = Bf(x_1, x_2) + \sigma(x_1, x_2) f(x_1) - \sigma(x_2, x_3) f(x_1)$ and in this case $\mathcal{L}$ is well defined. Let us define the space $S_1 = \{f = (f_1, f_2, \cdots) \in \sum_{k=1}^{\infty} \hat{C}(E^k) : \|f\|_\gamma \equiv \sup_{k \geq 1} \gamma^k \|f_k\| < \infty\}$. Denote $\varphi_f(\mu) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle$ for $f = f = (f_1, f_2, \cdots) \in S_1$. Let $\mathcal{C} = \{\varphi_f(\mu) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle : f_k \in \hat{C}(E^k), \|f\|_\gamma < \infty\}$, and $\mathcal{D} = \{\varphi_f(\mu) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle \in \mathcal{C} : f_k \in \mathcal{D}(A^{(k)})\}$. For $f = (f_1, f_2, \cdots) \in S_1$ and $\mu \in \mathcal{P}(E)$ define $\langle f, \mu^\infty \rangle = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle$

We will construct a semigroup $\{U(t)\}$ corresponding to $\hat{\mathcal{L}}$ on Banach space S_1 with the norm $\|\cdot\|_\gamma$.

Theorem 1. *Assume* E is a locally compact and assume above and $\mathcal{L}$ *of* (2) *defined on* $\mathcal{D}$ *is well defined, closable, and dissipative, and conservative, and generates a semigroup* $\{\mathcal{T}(t)\}$ *corresponding to a Markov process* (P_μ, μ_t) *then there exists a semigroup* $U(t)$ *on* S_1 *and constants* ρ *and* c_0 , *and it holds that*

$$\mathcal{T}(t)\varphi_f(\mu) = E_\mu[\langle f, \mu_t^\infty\rangle] = \langle U(t)f, \mu^\infty\rangle \tag{3}$$

for any $t \geq 0$ and $f \in S_1$ and

$$\|U(t)\| \leq (1-\rho)^{-1}e^{c_0 t}.$$

Proof. For $\varphi_f(\mu) = \sum_{k=1}^\infty \langle f_k, \mu^k\rangle \in \mathcal{D}$ and $\varphi_g(\mu) = \sum_{k=1}^\infty \langle g_k, \mu^k\rangle \in \mathcal{C}$, the equation $\mathcal{L}\varphi_f(\mu) = \varphi_g(\mu)$ follows from the formula

$$\hat{\mathcal{L}}f = g$$

where

$$(\hat{\mathcal{L}}f)_k \equiv \sum_{1\leq i<j\leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + (A^{(k)} - \binom{k}{2})f_k + \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}$$

for $k \geq 1$, and $B_l^{(k)} : \hat{C}(E^k) \to \hat{C}(E^{k+l-1})$ defined by

$$B_l^{(k)} f(x_1, \cdots, x_{k+l-1}) = \sum_{i=1}^{k} B_l f(x_1, \cdots, x_{i-1}, \cdot, x_i, \cdots, x_{k-1})(x_k, \cdots, x_{k+l-1})$$

for $f \in \bar{C}(E^k)$, and for $i < j$

$$\Phi_{ij}^{(k)} f_k(x_1, \cdots, x_{k-1}) = f_k(x_1, \cdots, x_{j-1}, x_i, x_j, \cdots, x_{k-1})$$

for $f_k \in \bar{C}(E^k)$.

Because $\|B_l^{(k)}\| \leq k\|B_l\|$, for any $\delta > 0$ let a positive constant be $L = L(\delta) = \frac{9\delta^2-10\delta+4}{8\delta}$ such that $k \leq L + \delta\binom{k-1}{2}$ and let $\lambda \geq 0$. Then

$$\frac{\binom{k}{2}\gamma^{k-1}}{(\lambda + \binom{k-1}{2})\gamma^k} + \sum_{l=1}^{\infty} \frac{\|B_l^{(k)}\|\gamma^{k+l-1}}{(\lambda + \binom{k+l-1}{2})\gamma^k} \leq \frac{\binom{k}{2}/\gamma + kd(\gamma)}{\lambda + \binom{k-1}{2}}$$

for any k where

$$d(\gamma) = \sum_{l=1}^{\infty} \|B_l\| \gamma^{l-1},$$

and put $\delta > 0$ so that $\rho = (1+\delta)/\gamma + \delta d(\gamma) < 1$.

For given $h \in S_1$ we consider $f(t) = (f_1(t), f_2(t), \ldots)$ with $f_k(t) \in \bar{C}(E^k)$ and $f(0) = h$ such that

$$\begin{aligned} (4) \qquad \frac{d}{dt} f_k(t) &= (\hat{\mathcal{L}} f(t))_k \\ &= \sum_{1 \le i < j \le k+1} \Phi_{ij}^{(k+1)} f_{k+1}(t) \\ &\quad + (A^{(k)} - \binom{k}{2}) f_k(t) + \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}(t) \end{aligned}$$

for $k \ge 1$ and $t > 0$. This is equivalent to

$$\begin{aligned} (5) \qquad f_k(t) &= e^{-\binom{k}{2}(t-u)} T_k(t-u) f_k(u) \\ &\quad + \int_u^t e^{-\binom{k}{2}(t-s)} T_k(t-s) \{ \sum_{1 \le i < j \le k+1} \Phi_{ij}^{(k+1)} f_{k+1}(s) \\ &\quad + \sum_{l=1}^{k} B_l^{(k-l+1)} f_{k-l+1}(s) \} ds \end{aligned}$$

for $k \ge 1$ and $t > u$, and we have that

$$\begin{aligned} \|f_k(t)\| &\le \|f_k(u)\| \\ &\quad + \int_u^t e^{-\binom{k}{2}(t-s)} (\binom{k+1}{2} \|f_{k+1}(s)\| + \sum_{l=1}^{k} \|B_l^{(k-l+1)}\| \|f_{k-l+1}(s)\|) ds. \end{aligned}$$

Let $m(t) = \sup_{k \ge 1, s \le t} \gamma^k e^{-\lambda s} \|f_k(s)\|$,
then $\|f_k(s)\| \le \gamma^{-k} e^{\lambda s} m(s)$ and $\sum_{l=1}^{k} \|B_l^{(k-l+1)}\| \gamma^{l-1} \le k d(\gamma)$, and we have

$$\begin{aligned} e^{-\lambda t} \gamma^k \|f_k(t)\| &\le e^{-\lambda t} \gamma^k \|f_k(u)\| \\ &\quad + \int_u^t e^{-\{\binom{k}{2} + \lambda\}(t-s)} (\binom{k+1}{2} / \gamma + k d(\gamma)) m(s) ds \\ &\le m(u) + \frac{(\binom{k+1}{2} / \gamma + k d(\gamma))}{\binom{k}{2} + \lambda} m(t). \end{aligned}$$

Let $\lambda \geq c_0 \equiv L(\gamma^{-1} + d(\gamma))/\rho$, then $m(t) \leq m(u) + \rho m(t)$. Therefore by $\rho < 1$, we have

$$m(t) \leq (1-\rho)^{-1} m(u).$$

Therefore

$$\gamma^k \|f_k(t)\| \leq (1-\rho)^{-1} e^{c_0 t} \sup_k \gamma^k \|f_k(0)\| \qquad \text{for} \qquad t > 0. \tag{6}$$

By this inequality $f(0) = 0$ implies $f(t) = 0$. So the equation (4) has a unique solution for $f(0) = h \in S_1$ and implies

$$\frac{d}{dt} \varphi_{f(t)}(\mu) = \mathcal{L} \varphi_{f(t)}(\mu).$$

Therefore $f(t)$ satisfies

$$\mathcal{T}(t) \varphi_h(\mu) = \langle f(t), \mu^\infty \rangle.$$

So we have

$$E_\mu[\langle h, \mu_t^\infty \rangle] = \sum_{k=1}^{\infty} \langle f_k(t), \mu^k \rangle.$$

By the inequality (6) there exists a semigroup $\{U(t)\}$ on S_1 corresponding to $\hat{\mathcal{L}}$ such that

$$\|U(t)\| \leq (1-\rho)^{-1} e^{c_0 t}.$$

Q.E.D.

Let us denote the semigroup $\{U(t)\}$ by $\{U_0(t)\}$ when $\tilde{B} = 0$. Then we have

Lemma 1. *Assume the assumption of Theorem 1 , then* $\{U_0(t)\}$ *and* $\{U(t)\}$ *on* S_1 *satisfies*

$$\|U(t) - U_0(t)\| \leq (1-\rho_0)^{-1} (1-\rho)^{-1} \beta d(\gamma) e^{c_0 t}.$$

where $\rho, \rho_0, \beta,$ *and* c_0 *are constants depends only on* $\gamma, d(\gamma)$.

Proof. For given $h \in S_1$ we consider $f^0(t) = (f_1^0(t), f_2^0(t), \ldots)$ with $f_k^0(t) \in \bar{C}(E^k)$ and $f(0) = h$ such that

$$\begin{aligned} \frac{d}{dt} f_k^0(t) &= (\hat{\mathcal{L}}_0 f^0(t))_k \\ &= \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}^0(t) + (A^{(k)} - \binom{k}{2}) f_k^0(t) \end{aligned} \tag{7}$$

for $k \geq 1$ and $t > 0$. This is equivalent to

$$(8) \quad \begin{aligned} f_k^0(t) &= e^{-\binom{k}{2}(t-u)} T_k(t-u) f_k^0(u) \\ &+ \int_u^t e^{-\binom{k}{2}(t-s)} T_k(t-s) \{ \sum_{1\leq i<j\leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}^0(s) \} ds \end{aligned}$$

for $k \geq 1$ and $t > u$, and we have that

$$\begin{aligned} \|f_k(t) - f_k^0(t)\| &\leq \|f_k(u) - f_k^0(u)\| \\ &+ \int_u^t e^{-\binom{k}{2}(t-s)} \{ \binom{k+1}{2} \|f_{k+1}(s) - f_{k+1}^0(s)\| + \\ &+ \sum_{l=1}^{k} \|B_l^{(k-l+1)}\| \|f_{k-l+1}(s)\| \} ds. \end{aligned}$$

Let $l(t) = \sup_{k\geq 1, s\leq t} \gamma^k e^{-\lambda s} \|f_k(s) - f_k^0(s)\|$, then $\|f_k(s) - f_k^0(s)\| \leq \gamma^{-k} e^{\lambda s} l(s)$ and $\sum_{l=1}^{k} \|B_l^{(k-l+1)}\| \gamma^{l-1} \leq kd(\gamma)$, and we have

$$\begin{aligned} e^{-\lambda t}\gamma^k \|f_k(t) - f_k^0(t)\| &\leq \int_u^t e^{-\{\binom{k}{2}+\lambda\}(t-s)} (\binom{k+1}{2} (1/\gamma) l(s) + kd(\gamma) m(s)) ds \\ &\leq m(u) + \frac{(\binom{k+1}{2}(1/\gamma) l(t) + kd(\gamma) m(t))}{\binom{k}{2} + \lambda}. \end{aligned}$$

Let $\lambda \geq c_0$, and put $\rho_0 = \sup \frac{\binom{k+1}{2}(1/\gamma)}{\binom{k}{2}+\lambda}$, $\beta = \sup_k \frac{k}{\binom{k}{2}+\lambda}$, then $l(t) \leq \rho_0 l(t) + \beta d(\gamma) m(t)$. Therefore by $\rho_0 < 1$, we have

$$l(t) \leq (1-\rho_0)^{-1} \beta d(\gamma) m(t).$$

Therefore

$$(9) \quad \gamma^k \|f_k(t) - f_k^0(t)\| \leq (1-\rho_0)^{-1}(1-\rho)^{-1} \beta d(\gamma) e^{c_0 t} \sup_k \gamma^k \|f_k(0)\|$$

for $t > 0$.

By the inequality (9) semigroups $\{U_0(t)\}$ and $\{U(t)\}$ on S_1 satisfies

$$\|U(t) - U_0(t)\| \leq (1-\rho_0)^{-1}(1-\rho)^{-1} \beta d(\gamma) e^{c_0 t}.$$

Q.E.D.

3 Ergodicity of semigroups

We define $\{T(t)\}$ is uniformly ergodic if there exist a stationary distribution π_0 such that $\|T(t) - \langle \cdot, \pi_0 \rangle 1\| \to 0 (t \to \infty)$.

Theorem 2. *Assume and that $\{T(t)\}$ is uniformly ergodic and that for some positive constants M and λ_0 and a stationary distribution π_0*

$$\|T(t)f - \langle f, \pi_0 \rangle 1\| \leq Me^{-\lambda_1 t} \|f\|.$$

Let $\lambda_1 = \min(\lambda_0, 1)$. Then there exists a stationary distribution Π such that for any $\epsilon > 0$ there exist constants $M_1 = M_1(\epsilon), \delta = \delta(\epsilon) > 0$ satisfying that

$$\|\mathcal{T}(t)\varphi_f(\mu) - \langle \varphi_f(\mu), \Pi \rangle 1\| \leq M_1 e^{-(\lambda_1 - \epsilon)t} \|f\|_\gamma$$

for $f \in S_1$ if $\|\sigma\| + \alpha < \delta$.

We denote $h_0 = (1, 0, 0, \cdots) \in S_1$

Theorem 3. *Under the assumption of Theorem 2 it holds that $\{U_0(t)\}$ corresponding to $\hat{\mathcal{L}}_0$ is ergodic in the sense that for a positive constant $M_2 > 0$ and $m \in S_1^*$ and $h_0 \in S_1$ such that*

$$\|U_0(t)f - \langle f, m \rangle h_0\|_\gamma \leq M_2 e^{-\lambda_1 t} \|f\|_\gamma.$$

where $m = (m_1, m_2, \cdots), \langle f, m \rangle = \sum_k \langle f_k, m_k \rangle, m_k \in \mathcal{P}(E^k)$.
Proof. Let $N(t)$ be a death process with rate $\binom{j}{2}$ from j to $j-1$ and τ_j be the hitting time of j. Put an operator $\Phi_k = \frac{1}{\binom{k}{2}} \sum_{i<j} \Phi_{ij}^{(k)}$, then by (5)

$$(U_0(t)f)_j = \sum_{k \geq j} E_k[T_j(t - \tau_j)\Phi_{j+1} \cdots T_k(\tau_{k-1})f_k; \tau_j \leq t < \tau_{j+1}].$$

Let $Y_k = \Phi_{j+1} \cdots T_k(\tau_{k-1})f_k$ on $\tau_j \leq t < \tau_{j+1}$, then

$$\begin{aligned} \|U_0(t)f - \langle f, m \rangle h_0\|_\gamma &\leq \sum_k |E[T(t - \tau_1)Y_k - \langle Y_k, \pi_0 \rangle; t > \tau_1]| \\ &+ 2(\gamma - 1)^{-1} P(\tau_1 \geq t) \|f\|_\gamma \\ &\leq \gamma(\gamma - 1)^{-1} (\|T(t - \tau_1) - \langle \cdot, \pi_0 \rangle 1\| + 2P(\tau_1 \geq t)) \|f\|_\gamma \end{aligned}$$

where $m = (m_1, m_2, \cdots)$ and m_k is defined by $\langle f, m_k \rangle = \int \langle f, \mu^k \rangle \Pi_0(d\mu)$ for $f \in C(E^k)$, $m_1 = \pi_0$, and $m_k = R_k(\binom{k}{2})^* (\sum_{i<j} \Phi_{ij}^{(k)})^* m_{k-1}$ $(k \geq 2)$. Here

$R_k(\lambda)$ is the resolvent of $T_k(t)$. By [3] $P(\tau_1 \geq t) \leq 3e^{-t}$, so the Theorem holds.

Q.E.D.

Lemma 2. *Let L be a Banach space and $h \in L$ and $m \in L^*$* with $\|h\| = a$ and $\|m\| = b$. *Assume B is a bounded operator on L with uniform norm $\|B\| < 1/(2+4ab)$ and $\langle h, m\rangle = 1$. Let $P_0 = \langle \cdot, m\rangle h$ and $U = P_0 + B$* , then we have

(a) *For $\zeta \in \Gamma \equiv \{\zeta \in \mathbf{C} : |\zeta - 1| = \frac{1}{2}\}$, $\zeta - U$ is invertible in L. Put*

$$P_1 = \frac{1}{2\pi i}\oint_\Gamma (\zeta - U)^{-1} d\zeta,$$

then $\dim P_1 L = \dim P_1^* L^* = 1$, $P_1 U = U P_1$, *and* $P_1^2 = P_1$. *$P_1 L$ is the eigenspace of U corresponding to the eigenvalue , contained in $D \equiv \{\zeta \in \mathbf{C} : |\zeta - 1| < 1/2\}$. It becomes that the eigenvalue in D is unique with multiplicity 1. Similar results hold as P_1^** and U^* .

(b) *Assume U has an eigenvalue ζ_0 with eigenvector φ_0* and $|\zeta_0 - 1| < 1/2$, *then we have that $\varphi_0 = c(\zeta_0 - B)^{-1}h$ and*

$$\langle \varphi_0, m\rangle = c$$

and

(10) $$P_1 = \langle (\zeta_0 - B)^{-2}h, m\rangle^{-1} \langle \cdot, (\bar{\zeta_0} - B^*)^{-1} m\rangle (\zeta_0 - B)^{-1} h,$$

$$U P_1 = P_1 U = P_1,$$

(c) *Under the assumption of (b), the next relation holds.*

$$\|U - \zeta_0 P_1\| \leq 8\|B\|$$

if $\|B\| < 1/(4+8ab)$.

Lemma 3. *Under the assumption of theorem 1 for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $d(\gamma) < \delta$, then there exists $h_1 \in S_1$ and $m_1 \in S_1^*$ and $M_1 > 0$ such that*

$$\|U(t)f - \langle f, m_1\rangle h_1\|_\gamma \leq M_1 e^{-(\lambda_1 - \epsilon)t} \|f\|_\gamma,$$

and $\langle h_0, m_1\rangle \langle h_1, \mu^\infty\rangle = 1$.

Proof. By Theorem 3 we have that for any $0 < \epsilon < \lambda_1$ there exist h_0, m, and t_0 such that

$$\|U(t_0)f - \langle f, m\rangle h_0\|_\gamma \leq \frac{1}{16} e^{-(\lambda_1-\epsilon)t_0}\|f\|_\gamma.$$

By Lemma 1 we have that there exists $\delta > 0$ such that for $d(\gamma) < \delta$

$$\|U(t_0)f - U_0(t_0)f\|_\gamma \leq \frac{1}{16} e^{-(\lambda_1-\epsilon)t_0}\|f\|_\gamma.$$

According to Lemma 3 we have that there exist m_1 , h_1, and ζ_0 such that

$$\|U(t_0)f - \zeta_0\langle f, m_1\rangle h_1\|_\gamma \leq e^{-(\lambda_1-\epsilon)t_0}\|f\|_\gamma.$$

So we have for any $n > 0$

$$\|U(nt_0)f - \zeta_0^n\langle f, m_1\rangle h_1\|_\gamma \leq e^{-(\lambda_1-\epsilon)nt_0}\|f\|_\gamma.$$

By Theorem 1 there exists $M' > 0$ such that $\|U(s)\| \leq M'$ for $0 \leq s \leq t_0$. We have that

$$\|U(nt_0+s)f - \zeta_0^n\langle U(s)f, m_1\rangle h_1\|_\gamma \leq M'e^{-(\lambda_1-\epsilon)t}\|f\|_\gamma$$

and

$$\|U(nt_0+s)f - \zeta_0^n\langle f, m_1\rangle U(s)h_1\|_\gamma \leq M'e^{-(\lambda_1-\epsilon)t}\|f\|_\gamma$$

for $0 \leq s \leq t_0$. Then $|\zeta_0| \leq 1$ and if $|\zeta_0| = 1$, then

$$\langle U(s)f, m_1\rangle h_1 = \langle f, m_1\rangle U(s)h_1 = c(s)\langle f, m_1\rangle h_1$$

with some constant $c(s)$. Because $\mathcal{T}(t)1 = 1$, by the above equations and (3) we have

$$1 = (\mathcal{T}(nt_0+s)1)(\mu) = \langle U(nt_0+s)h_0, \mu^\infty\rangle = c(s)\langle h_0, m_1\rangle\langle h_1, \mu^\infty\rangle \lim_{n\to\infty} \zeta_0^n.$$

Therefore $\zeta_0 = 1$. Because $U(0) = I$, $c(s) = c(0) = 1$ holds. Therefore let $M_1 = M'e^{(\lambda_1-\epsilon)t_0}$, then the inequality of the Theorem holds.

Q.E.D.

Proof of Theorem 2. Because $\mathcal{T}(t)1 = 1$, by Lemma 3

$$1 = (\mathcal{T}(t)1)(\mu) = \langle U(t)h_0, \mu^\infty\rangle = \langle h_0, m_1\rangle\langle h_1, \mu^\infty\rangle.$$

Let $m_2 = (m_2^{(1)}, m_2^{(2)}, \cdots) = \frac{1}{\langle h_0, m_1 \rangle} m_1$ and $h_2 = \langle h_0, m_1 \rangle h_1$, then $m_2^{(k)} \in \mathcal{P}(E^k)$ and $\langle h_2, \mu^\infty \rangle = 1$. Because $\varphi_f(\mu) = \langle f, \mu^\infty \rangle$, Lemma 3 implies that

$$\begin{aligned} |\mathcal{T}(t)\varphi_f(\mu) - \langle f, m_2 \rangle \langle h_2, \mu^\infty \rangle| &= |\langle U(t)f - \langle f, m_2 \rangle h_2, \mu^\infty \rangle| \\ &\leq M_1 \gamma (\gamma - 1)^{-1} e^{-(\lambda_1 - \epsilon)t} \|f\|_\gamma \end{aligned}$$

so Theorem 2 holds.

Q.E.D.

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