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Ergodic properties of Fleming-Viot processes with selection and recombination

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1 Introduction

Let $E$ be a locally compact separable metric space and $\mathcal{P}(E)$ be the space of all probability measures on $E$. For $\mu \in \mathcal{P}(E)$ let us denote $(f, \mu) = \int_E f d\mu$. For any $f_1, \cdots, f_m \in \mathcal{D}(A)$ and $F \in C^2(\mathbb{R}^m)$ let $\varphi(\mu) = F((f_1, \mu), \cdots, (f_m, \mu)) = F((f, \mu))$.

\[
\mathcal{L} \varphi(\mu) = \frac{1}{2} \sum_{i,j=1}^{m} \langle f_i f_j, \mu \rangle - \langle f_i, \mu \rangle \langle f_j, \mu \rangle F_{zz_i z_j}((f, \mu)) \\
+ \sum_{i=1}^{m} (\langle A f_i, \mu \rangle + \langle F f_i, \mu^2 \rangle) F_{z_i}((f, \mu)) \\
+ \sum_{i=1}^{m} \{ \langle (f_i \otimes 1) \sigma, \mu^2 \rangle - \langle f_i, \mu \rangle \langle \sigma, \mu^2 \rangle \} F_{z_i}((f, \mu)).
\]

Here $E$ is the space of genetic types and $A$ is a mutation operator in $\hat{C}(E)$ (≡ the space of bounded continuous functions on $E$) which is the generator for a Feller semigroup $\{T(t)\}$ on $\hat{C}(E)$ (≡ the space of continuous functions vanishing at infinity). Here $\sigma = \sigma(x, y)$ is a bounded symmetric function on $E \times E$ which is selection parameters for types $x, y \in E$ $B$ is a recombination operator defined by

\[
Bf(x, y) = \alpha \int_E (f(x') - f(x)) R((x, y), dx')
\]

where $\alpha \geq 0$ and $R((x, y), dx')$ is a one step transition function on $E^2 \times \mathcal{B}(E)$, and we denote $\mu^n$ the $n$-fold product of $\mu$. According to [3], this operator defines a generator corresponding to a Markov process on $\mathcal{P}(E)$ in the sense that the $C_{\mathcal{P}(E)}[0, \infty)$ martingale problem for $\mathcal{L}$ is well posed. This process
is called the Fleming-Viot process. The aim of this paper is to consider ergodicity for this process by using the duality in the form

$$E_{\mu}[\langle f, \mu^n \rangle] = \sum_{k=1}^{\infty} \langle f_k(t), \mu^k \rangle$$

for any $t \geq 0$, $n \in \mathbb{N}$ and $f \in \overline{C}(E^n)$ with sup-norm $\| \cdot \|$. Here $f_k(t) \in \overline{C}(E^k)$ and satisfy $\sum_{k=1}^{\infty} \gamma^k \| f_k(t) \| < \infty$ for some $\gamma > 1$ and $f_n(0) = f$ and $f_k(0) = 0$ for $k \neq n$, and we consider a semigroup for this process.

## 2 Construction of a semigroup

We consider that $E$ is a locally compact separable metric space, and treat the case of the formula (1) and assume $\{ T(t) \}$ is a Feller semigroup on $\hat{C}(E)$ with the generator $A$. Denote the semigroup $T_k(t) = T(t) \otimes \cdots \otimes T(t)$ on $\overline{C}(E^k)$ and its generator $A^{(k)}$.

We now consider duality under general condition for the diffusion. In this section we consider the operator of the form

$$(2) \quad \mathcal{L} \varphi(\mu) = \frac{1}{2} \sum_{i,j=1}^{m} (\langle f_i f_j, \mu \rangle - \langle f_i, \mu \rangle \langle f_j, \mu \rangle) F(z_i z_j \langle \varphi, \mu \rangle) + \sum_{i=1}^{m} (\langle A f_i, \mu \rangle + \langle \tilde{B} f_i, \mu^\infty \rangle) F(z_i \langle \varphi, \mu \rangle).$$

Here $\tilde{B}$ is an operator from $\hat{C}(E)$ to $\overline{C}(E^\infty)$ with $\tilde{B} f = \sum_{l=1}^{\infty} B_l f$ and $B_l: \hat{C}(E) \rightarrow \hat{C}(E^l)$ a bounded operator and $\sum_{l=1}^{\infty} \| B_l \|\gamma^{l-1} < \infty$ for some $\gamma > 1$ and $\langle \tilde{B} f_i, \mu^\infty \rangle = \sum_{k=1}^{\infty} \langle B_k f_i, \mu^k \rangle$. In the formula (1) we consider $\tilde{B} f(x) = B f(x_1, x_2) + \sigma(x_1, x_2) f(x_1) - \sigma(x_2, x_3) f(x_1)$ and in this case $\mathcal{L}$ is well defined. Let us define the space $S_1 = \{ f = (f_1, f_2, \cdots) \in \sum_{k=1}^{\infty} \overline{C}(E_k) : \| f \|_{\gamma} \equiv \sup_{k \geq 1} \gamma^k \| f_k \| < \infty \}$. Denote $\varphi_f(\mu) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle$ for $f = f = (f_1, f_2, \cdots) \in S_1$. Let $\mathcal{C} = \{ \varphi_f(\mu) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle : f_k \in \overline{C}(E^k), \| f \|_{\gamma} < \infty \}$, and $\mathcal{D} = \{ \varphi_f(\mu) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle \in \mathcal{C} : f_k \in \mathcal{D}(A^{(k)}) \}$. For $f = (f_1, f_2, \cdots) \in S_1$ and $\mu \in \mathcal{P}(E)$ define $(f, \mu^\infty) = \sum_{k=1}^{\infty} \langle f_k, \mu^k \rangle$

We will construct a semigroup $\{ U(t) \}$ corresponding to $\mathcal{L}$ on Banach space $S_1$ with the norm $\| \cdot \|_\gamma$. 
Theorem 1. Assume $E$ is a locally compact and assume above and $\mathcal{L}$ of (2) defined on $\mathcal{D}$ is well defined, closable, and dissipative, and conservative, and generates a semigroup \( \{T(t)\} \) corresponding to a Markov process \((P_{\mu}, \mu_{t})\) then there exists a semigroup $U(t)$ on $S_1$ and constants $\rho$ and $c_0$, and it holds that

\[
T(t)\varphi_f(\mu) = E_{\mu}[(f, \mu_t\infty)] = (U(t)f, \mu^{\infty})
\]

for any $t \geq 0$ and $f \in S_1$ and

\[
\|U(t)\| \leq (1 - \rho)^{-1}e^{c_0t}.
\]

Proof. For $\varphi_f(\mu) = \sum_{k=1}^{\infty} (f_k, \mu^k) \in \mathcal{D}$ and $\varphi_g(\mu) = \sum_{k=1}^{\infty} (g_k, \mu^k) \in \mathcal{C}$, the equation $\mathcal{L}\varphi_f(\mu) = \varphi_g(\mu)$ follows from the formula

\[
\hat{\mathcal{L}}f = g
\]

where

\[
(\hat{\mathcal{L}}f)_k \equiv \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1} + (A^{(k)} - \binom{k}{2}) f_k + \sum_{l=1}^{k} B^{(k-l+1)}_l f_{k-l+1}
\]

for $k \geq 1$, and $B^{(k)}_l : \hat{\mathcal{C}}(E^k) \to \hat{\mathcal{C}}(E^{k+l-1})$ defined by

\[
B^{(k)}_l f(x_1, \cdots, x_{k+l-1}) = \sum_{i=1}^{k} B_l f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{k-1})(x_k, \cdots, x_{k+l-1})
\]

for $f \in \hat{\mathcal{C}}(E^k)$, and for $i < j$

\[
\Phi_{ij}^{(k)} f_k(x_1, \cdots, x_{k-1}) = f_k(x_1, \cdots, x_{j-1}, x_i, x_j, \cdots, x_{k-1})
\]

for $f_k \in \hat{\mathcal{C}}(E^k)$.

Because $\|B^{(k)}_l\| \leq k \|B_l\|$, for any $\delta > 0$ let a positive constant be $L = L(\delta) = \frac{9\delta^2 - 10\delta + 4}{8\delta}$ such that $k \leq L + \delta \binom{k-1}{2}$ and let $\lambda \geq 0$. Then

\[
\frac{\binom{k}{2}}{(\lambda + \binom{k-1}{2})} \gamma^{k-1} + \sum_{l=1}^{\infty} \frac{\|B^{(k)}_l\| \gamma^{k+l-1}}{(\lambda + \binom{k+l-1}{2})} \gamma^k \leq \frac{\binom{k}{2}}{\lambda + \binom{k-1}{2}}
\]

for any $k$ where
\[ d(\gamma) = \sum_{l=1}^{\infty} \|B_l\| \gamma^{l-1}, \]

and put \( \delta > 0 \) so that \( \rho = (1 + \delta)/\gamma + \delta d(\gamma) < 1. \)

For given \( h \in S_1 \) we consider \( f(t) = (f_1(t), f_2(t), \ldots) \) with \( f_k(t) \in \bar{C}(E^k) \) and \( f(0) = h \) such that

\[
\frac{d}{dt} f_k(t) = (\mathcal{L} f(t))_k
= \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(t)
+ \left( A^{(k)} - \frac{k}{2} \right) f_k(t) + \sum_{l=1}^{k} B_{l}^{(k-l+1)} f_{k-l+1}(t)
\]

for \( k \geq 1 \) and \( t > 0. \) This is equivalent to

\[
f_k(t) = e^{-(t-u)} T_k(t-u) f_k(u)
+ \int_{u}^{t} e^{-\frac{k}{2}(t-s)} T_k(t-s) \left\{ \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(s) \right\} ds
+ \sum_{l=1}^{k} B_{l}^{(k-l+1)} f_{k-l+1}(s) \right\} ds
\]

for \( k \geq 1 \) and \( t > u, \) and we have that

\[
\|f_k(t)\| \leq \|f_k(u)\|
+ \int_{u}^{t} e^{-\frac{k}{2}(t-s)} \left( \frac{k+1}{2} \right) \|f_{k+1}(s)\| + \sum_{l=1}^{k} \|B_{l}^{(k-l+1)}\| \|f_{k-l+1}(s)\| ds.
\]

Let \( m(t) = \sup_{k \geq 1, s \leq t} \gamma^{-k} \|f_k(s)\|, \)
then \( \|f_k(s)\| \leq \gamma^{-k} e^{\lambda s} m(s) \) and \( \sum_{l=1}^{k} \|B_{l}^{(k-l+1)}\| \|f_{l-1}^{(k-l+1)}\| \leq kd(\gamma), \) and we have

\[
e^{-\lambda t} \gamma^k \|f_k(t)\| \leq e^{-\lambda t} \gamma^k \|f_k(u)\|
+ \int_{u}^{t} e^{-\frac{k}{2}(t-s) + \lambda s} \left( \frac{k+1}{2} \right) / \gamma + kd(\gamma) m(s) ds
\leq m(u) + \frac{\left( \frac{k+1}{2} / \gamma + kd(\gamma) \right)}{\left( \frac{k}{2} + \lambda \right)} m(t).
\]
Let $\lambda \geq c_0 \equiv L(\gamma^{-1} + d(\gamma))/\rho$, then $m(t) \leq m(u) + \rho m(t)$. Therefore by $\rho < 1$, we have 

$$m(t) \leq (1 - \rho)^{-1}m(u).$$

Therefore

$$\gamma^k \|f_k(t)\| \leq (1 - \rho)^{-1}e^{\gamma t}\sup_k \|f_k(0)\| \quad \text{for} \quad t > 0. \tag{6}$$

By this inequality $f(0) = 0$ implies $f(t) = 0$. So the equation (4) has a unique solution for $f(0) = h \in S_1$ and implies

$$\frac{d}{dt} \varphi_{f(t)}(\mu) = \mathcal{L} \varphi_{f(t)}(\mu).$$

Therefore $f(t)$ satisfies

$$\mathcal{T}(t) \varphi_{h}(\mu) = \langle f(t), \mu^\infty \rangle.$$ 

So we have

$$E_{\mu}[\langle h, \mu_t^\infty \rangle] = \sum_{k=1}^{\infty} \langle f_k(t), \mu^k \rangle.$$ 

By the inequality (6) there exists a semigroup $\{U(t)\}$ on $S_1$ corresponding to $\hat{\mathcal{L}}$ such that

$$\|U(t)\| \leq (1 - \rho)^{-1}e^{ct}.$$ 

Q.E.D.

Let us denote the semigroup $\{U(t)\}$ by $\{U_0(t)\}$ when $\tilde{B} = 0$. Then we have

**Lemma 1.** Assume the assumption of Theorem 1, then $\{U_0(t)\}$ and $\{U(t)\}$ on $S_1$ satisfies

$$\|U(t) - U_0(t)\| \leq (1 - \rho_0)^{-1}(1 - \rho)^{-1}\beta d(\gamma)e^{ct}.$$ 

where $\rho, \rho_0, \beta$, and $c_0$ are constants depends only on $\gamma, d(\gamma)$.

**Proof.** For given $h \in S_1$ we consider $f^0(t) = (f_1^0(t), f_2^0(t), \ldots)$ with $f_k^0(t) \in \overline{C}(E^k)$ and $f(0) = h$ such that

$$\frac{d}{dt} f_k^0(t) = (\hat{\mathcal{L}} f_k^0(t))_k = \sum_{1 \leq i < j \leq k+1} \Phi^{(k+1)}_{ij} f_{k+1}^0(t) + (A^{(k)} - \binom{k}{2}) f_k^0(t).$$ 

where $\Phi^{(k+1)}_{ij}$ are the elements of the $(k+1)$th power of the matrix $A^{(k)}$. 

By the definition of $f^0(t)$, we have

$$f^0(t) = (f_1^0(t), f_2^0(t), \ldots, f_k^0(t), \ldots)$$

and

$$\frac{d}{dt} f_k^0(t) = (\hat{\mathcal{L}} f_k^0(t))_k.$$ 

This completes the proof.
for $k \geq 1$ and $t > 0$. This is equivalent to

\begin{align*}
(8) \quad f_k^0(t) &= e^{-\binom{k}{2}(t-u)}T_k(t-u)f_k^0(u) \\
&+ \int_u^t e^{-\binom{k}{2}(t-s)}T_k(t-s)\left\{ \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} f_{k+1}(s) \right\} ds
\end{align*}

for $k \geq 1$ and $t > u$, and we have that

\[
\|f_k(t) - f_k^0(t)\| \leq \|f_k(u) - f_k^0(u)\| + \int_u^t e^{-(\binom{k}{2})(t-s)}\left\{ \sum_{1 \leq i < j \leq k+1} \Phi_{ij}^{(k+1)} \right\} ds.
\]

Let $l(t) = \sup_{k \geq 1, s \leq t} \gamma^k e^{-\lambda s} \|f_k(s) - f_k^0(s)\|$, then $\|f_k(s) - f_k^0(s)\| \leq \gamma^{-k} e^{\lambda s} l(s)$ and $\sum_{l=1}^{k} \|B(k-l+1)\|^l \gamma^{k-l} \leq k d(\gamma)$, and we have

\[
e^{-\lambda t} \gamma^k \|f_k(t) - f_k^0(t)\| \leq \int_u^t e^{-\binom{k}{2}(t-s)}\left( \frac{k+1}{2} \right) (1/\gamma) l(t) + kd(\gamma) m(s) ds
\]

\[
\leq m(u) + \frac{\left( \frac{k+1}{2} \right) (1/\gamma) l(t) + kd(\gamma) m(t)}{\frac{k}{2} + \lambda}.
\]

Let $\lambda \geq c_0$, and put $\rho_0 = \sup \frac{\left( \frac{k+1}{2} \right) (1/\gamma)}{\frac{k}{2} + \lambda}$, then $l(t) \leq \rho_0 l(t) + \beta d(\gamma) m(t)$. Therefore by $\rho_0 < 1$, we have

\[
l(t) \leq (1 - \rho_0)^{-1} \beta d(\gamma) m(t).
\]

Therefore

\[
(9) \quad \gamma^k \|f_k(t) - f_k^0(t)\| \leq (1 - \rho_0)^{-1} (1 - \rho)^{-1} \beta d(\gamma) e^{\rho_0 t} sup \gamma^k \|f_k(0)\|
\]

for $t > 0$.

By the inequality (9) semigroups $\{U_0(t)\}$ and $\{U(t)\}$ on $S_1$ satisfies

\[
\|U(t) - U_0(t)\| \leq (1 - \rho_0)^{-1} (1 - \rho)^{-1} \beta d(\gamma) e^{\rho_0 t}.
\]

Q.E.D.
3 Ergodicity of semigroups

We define \( \{T(t)\} \) is uniformly ergodic if there exist a stationary distribution \( \pi_0 \) such that \( \|T(t) - \langle \cdot, \pi_0 \rangle 1 \| \to 0 (t \to \infty) \).

**Theorem 2.** Assume and that \( \{T(t)\} \) is uniformly ergodic and that for some positive constants \( M \) and \( \lambda_0 \) and a stationary distribution \( \pi_0 \)

\[
\|T(t)f - \langle f, \pi_0 \rangle 1\| \leq Me^{-\lambda_1 t}\|f\|.
\]

Let \( \lambda_1 = \min(\lambda_0, 1) \). Then there exists a stationary distribution \( \Pi \) such that for any \( \epsilon > 0 \) there exist constants \( M_1 = M_1(\epsilon) \), \( \delta = \delta(\epsilon) > 0 \) satisfying that

\[
\|T(t)\varphi_f(\mu) - \langle \varphi_f(\mu), \Pi \rangle 1\| \leq M_1e^{-\lambda_1 - \epsilon}t\|f\|_\gamma
\]

for \( f \in S_1 \) if \( \|\sigma\| + \alpha < \delta \).

We denote \( h_0 = (1, 0, 0, \cdots) \in S_1 \)

**Theorem 3.** Under the assumption of Theorem 2 it holds that \( \{U_0(t)\} \) corresponding to \( \hat{L}_0 \) is ergodic in the sense that for a positive constant \( M_2 > 0 \) and \( m \in S_1^* \) and \( h_0 \in S_1 \) such that

\[
\|U_0(t)f - \langle f, m \rangle h_0\|_\gamma \leq M_2e^{-\lambda_1 t}\|f\|_\gamma.
\]

where \( m = (m_1, m_2, \cdots), \langle f, m \rangle = \sum_k \langle f_k, m_k \rangle, m_k \in \mathcal{P}(E^k) \).

**Proof.** Let \( N(t) \) be a death process with rate \( \binom{t}{2} \) from \( j \) to \( j - 1 \) and \( \tau_j \) be the hitting time of \( j \). Put an operator \( \Phi_k = \frac{1}{(j^2)} \sum_{i<j} \Phi^{(k)}_{ij} \), then by (5)

\[
(U_0(t)f)_j = \sum_{k \geq j} E_k[T_j(t - \tau_j)\Phi_{j+1}\cdots T_k(\tau_{k-1})f_k; \tau_j \leq t < \tau_{j+1}].
\]

Let \( Y_k = \Phi_{j+1}\cdots T_k(\tau_{k-1})f_k \) on \( \tau_j \leq t < \tau_{j+1} \), then

\[
\|U_0(t)f - \langle f, m \rangle h_0\|_\gamma \leq \sum_k |E[T(t - \tau_1)Y_k - \langle Y_k, \pi_0 \rangle; t > \tau_1]|
\]

\[
+ 2(\gamma - 1)^{-1}P(\tau_1 \geq t)\|f\|_\gamma
\]

\[
\leq \gamma(\gamma - 1)^{-1}(\|T(t - \tau_1) - \langle \cdot, \pi_0 \rangle 1\| + 2P(\tau_1 \geq t))\|f\|_\gamma
\]

where \( m = (m_1, m_2, \cdots) \) and \( m_k \) is defined by \( \langle f, m_k \rangle = \int \langle f, \mu^k \rangle \Pi_0(d\mu) \) for \( f \in C(E^k), m_1 = \pi_0, \) and \( m_k = R_k(\binom{k}{2})^* (\sum_{i<j} \Phi^{(k)}_{ij})^* m_{k-1} \) \( (k \geq 2) \). Here
$R_k(\lambda)$ is the resolvent of $T_k(t)$. By [3] $P(\tau_1 \geq t) \leq 3e^{-t}$, so the Theorem holds.

Q.E.D.

**Lemma 2.** Let $L$ be a Banach space and $h \in L$ and $m \in L^*$ with $\|h\| = a$ and $\|m\| = b$. Assume $B$ is a bounded operator on $L$ with uniform norm $\|B\| < 1/(2 + 4ab)$ and $\langle h, m \rangle = 1$. Let $P_0 = \langle \cdot, m \rangle h$ and $U = P_0 + B$, then we have

(a) For $\zeta \in \Gamma \equiv \{ \zeta \in \mathbb{C} : |\zeta - 1| = \frac{1}{2} \}$, $\zeta - U$ is invertible in $L$. Put

$$P_1 = \frac{1}{2\pi i} \oint_{\Gamma} (\zeta - U)^{-1} d\zeta,$$

then $\dim P_1 L = \dim P_1^* L^* = 1$, $P_1 U = U P_1$, and $P_1^2 = P_1$. $P_1 L$ is the eigenspace of $U$ corresponding to the eigenvalue, contained in $D \equiv \{ \zeta \in \mathbb{C} : |\zeta - 1| < 1/2 \}$. It becomes that the eigenvalue in $D$ is unique with multiplicity 1. Similar results hold as $P_1^*$ and $U^*$.

(b) Assume $U$ has an eigenvalue $\zeta_0$ with eigenvector $\varphi_0$ and $|\zeta_0 - 1| < 1/2$, then we have that $\varphi_0 = c(\zeta_0 - B)^{-1}h$ and

$$\langle \varphi_0, m \rangle = c$$

and

(10) \hspace{1cm} \hspace{1cm} P_1 = \langle (\zeta_0 - B)^{-2} h, m \rangle^{-1} \langle \cdot, (\bar{\zeta_0} - B^*)^{-1} m \rangle (\zeta_0 - B)^{-1} h,

$$U P_1 = P_1 U = P_1,$$

(c) Under the assumption of (b), the next relation holds.

$$\|U - \zeta_0 P_1\| \leq 8\|B\|$$

if $\|B\| < 1/(4 + 8ab)$.

**Lemma 3.** Under the assumption of theorem 1 for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $d(\gamma) < \delta$, then there exists $h_1 \in S_1$ and $m_1 \in S_1^*$ and $M_1 > 0$ such that

$$\|U(t)f - \langle f, m_1 \rangle h_1\|_{\gamma} \leq M_1 e^{-\langle \lambda_1 - \epsilon \rangle t} \|f\|_{\gamma},$$

and $\langle h_0, m_1 \rangle \langle h_1, \mu^\infty \rangle = 1$. 

Proof. By Theorem 3 we have that for any $0 < \epsilon < \lambda_1$ there exist $h_0, m$, and $t_0$ such that

$$\|U(t_0)f - \langle f, m \rangle h_0\|_{\gamma} \leq \frac{1}{16} e^{-(\lambda_1 - \epsilon)t_0} \|f\|_{\gamma}.$$ 

By Lemma 1 we have that there exists $\delta > 0$ such that for $d(\gamma) < \delta$

$$\|U(t_0)f - U_0(t_0)f\|_{\gamma} \leq \frac{1}{16} e^{-(\lambda_1 - \epsilon)t_0} \|f\|_{\gamma}.$$ 

According to Lemma 3 we have that there exist $m_1$, $h_1$, and $\zeta_0$ such that

$$\|U(t_0)f - \zeta_0 \langle f, m_1 \rangle h_1\|_{\gamma} \leq e^{-(\lambda_1 - \epsilon)t_0} \|f\|_{\gamma}.$$ 

So we have for any $n > 0$

$$\|U(nt_0)f - \zeta_0^n \langle f, m_1 \rangle h_1\|_{\gamma} \leq e^{-(\lambda_1 - \epsilon)n t_0} \|f\|_{\gamma}.$$ 

By Theorem 1 there exists $M' > 0$ such that $\|U(s)\| \leq M'$ for $0 \leq s \leq t_0$. We have that

$$\|U(nt_0 + s)f - \zeta_0^n \langle U(s)f, m_1 \rangle h_1\|_{\gamma} \leq M' e^{-(\lambda_1 - \epsilon)t} \|f\|_{\gamma}$$

and

$$\|U(nt_0 + s)f - \zeta_0^n \langle f, m_1 \rangle U(s)h_1\|_{\gamma} \leq M' e^{-(\lambda_1 - \epsilon)t} \|f\|_{\gamma}$$

for $0 \leq s \leq t_0$. Then $|\zeta_0| \leq 1$ and if $|\zeta_0| = 1$, then

$$\langle U(s)f, m_1 \rangle h_1 = \langle f, m_1 \rangle U(s)h_1 = c(s) \langle f, m_1 \rangle h_1$$

with some constant $c(s)$. Because $T(t)1 = 1$, by the above equations and (3) we have

$$1 = (T(nt_0 + s)1)(\mu) = \langle U(nt_0 + s)h_0, \mu^\infty \rangle = c(s) \langle h_0, m_1 \rangle \langle h_1, \mu^\infty \rangle \lim_{n \to \infty} \zeta_0^n.$$ 

Therefore $\zeta_0 = 1$. Because $U(0) = I$, $c(s) = c(0) = 1$ holds. Therefore let $M_1 = M'e^{(\lambda_1 - \epsilon)t_0}$, then the inequality of the Theorem holds.

Q.E.D.

Proof of Theorem 2. Because $T(t)1 = 1$, by Lemma 3

$$1 = (T(t)1)(\mu) = \langle U(t)h_0, \mu^\infty \rangle = \langle h_0, m_1 \rangle \langle h_1, \mu^\infty \rangle.$$
Let $m_2 = (m_2^{(1)}, m_2^{(2)}, \cdots) = \frac{1}{\langle h_0, m_1 \rangle} m_1$ and $h_2 = \langle h_0, m_1 \rangle h_1$, then $m_2^{(k)} \in \mathcal{P}(E^k)$ and $\langle h_2, \mu^{\infty} \rangle = 1$. Because $\varphi_f(\mu) = \langle f, \mu^{\infty} \rangle$, Lemma 3 implies that

$$\left| T(t) \varphi_f(\mu) - \langle f, m_2 \rangle \langle h_2, \mu^{\infty} \rangle \right| = \left| \langle U(t)f - \langle f, m_2 \rangle h_2, \mu^{\infty} \rangle \right| \leq M_1 \gamma (\gamma - 1)^{-1} e^{-(\lambda_1 - \epsilon)t} \| f \|_\gamma$$

so Theorem 2 holds.

Q.E.D.

References


