OPTIMAL STOPPING GAMES WHERE PLAYERS HAVE WEIGHTED PRIVILEGE

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Abstract

A non-zero-sum n-stage game version of a full-information best-choice problem under ENV maximization is analyzed and solutions are obtained in some special cases of 2-person and 3-person games. The essential feature contained in this multistage game is the fact that the players have their own weights by which at each stage one player's desired decision is preferred to the opponent's one by drawing a lottery.

1. A Two Person Optimal Stopping Game

A non-zero-sum game version of the discrete-time, full-information best-choice problem under ENV-maximization is considered in this section. We first state the problem as follows:

(1°) There are two players I and II and a sequence of n iid r.v.s. \( \{X_i\}_{i=1}^n \) with a common cdf \( F(x) \), \( 0 \leq x < \infty \). Both players observe \( X_i \)'s sequentially one by one.

(2°) Observing each \( X_i \), both players select, simultaneously and independently, either to accept (A) or to reject (R) the \( X_i \). If I - II choice is A - A, then player I (II) accepts to receive \( X_i \) with probability \( \omega^I(\omega^I|\omega^I) \), \( \frac{1}{2} < \omega^I < 1 \), and drops out from the play thereafter. The player remained continues his \( c_{n\in\mathbb{N}} \)-person game. If I-II choice is A-R (R-A), then I (II) accepts \( X_i \) and drops out and his opponent continues the remaining one-person game. If I-II choice is R - R, then \( X_i \) is rejected and then the players face the next \( X_{i+1} \).

(3°) The aim of each player in the game is to determine his acceptance strategy under which he maximizes the expected net value (ENV) he obtains.

Define state \( (x|n) \) to mean that (1) both players remain in the game, and (2) there remains \( n \) r.v.s to be observed and the players currently face the first observation
Let $V_n$ be the value of the game for player $i$ for the $n$-problem. Then we have

\begin{equation}
(1.1)
\begin{align*}
\nabla_n^i &= \mathbb{E}_{\mathcal{F}}\left[ (\psi^i(x, x) + \omega^i \mathbb{1}_{n-1}) \varphi^i(x, n) + X \varphi^i(x, n) + \nabla_{n-1}^i \bar{\varphi}^i(x, n) \right] \\
&+ \mathbb{1}_{n-1} \mathbb{1}_{n-1} \varphi^i(x, n) + \nabla_{n-1}^i \bar{\varphi}^i(x, n) + \nabla_{n-1}^i \bar{\varphi}^i(x, n) \\
& \quad \left( j = 1, 2, \quad j = 1, 2, \quad n = 1, 2, \ldots, \quad \nabla_0^i = \nabla_0^i \equiv 0 \right)
\end{align*}
\end{equation}

Here $\mathbb{1}_{n-1}$ is the value of the game for the remaining player when his opponent has already dropped out with $n-1$ unobserved r.v.s thereafter. The optimal strategy for the player in this state is evidently to accept (reject) if $x > \psi(x, n)$, since the sequence $\{U_n\}$ satisfies the recursion

\begin{equation}
(1.2) \quad U_n = \mathbb{E}_\mathcal{F}(X \mathbb{1}_{n-1})(n = 1, 2, \ldots; \ U_0 \equiv 0)
\end{equation}

Our problem is to find the Nash equilibrium

\begin{equation}
(1.3) \quad (\nabla_n^1, \nabla_n^2) \rightarrow \text{Nash eq. in } (\psi^1(x, n), \psi^2(x, n))
\end{equation}

for each $n = 1, 2, \ldots$.

We hereafter write $\langle w, w^2 \rangle$ as $\langle w, w \rangle$. We consider the problem (1.1)–(1.3) as the optimal stopping game described by the Optimality Equation

\begin{equation}
(1.4) \quad (\nabla_n^1, \nabla_n^2) = \mathbb{E}_\mathcal{F}\left[ \text{eq. val } M_n(X) \right]
\end{equation}

\begin{equation}
(1.5) \quad M_n(x) = \begin{array}{c|c|c}
\hline
& R & A \hline
\nabla_{n-1}^1 & U_{n-1} & x \\
\nabla_{n-1}^2 & w_x + w U_{n-1} & \bar{w}_x + w U_{n-1} \\
\hline
\end{array}
\end{equation}

where $M_n(x)$ is the bimatrix game which the players face in state $(x | n)$, and we assume that $M_n(x)$ has a unique equilibrium for every $x$ satisfying $0 < \gamma(x) < 1$.

The problems we consider in this paper belong to a class of best-choice problems
combined with sequential games. Recent works related to this area of problems are Enns and Ferstein [1, 2], Mazalov [4] and Sakaguchi [5, 7, 8]. A very important and now classical literature in full-information best-choice problems is Gilbert and Mosteller [3]. Also a recent look for the optimal stopping games in various phases can be found in Sakaguchi [6].

2. Two Special Cases in Two-Person Games.

2.1. The case where \( F(x) = x(0 \leq x \leq 1) \) and \( w = \frac{1}{2} \).

We prove

Theorem 1. The solution to OSG described by Optimality Equation (1.4)-(1.5) for \( F(x) = x(0 \leq x \leq 1) \) and \( w = \frac{1}{2} \), is as follows. The common equilibrium strategy for both player is, in state \((x|n)\),

\[
\begin{align*}
&\text{Choose } R, & \text{if } 0 \leq x < V_{n-1}, \\
&\text{Randomize } R \text{ and } A \text{ with probability } \frac{2}{\frac{2}{n-1}} & \text{for } A, & \text{if } V_{n-1} < x < U_{n-1}, \\
&\text{Choose } A. & \text{if } U_{n-1} < x \leq 1.
\end{align*}
\]

where \( z_{n-1} = (1 - V_{n-1})/(U_{n-1} - V_{n-1}) \) and the equilibrium values are \( (V_n, U_n) \). The sequences \( \{U_n\} \) and \( \{V_n\} \) are given by the recursions 
\[
U_n = \frac{1}{2}(U_{n-1} + z_{n-1}), (n \geq 1), U_0 = 0 \text{ and } V_n = 2(1-z_{n-1} U_n + U_n)^2 + U_n V_n + \frac{1}{4}(1-U_n)(1+3U_n)
\]

\( \forall n = 1, 2, \ldots \). \( \forall n \rightarrow 1 \) as \( n \rightarrow \infty \).

The sequences satisfy \( 0 < V_n < U_n < 1, (n \geq 1) \) and \( V_n \uparrow 1 \) as \( n \rightarrow \infty \).

Numerical values of \( V_n, n = 1(1)2 \), are given in Table 1 of Section 5.

By using these values the common equilibrium strategy for the equal-weight game in state \((x|n)\), \( n = 1(1)2 \) are shown by Figure 1(a).

The shaded region means that the player here should randomize the two decision R and A, as mentioned in Theorem 1.

3. A Three-Person Optimal Stopping Game.

The analysis made in the previous two sections can be extended to three-person games. We state the problem in correspondence to \((1') \sim (3') \) in Section 1, as follows:

(1') There are three persons I, II, and III. These players have their weights \( w_1, w_2, \) and \( w_3 \), respectively. Let \( 1 \geq w_1 \geq w_2 \geq w_3 \geq 0, w_1 + w_2 + w_3 = 1, \) and \( w_{(j,i)} = w_j/(w_i + w_j) \), \( j + i \).

(2') If three-players choice is A-A-A, then player 1 (II, III) accepts \( X_\epsilon \) with probability \( w_j \) \((w_2, w_3)\) and drops out from the play thereafter. The two players remained continue their two-person game with their revised weights. If three players' choice is R-A-A, then II (III) accepts \( X_\epsilon \) with probability \( w_{(j,i)} \) \((w_{(2,3)}, w_{(3,2)})\) dropping out from the game, and the remaining players III (II) and I continue their two-person game with their revised
new weights. If three-players choice is R-R-A, then III accepts \( X_t \) and drops out and his opponents I and II continue the remaining two-person game. If players' choice-triple is R-R-R, then \( X_t \) is rejected and the players face the next \( X_{t+1} \). In cases of other four choice-triples A-R-A, A-A-R, R-A-R, and A-R-R, the game is played similarly as mentioned above.

(3\( \dagger \)). The aim of each player is the same as in (3\( \circ \)).

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4. Three Special Cases of Three-Person Games.

4.1. The case where \( F(x) = x \) (\( 0 \leq x \leq 1 \)) and \( w_1 = w_2 = w_3 = 1/3 \).

Let \( W_n \) (\( V_n \)) be the value of the game for each player in the equal-weight 3-person (2-person ), n-problem. The detail about \( \{ V_n \} \) is already given by Theorem 1 in Section 2.

Considering symmetry in the role of three players, Eq. (3.1) in state \( (x \mid n) \) becomes

\[
(4.1) \quad M_n(x) = \begin{array}{ccc}
R & M_{n,R}(x) & \\
A & M_{n,A}(x) & \\
\end{array}
\]

where

\[
M_{n,R}(x) = \begin{array}{ccc}
R & x & V \\
A & \frac{1}{3}(x+V), & \frac{1}{3}(x+V) \\
\end{array}
\]

\[
M_{n,A}(x) = \begin{array}{ccc}
R & \frac{1}{3}(x+V), & \frac{1}{3}(x+V) \\
A & \frac{1}{3}(x+V), & \frac{1}{3}(x+V) \\
\end{array}
\]

In these two matrices the subscripts \( n-1 \) in \( W \) and \( V \) are omitted for simplicity.

The Optimality Equation is

\[
(4.2) \quad (W_n, W_n, W_n) = E_F [ \text{eq. val. } M_n(x) ] \quad (n \geq 1, W_0 \equiv 0)
\]

provided that \( \text{eq. val. } M_n(x) \) exists uniquely for \( \forall x \). See Remark 2 in Section 5.

As our common sense suggests we assume that \( 0 < W_n < V_n < 1 \). Then it is easily found from \( M_{n,R}(x) \) and \( M_{n,A}(x) \) that R-R-R (A-A-A) choice-triple is the
pure-strategy equilibrium if \( x < W_{n-1} \) \( (x > V_{n-1}) \). \( \text{We prove} \)

**Theorem 3.** The solution to OSG described by Optimality Equation (4.1)-(4.2) for \( F(x) = x \) \( (0 \leq x \leq 1) \) and \( w_I = w_2 = w_3 = \frac{1}{3} \) is as follows. The common equilibrium strategy for each player, i.e., in state \((x | n)\),

\[
\begin{cases} 
\text{Choose } R, & \text{if } 0 \leq x < W_{n-1} \\
\text{Randomize } R \text{ and } A \text{ with prob.} & \\
\frac{3}{2} \left[ 1 - \frac{1}{2 + z_{n-1}} - \frac{\sqrt{(1 - z_{n-1})(3 + z_{n-1})}}{\sqrt{3(2 + z_{n-1})}} \right] & \text{for } A, \text{ if } W_{n-1} \leq x \leq V_{n-1} \\
\text{Choose } A, & \text{if } V_{n-1} < x \leq 1,
\end{cases}
\]

where \( z_{n-1} = (x - W_{n-1}) / (V_{n-1} - W_{n-1}) \), and the common eq. value is \( W_n \).

The sequences \( \{V_n\} \) and \( \{W_n\} \) are determined by the recursions

\[
V_n = 2(1 - \log_2 2)(U - V)^2 + U V + \frac{1}{4}(1 - U)(1 + 3U)
\]
and

\[
W_n = \mu V - W)^2 + V W + \frac{1}{6}(1 - V)(1 + 3V) \\
= \frac{1}{6}(1 + 4V + V^2) - (V - W)(V + \mu W)
\]

where

\[
\mu \equiv \int_0^1 \lambda d \lambda = \frac{3}{4} - \frac{\sqrt{3}}{2} \int_0^1 \frac{(1 - 2\sqrt{3})^2}{(2 + z)^2} dz.
\]

\[
\Rightarrow \frac{3}{4} - \frac{\sqrt{23}}{2} \times 0.6381 = 0.60814
\]

(Subscripts \( n \rightarrow 1 \) in \( V \) and \( W \) are omitted in the r.h.s. in (4.7)-(4.9))

Moreover \( 0 < W_n < V_n < 1 \) \( (\forall n \geq 1) \) and \( W_n \uparrow 1 \) as \( n \to \infty \).

4.3. The case where \( F(x) = x \) \( (0 \leq x \leq 1) \) and \( \langle w_I, w_2, w_3 \rangle = \langle l, 0, 0 \rangle \).

Since \( w_2 = w_3 = 0 \), player I can behave as if he has no rival, and can get the reward \( U_n \), which is determined by the recursion (1.2). Let \( W_n \) be the weak players' common eq. value in three-person \( \langle I, 0, 0 \rangle \) -weight game for \( n \)-problem. \( V_n^{IE} \) and \( V_n \) are the same as in Subsection 4.2.

Now we consider the II-III behavior in state \((x | n)\), where \( 0 \leq x < U_{n-1} \). Eq.(3.1) becomes

\[
M_n(x) = \begin{cases} 
R \text{ by } I & M_n, R(x) \\
A \text{ by } I & M_n, A(x)
\end{cases}
\]
where

\[
M_{n,R}(x) = \begin{cases} 
R & \begin{array}{|c|c|c|} 
\hline 
U & W & W \\
\hline 
U & x & \nabla \\
\hline 
\end{array} \\
\hline 
A & \begin{array}{|c|c|c|} 
\hline 
U & \nabla & x \\
\hline 
U & \nabla & \nabla \\
\hline 
\end{array} \\
\hline 
\end{cases}
\]

and

\[
M_{n,A}(x) = \begin{cases} 
(x, V^E, V^E), & \text{for } \forall \text{ choice-pair by II-III} \\
\end{cases}
\]

Here the subscripts \(n-1\) in \(U, W, V\) and \(V_E\) in the matrices are omitted for simplicity.

We prove

\[\textbf{Theorem 5.}\] The solution to our 3-person OSG, when \(F(x) = x\) and \((w_1, w_2, w_3) = (1, 0, 0)\) is as follows. In state \((x \mid n)\), player I chooses \(R(A)\) if \(x < (>) U_{n-1}\). Players II and III behave as mentioned in (4.16), if \(0 < x < U_{n-1}\) and follow the common eq. strategy in the two-person equal-weight game for \((n-1)\)-problem, if \(U_{n-1} < x \leq 1\). The eq. payoffs are \((U_n, W_n, W_n)\), where \(W_n\) is determined by the recursion

\[W_n = 2(1 - \log_2(V_n - W_n) + V_n + \frac{1}{4}(U_n^2 + 2U_n V_n - 3V_n^2) + (1 - U)V_n^E) \quad (n \geq 1; W_0 = W_1 = 0)\]

where the sequences \(\{V_n\}\) and \(\{V_n\}\) are determined by the recurrences (4.11) and (2.6), respectively.

\textbf{REFERENCES}


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