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OPTIMAL STOPPING GAMES WHERE PLAYERS HAVE WEIGHTED PRIVILEGE

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Abstract

A non-zero-sum n-stage game version of a full-information best-choice problem under ENV maximazation is analysed and solutions are obtained in some special cases of 2-person and 3-person games. The essential feature contained in this multistage game is the fact that the players have their own weights by which at each stage one player's desired decision is preferred to the opponent's one by drawing a lottery.

1. A Two Person Optimal Stopping Game

A non-zero-sum game version of the discrete-time, full-information best-choice problem under ENV-maximization is considered in this section. We first state the problem as follows:

(1°) There are two players I and II and a sequence of n *iid* r.v.s. $\{X_i\}_{i=1}$ with a common cdf F(x) , $0 \le x < \infty$. Both players observe X_i s sequentially one by one.

(2°) Observing each X_t, both players select, simultaneously and independently, either to accept (A) or to reject (R) the X_t. If I - II choice is A – A, then player I (II) accepts to receive X_t with probability $w^1 (w^2 \equiv 1-w^1)$, $\frac{1}{2} < w^1 < 1$, and drops out from the play thereafter. The player remained continues his one-person game. If I-II choice is A-R (R-A), then I (II) accepts X_t and drops out and his opponent continues the remaining one-person game. If I-II choice is R- R, then X_t is rejected and then the players face the next X_{t+1}.

(3°) The aim of each player in the game is to determine his acceptance strategy under which he maximizes the expected net value (ENV) he obtains.

Define state (x|n) to mean that (1) both players remain in the game, and (2) there remains n r.v.s to be observed and the players currently face the first observation

$X_1 = x$. Player i's strategy, $i = 1, 2$, in state $(x| n)$ is to choose A with probability $\varphi^i(x,n) \in [0,1]$, and R with probability $\bar{\varphi}^i(x,n)$. Evidently $\varphi^i(x|1) \equiv 1$, $\forall x \in (0,\infty)$ $i = 1, 2$.

Let V_n be the value of the game for player i for the n-problem. Then we have

$$(1.1)\quad V_n^i = E_F\Big[(w^i X + w^j U_{n-1})\varphi^i(X,n)\varphi^j(X,n) + X\varphi^i(X,n)\bar{\varphi}^j(X,n) + U_{n-1}\bar{\varphi}^i(X,n)+\varphi^j(X,n) + V_{n-1}^i\bar{\varphi}^i(X,n)\bar{\varphi}^j(X,n)\Big]$$
$$(\, j=3-i,\ i=1,2,\ n=1,2,\cdots,\ V_0^1=V_0^2\equiv 0\,)$$

Here U_{n-1} is the value of the game for the remaining player when his opponent has already dropped out with $n-1$ unobserved r.v.s thereafter. The optimal strategy for the player in this state is evidently to accept (reject) if $x >(<) U_{n-1}$, since the sequence $\{U_n\}$ satisfies the recursion

$$(1.2)\qquad U_n = E_F(X \vee U_{n-1}) \qquad (\, n=1,2,\cdots;\ U_0\equiv 0\,)$$

Our problem is to find the Nash equilibrium

$$(1.3)\qquad (V_n^1, V_n^2) \rightarrow \text{Nash eq. in } (\varphi^1(\cdot,n), \varphi^2(\cdot,n))$$

for each $n = 1, 2, \cdots$.

We hereafter write $\langle w^1, w^2\rangle$ as $\langle w,\bar{w}\rangle$. We consider the problem $(1.1)\sim(1.3)$ as the optimal stopping game described by the Optimality Equation

$$(1.4)\qquad (V_n^1, V_n^2) = E_F\{\text{eq. val } M_n(X)\}$$

$$(1.5)\qquad M_n(x) = \begin{array}{c|c|c} & R & A \\ \hline R & V_{n-1}^1,\ V_{n-1}^2 & U_{n-1},\quad x \\ \hline A & x,\quad U_{n-1} & wx+\bar{w}U_{n-1},\ \bar{w}x+wU_{n-1} \end{array}$$

where $M_n(x)$ is the bimatrix game which the players face in state $(x \mid n)$, and we assume that $M_n(x)$ has a unique equilibrium for every x satisfying $0<\bar{F}(x)<1$.

The problems we consider in this paper belong to a class of best-choice problems combined with sequential games. Recent works related to this area of problems are Enns and Ferenstein [1,2], Mazalov [4] and Sakaguchi [5,7,8]. A very important and now classical literature in full-information best-choice problems is Gilbert and Mosteller [3]. Also a recent look for the opt imal stopping games in various phases can be found in Sakaguchi [6].

2. Two Special Cases in Two-Person Games.

2.1 The case where $F(x)=x\ (0\leqq x\leqq 1)$ and $w=\frac{1}{2}$.

We prove

Theorem 1. The solution to OSG descrived by Optimality Equation (1.4)-(1.5), for $F(x)=x\ (0\leqq x\leqq 1)$ and $w=\frac{1}{2}$, is as follows. The common equilibrium strategy for both player is, in state $(x\mid n)$,

$$\begin{cases} \text{Choose } R, & \text{if } 0\leq x<V_{n-1}, \\ \text{Randomize } R \text{ and } A \text{ with probability } \dfrac{2z_{n-1}}{1+z_{n-1}} \text{ for } A. & \text{if } V_{n-1}<x<U_{n-1}, \\ \text{Choose } A. & \text{if } U_{n-1}<x\leq 1. \end{cases}$$

where $z_{n-1}=(x-V_{n-1})/(U_{n-1}-V_{n-1})$ and the equilibrium values are (V_n, V_n), where the sequences $\{U_n\}$ and $\{V_n\}$ are given by the recursions $U_n=\frac{1}{2}(1+U_{n-1}^2)$, $(n\geqq 1, U_0=0)$ and

(2.4) $$V_n=2(1-\log 2)(U-V)^2+UV+\tfrac{1}{4}(1-U)(1+3U)$$
$$(n=1,2,\cdots;\ U_0=V_0=0)$$

The sequences satisfy $0<V_n<U_n<1$, $(n\geqq 1)$ and $V_n\uparrow 1$ as $n\to\infty$.

Numerical values of V_n, $n=1(1)12$, are given in Table 1 of Section 5.

By using these values the common equilibrium strategy for the equal-weight game in state $(x\mid n)$, $n=1(1)12$ are shown by Figure 1(略). The shaded region means that the player here should randomize the two decision R and A, as mentioned in Thorem 1.

3. A Three-Person Optimal Stopping Game.

The analysis made in the previous two sections can be extended to three-person games. We state the problem in correspondence to $(1^\circ)\sim(3^\circ)$ in Section 1, as follows:

$(1^\dagger)$ There are three persons I,II,and III. These players have their weights w_1, w_2 and w_3, respectively. Let $1\geqq w_1\geqq w_2\geqq w_3\geqq 0$, $w_1+w_2+w_3=1$, and $w_{(i,j)}\equiv w_i/(w_i+w_j)$, $i\neq j$.

$(2^\dagger)$ If three-players choice is A-A-A, then player I (II,III) accepts X_t with probability w_1. (w_2, w_3) and drops out from the play thereafter. The two players remained continue their two-person game with their "revised" new weights. If three players' choice is R-A-A, then II (III) accepts X_t with probability $w_{(2,3)}$ ($w_{(3,2)}$) dropping out grom the game, and the remaining players III (II) and I continue their two-person game with their revised new weithts. If three-players choice is R-R-A, then III accepts X_t and drops out and his opponents I and II continue the remaining two-person game. If players' choice-triple is R-R-R, then X_t is rejected and the players face the next X_{t+1}. In cases of other four choice-triples A-R-A, A-A-R, R-A-R. and A-R-R, the game is played similarly as mentioned above.

($3^{\dagger}$). The aim of each player is the same as in (3°).

4. Three Special Cases of Three-Person Games.

4.1 The case where $F(x) = x$ $(0 \leqq x \leqq 1)$ and $w_1 = w_2 = w_3 = 1/3$.

Let W_n (V_n) be the value of the game for each player in the equal-weight 3-person (2-person), n-problem. The detail about $\{V_n\}$ is already given by Theorem 1 in Section 2.

Considering symmetry in the role of three players, Eq. (3.1) in state $(x \mid n)$ becomes

$$(4.1) \qquad M_n(x) = \begin{array}{l} \xrightarrow{\text{R by III}} M_{n,R}(x) \\ \xrightarrow{\text{A by III}} M_{nA}(x) \end{array}$$

where $M_{n,R}(x) =$

	(II) R			(II) A		
(I) R	$\overline{W}$,	$\overline{W}$,	$\overline{W}$	$\overline{V}$,	x,	$\overline{V}$
(I) A	x,	$\overline{V}$,	$\overline{V}$	$\frac{1}{2}(x+\overline{V})$,	$\frac{1}{2}(x+\overline{V})$,	$\overline{V}$

and $M_{n,A}(x) =$

	R			A		
R	$\overline{V}$,	$\overline{V}$,	x	$\overline{V}$,	$\frac{1}{2}(x+\overline{V})$,	$\frac{1}{2}(x+\overline{V})$
A	$\frac{1}{2}(x+\overline{V})$,	$\overline{V}$,	$\frac{1}{2}(x+\overline{V})$	$\frac{1}{3}(x+2\overline{V})$,	$\frac{1}{3}(x+2\overline{V})$,	$\frac{1}{3}(x+2\overline{V})$

In these two matrices the subscripts $n-1$ in W and V are omitted for simplicity. The Optimality Equation is

$$(4.2) \qquad (W_n,\ W_n,\ W_n) = E_F\left[eq.\ val.\ M_n(X)\right] \qquad (n \geqq 1,\ \overline{W}_0 \equiv 0)$$

provided that eq. val. M_n (x) exists uniquely for $\forall$ x. See Remark 2 in Section 5.

As our common sense suggests we assume that $0 < W_n < V_n < 1$. Then it is easily found from $M_{n,R}(x)$ and $M_{n,A}(x)$ that R-R-R (A-A-A) choice-triple is the pure- strategy equilibrium if $x < W_{n-1}$ $(x > V_{n-1})$. We prove

Theorem 3. The solution to OSG described by Optimality Equation (4.1)-(4.2) for $F(x)=x$ $(0 \leqq x \leqq 1)$ and $w_1 = w_2 = w_3 = \frac{1}{3}$ is as follows. The common equilibrium strategy for each player, is, in state $(x \mid n)$,

$$\begin{cases} \text{Choose R}, & \text{if } 0 \leqq x < W_{n-1} \\ \text{Randomize R and A with prob.} & \\ \dfrac{3}{2}\left[1 - \dfrac{1}{2+z_{n-1}} - \dfrac{\sqrt{(1-z_{n-1})(3+z_{n-1})}}{\sqrt{3}(2+z_{n-1})}\right] \text{ for A}, & \text{if } W_{n-1} \lessdot x < V_{n-1} \\ \text{Choose A}, & \text{if } V_{n-1} < x \leqq 1, \end{cases}$$

where $z_{n-1} = (x - W_{n-1})/(V_{n-1} - W_{n-1})$, and the common eq. value is W_n .

The sequences $\{V_n\}$ and $\{W_n\}$ are determined by the recursions

(4.7) $$V_n = 2(1 - \log 2)(U-V)^2 + UV + \frac{1}{4}(1-U)(1+3U)$$

and

(4.9) $$\begin{aligned} W_n &= \mu(V-W)^2 + VW + \frac{1}{6}(1-V)(1+5V) \\ &= \frac{1}{6}(1+4V+V^2) - (V-W)(\bar{\mu}V + \mu W) \end{aligned}$$

where

$$\begin{aligned} \mu \equiv \int_0^1 \lambda\, dz &= \frac{3}{4} - \frac{\sqrt{3}}{2}\int_0^1 \frac{(1-z)^{3/2}(3+z)^{1/2}}{(2+z)^2}\, dz. \\ &\fallingdotseq \frac{3}{4} - \frac{\sqrt{3}}{2} \times 0.16381 = 0.60814 \end{aligned}$$

(Subscripts $n-1$ in V and W are ommited in the r.h.s. in (4.7)-(4.9))

Moreover $0 < W_n < V_n < 1$ $(\forall n \geqq 1)$ and $W_n \uparrow 1$ as $n \to \infty$.

4.3 The case where $F(x)=x$ $(0 \leqq x \leqq 1)$ and $\langle w_1, w_2, w_3 \rangle = \langle 1, 0, 0 \rangle$.

Since $w_2 = w_3 = 0$, player I can behave as if he has no rival, and can get the reward U_n , which is determined by the recursion (1.2). Let W_n be the weak players' common eq. value in three-person $\langle 1, 0, 0 \rangle$ -weight game for n-problem. V_n^E and V_n are the same as in Subsection 4.2.

Now we consider the II-III behavior in state $(x \mid n)$, where $0 \leqq x < U_{n-1}$. Eq.(3.1) becomes

(4.14) $$M_n(x) = \begin{tikzcd}[row sep=small] & M_{n,R}(x) \\ \bullet \arrow[ur, "\text{R by I}"] \arrow[dr, "\text{A by I}"'] & \\ & M_{n,A}(x) \end{tikzcd}$$

where

$$
M_{n,R}(x) = (\mathrm{II}) \begin{array}{c|c|c|}
 & \multicolumn{2}{c}{\overbrace{\quad R \qquad\qquad\qquad A \quad}^{(\mathrm{III})}} \\
R & U,\ W,\ W & U,\ V,\ x \\
A & U,\ x,\ V & U,\ \frac{1}{2}(x+V),\ \frac{1}{2}(x+V)
\end{array}
$$

and

$$M_{n,A}(x) = \left[\ (x,\ V^E, V^E),\ \text{for } \forall \text{ choice-pair by II-III}\ \right]$$

Here the subscripts $n-1$ in U,W, V and V^E in the matrices are omitted for simplicity.

We prove

Theorem 5. The solution to our 3-person OSG, when $F(x)=x$ and $\langle w_1, w_2, w_3\rangle = \langle 1, 0, 0\rangle$ is as follows. In state $(x \mid n)$, player I chooses R(A) if $x<(>)\, U_{n-1}$. Players II and III behave as mentioned in (4.16), if $0 \leqq x < U_{n-1}$ and follow the common eq.strategy in the two-person equal-weight game for $(n-1)$-problem, if $U_{n-1} < x \leqq 1$. The eq.payoffs are (U_n, W_n, W_n), where $\{W_n\}$ is determined by the recursion

$$(4.17) \qquad W_n = 2(1-\log 2)(V-W)^2 + VW + \frac{1}{4}(U^2+2UV-3V^2) + (1-U)V^E$$
$$(\ n \geqq 1\ ;\ W_0 = W_1 = 0\ )$$

where the sequences $\{V_n^E\}$ and $\{V_n\}$ are determined by the recurences (4.11) and (2.6), respectively.

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