<table>
<thead>
<tr>
<th>Title</th>
<th>Degree of Order regarding Multidimensional Fuzzy Sets</th>
<th>Mathematical Modeling and Optimization under Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yoshida, Yuji; Kerre, E.E.</td>
<td></td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2001), 1194: 80-86</td>
<td></td>
</tr>
<tr>
<td>Issue Date</td>
<td>2001-03</td>
<td></td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64815">http://hdl.handle.net/2433/64815</a></td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
<td></td>
</tr>
</tbody>
</table>
Degree of Order regarding Multidimensional Fuzzy Sets

North九州大学経済学部 吉田祐治 (Yuji YOSHIDA)
ゲンツ大学応用数学科 ケアー (E.E.Kerre)

1. Introduction and notations

Kurano et al. [4] studied a pseudo order regarding fuzzy sets on $\mathbb{R}^n$ on the basis of a set-relation in $\mathbb{R}^n$ studied by Kuroiwa et al. [2] and Kuroiwa [3] for multi-criteria crisp set-valued optimizations in mathematical programming. First, we introduce a fuzzy relation, which is fuzzy partial ordering, induced by closed convex cones. Next, a pseudo order regarding fuzzy sets on $\mathbb{R}^n$ is given by inclusions defined from the fuzzy relation, and it is also a reasonable multi-dimensional extension of the fuzzy max order regarding fuzzy numbers. For incomparable fuzzy sets on $\mathbb{R}^n$, we present a degree of order, using a subsethood degree. This method is flexible and can be applied to various types of fuzzy decision-making and optimization problems in multi-criteria. For example, we can apply the method to problems where components on $\mathbb{R}^n$ are related to each other in multi-criteria.

In the rest of this section, we give some notations and introduce some results regarding vector ordering on $\mathbb{R}^n$ by convex cones. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^n$ be an n-dimensional Euclidean space, where $n$ is a positive integer. We write fuzzy sets on $\mathbb{R}^n$ and their membership functions by $\tilde{A}$ and $\mu_{\tilde{A}} : \mathbb{R}^n \rightarrow [0, 1]$ respectively. The $\alpha$-cut ($\alpha \in [0, 1]$) of the fuzzy set $\tilde{A}$ on $\mathbb{R}^n$ is defined as

$$\tilde{A}_\alpha := \{x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) \geq \alpha\} \quad (\alpha > 0) \quad \text{and} \quad \tilde{A}_0 := \text{cl}\{x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{A}$ is called convex if the $\alpha$-cut $\tilde{A}_\alpha$ is a convex set for all $\alpha \in [0, 1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets $\tilde{A}$ whose membership functions $\mu_{\tilde{A}} : \mathbb{R}^n \rightarrow [0, 1]$ are upper-semicontinuous and normal (sup$_{x \in \mathbb{R}^n} \mu_{\tilde{A}}(x) = 1$) and have a compact 0-cut. When one-dimensional case ($n = 1$), the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. In this paper, we deal with fuzzy sets on $\mathbb{R}^n$ as a multi-dimensional extension of fuzzy numbers. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all non-empty compact convex subsets of $\mathbb{R}^n$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and a scalar $\lambda \geq 0$, the sum $\tilde{A} + \tilde{B}$ and scalar multiplication $\lambda \tilde{A}$ are given by applying Zaheh's extension principle:

$$\mu_{\tilde{A} + \tilde{B}}(x) := \sup_{y, z \in \mathbb{R}^n : y + z = x} \min\{\mu_{\tilde{A}}(y), \mu_{\tilde{B}}(z)\}, \quad (1.1)$$

$$\mu_{\lambda \tilde{A}}(x) := \begin{cases} \mu_{\tilde{A}}(x/\lambda) & \text{if } \lambda > 0 \\ 1_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (1.2)$$

for $x \in \mathbb{R}^n$, where $1_{\{\cdot\}}(\cdot)$ is an indicator and $\{0\}$ denotes the crisp set of zero in $\mathbb{R}^n$. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for
Let $A, B \in C(\mathbb{R}^{n})$, the following holds immediately:

$$(\tilde{A} + \tilde{B})_{\alpha} := \tilde{A}_{\alpha} + \tilde{B}_{\alpha} \quad \text{and} \quad (\lambda \tilde{A})_{\alpha} = \lambda \tilde{A}_{\alpha} \quad (\alpha \in [0, 1]). \quad (1.3)$$

Let $K$ be a non-empty convex cone of $\mathbb{R}^{n}$, i.e. $x + y \in K$ and $\lambda x \in K$ hold for all $\lambda \geq 0$ and all $x, y \in K$. Using a convex cone $K$, we can define a pseudo order $\preceq_{K}$ on $\mathbb{R}^{n}$ by

(K.1) $x \preceq_{K} y$ means that $y - x \in K$.

Let $\mathbb{R}_{+}^{n} = \{x = (x^{1}, x^{2}, \cdots, x^{n}) \in \mathbb{R}^{n} | x^{i} \geq 0 (i = 1, 2, \cdots, n)\}$ be the subset of entrywise nonnegative elements in $\mathbb{R}^{n}$. When $K = \mathbb{R}_{+}^{n}$, the order $x \preceq_{\mathbb{R}_{+}^{n}} y$ means that $x^{i} \leq y^{i}$ for all $i = 1, 2, \cdots, n$, where $x = (x^{1}, x^{2}, \cdots, x^{n})$ and $y = (y^{1}, y^{2}, \cdots, y^{n}) \in \mathbb{R}^{n}$.

2. Fuzzy partial ordering by convex cones

In this section, we discuss a fuzzy relation induced by closed convex cones. Let $R_{\alpha} \ (\alpha \in [0, 1])$ be non-empty closed convex cones on $\mathbb{R}^{n}$ such that $\cap_{\alpha' \in (0, 1]} R_{\alpha'} = R_{\alpha}$ for $\alpha \in (0, 1]$. To avoid meaningless ordering, we assume $R_{\alpha} \ (\alpha \in (0, 1])$ are acute, i.e. there exists $a \in \mathbb{R}^{n}$ satisfying $a \cdot x > 0$ for all $x \in R_{\alpha}$ with $x \neq 0$, where $\cdot$ means the inner product of vectors on $\mathbb{R}^{n}$. We also put the support set by $R_{0} := \text{cl}(\cup_{\alpha \in (0, 1]} R_{\alpha})$. Define a fuzzy relation $\tilde{R}$ on $\mathbb{R}^{n} \times \mathbb{R}^{n}$ by closed convex cones $R_{\alpha}$ as follows.

$$\mu_{\tilde{R}}(x, y) := \sup\{\alpha \in [0, 1] | y - x \in R_{\alpha}\} \quad ((x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{n}), \quad (2.1)$$

where $\sup \emptyset := 0$. Then the $\alpha$-cut $\tilde{R}_{\alpha}$ is

$$\tilde{R}_{\alpha} = \{(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{n} | \mu_{\tilde{R}}(x, y) \geq \alpha\} = \{(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{n} | y - x \in R_{\alpha}\} \quad (2.2)$$

for $\alpha \in (0, 1]$. The fuzzy relation $\tilde{R}$ has the following properties.

**Theorem 2.1.** $\tilde{R}$ is a fuzzy partial ordering (\cite{[1]}), i.e., it satisfies the following (i) – (iii):

(i) $\mu_{\tilde{R}}(x, x) = 1$ for all $x \in \mathbb{R}^{n}$.

(ii) $\mu_{\tilde{R}}(x, z) \geq \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{R}}(y, z)\}$ for all $x, y, z \in \mathbb{R}^{n}$.

(iii) If $\mu_{\tilde{R}}(x, y) > 0$ and $\mu_{\tilde{R}}(y, x) > 0$, then $x = y$.

In Theorem 2.1, the property (i), (ii) and (iii) means reflexivity, transitivity and antisymmetry respectively.
3. Ordering of fuzzy quantities by the fuzzy relation $\tilde{R}$

In this section, we introduce an order for fuzzy sets on $\mathbb{R}^n$, using the fuzzy relation $\tilde{R}$ from Section 2. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, using the sup-min composition operation with $\tilde{R}$ ([1]), we define fuzzy sets $\tilde{A} \circ \tilde{R}, \tilde{R} \circ \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ by

$$
\mu_{\tilde{R} \circ \tilde{B}}(x) := \sup_{y \in \mathbb{R}^n} \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{B}}(y)\} \quad (x \in \mathbb{R}^n),
$$

$$
\mu_{\tilde{A} \circ \tilde{R}}(y) := \sup_{x \in \mathbb{R}^n} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R}}(x, y)\} \quad (y \in \mathbb{R}^n).
$$

**Definition 1.** Consider fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ means that

$$
\tilde{A} \subseteq \tilde{R} \circ \tilde{B} \quad \text{and} \quad \tilde{B} \subseteq \tilde{A} \circ \tilde{R}.
$$

Further, the equivalence $\tilde{A} \sim_{\tilde{R}} \tilde{B}$ means that $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{A}$. We also write $\tilde{A} \prec_{\tilde{R}} \tilde{B}$ when $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{A} \neq_{\tilde{R}} \tilde{B}$.

**Lemma 3.1.** The order $\preceq_{\tilde{R}}$ is a pseudo order: For fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n)$, the following (i) and (ii) hold.

(i) $\tilde{A} \preceq_{\tilde{R}} \tilde{A}$.

(ii) If $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{C}$, then $\tilde{A} \preceq_{\tilde{R}} \tilde{C}$.

In Lemma 3.1, the property (i) and (ii) means reflexivity and transitivity respectively.

In the one-dimensional case ($n = 1$), the order $\preceq_{\tilde{R}}$ coincides with the fuzzy max order, and then $\sim$ is replaced with $=$. The following example gives the order $\preceq_{\tilde{R}}$ induced from natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$ and shows that the order $\preceq_{\tilde{R}}$ is not a total order.

**Example 3.1** (The order $\preceq_{\tilde{R}}$ corresponding to natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$). We consider a case when $n = 2$ and we put an acute closed convex cone $R_\alpha = \mathbb{R}_+ = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^i \geq 0 (i = 1, 2)\}$ for all $\alpha \in [0, 1]$. Then the corresponding fuzzy relation is

$$
\mu_{\tilde{R}}((x^1, x^2), (y^1, y^2)) = \begin{cases} 
1 & \text{if } y^1 \geq x^1 \text{ and } y^2 \geq x^2 \\
0 & \text{otherwise.}
\end{cases}
$$

(3.3)

For $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, the definition of the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ is reduced to the following conditions (3.4) and (3.5):

$$
\mu_{\tilde{A}}(x^1, x^2) \leq \sup_{(y^1, y^2) : y^1 \geq x^1, y^2 \geq x^2} \mu_{\tilde{B}}(y^1, y^2) \quad \text{for all } (x^1, x^2) \in \mathbb{R}^2 \times \mathbb{R}^2; \quad (3.4)
$$

$$
\mu_{\tilde{B}}(y^1, y^2) \leq \sup_{(x^1, x^2) : x^1 \geq y^1, x^2 \geq y^2} \mu_{\tilde{A}}(x^1, x^2) \quad \text{for all } (y^1, y^2) \in \mathbb{R}^2 \times \mathbb{R}^2. \quad (3.5)
$$

Take pyramid-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined by

$$
\mu_{\tilde{A}}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1]\}, 0\}, \quad (3.6)
$$

$$
\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{1 - |x^1 - 1|, 1 - |x^2 + 1]\}, 0\} \quad (3.7)
$$

for $(x^1, x^2) \in \mathbb{R}^2$. Then we can easily check that neither $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ nor $\tilde{B} \preceq_{\tilde{R}} \tilde{A}$ hold.
4. Degree of the fuzzy order $\preceq_{\overline{R}}$

In this section, by using a subsethood degree, we present a method to evaluate the degree of satisfaction of the fuzzy order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ for all fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^{n})$. The fuzzy order $\preceq_{\overline{R}}$ is not a total order (Example 3.1), however we can apply this method to fuzzy sets which are incomparable by the order $\preceq_{\overline{R}}$. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^{n})$, the subsethood degree is defined by ([5],[1])

\[
\text{Sub}(\tilde{A} \subseteq \tilde{B}) := \frac{|\tilde{A} \cap \tilde{B}|}{|\tilde{A}|} \quad (4.1)
\]

if $|\tilde{A}| > 0$, where

\[
|\tilde{A}| := \int_{\mathbb{R}^{n}} \mu_{\overline{A}}(x) \, dx \quad (\tilde{A} \in \mathcal{F}(\mathbb{R}^{n})). \quad (4.2)
\]

For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^{n})$, in spirit of Definition 1 we define the degree of order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ by

\[
D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) := \min \{\text{Sub}(\tilde{A} \subseteq \tilde{R} \circ \tilde{B}), \text{Sub}(\tilde{B} \subseteq \tilde{A} \circ \tilde{R})\}. \quad (4.3)
\]

Then, the degree of order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ is written as

\[
D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = \min \left\{ \frac{|\tilde{A} \cap (\tilde{R} \circ \tilde{B})|}{|\tilde{A}|}, \frac{|\tilde{B} \cap (\tilde{A} \circ \tilde{R})|}{|\tilde{B}|} \right\} = \min \left\{ \int_{\mathbb{R}^{n}} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R} \circ \tilde{B}}(x)\} \, dx, \int_{\mathbb{R}^{n}} \min\{\mu_{\tilde{B}}(x), \mu_{\tilde{A} \circ \tilde{R}}(x)\} \, dx \right\}. \quad (4.4)
\]

The following lemma implies a correspondence between the order $\preceq_{\overline{R}}$ in Definition 1 and the degree of order $\preceq_{\overline{R}}$ in (4.4).

**Theorem 4.1.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^{n})$. The order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ holds if and only if $D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = 1$.

The degree of order $\preceq_{\overline{R}}$ has the following properties.

**Lemma 4.1.** Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^{n})$ and $\lambda \geq 0$. Then, the following (i) - (ii) holds:

(i) $D(\tilde{A} \preceq_{\overline{R}} \tilde{A}) = 1$.

(ii) If $D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = 1$ and $D(\tilde{B} \preceq_{\overline{R}} \tilde{C}) = 1$, then $D(\tilde{A} \preceq_{\overline{R}} \tilde{C}) = 1$.

The following results are related to Theorem 3.1.

**Theorem 4.2.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^{n})$, $z \in \mathbb{R}^{n}$ and $\lambda > 0$. Then

\[
D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = D(\tilde{A} + \{z\} \preceq_{\overline{R}} \tilde{B} + \{z\}) = D(\lambda \tilde{A} \preceq_{\overline{R}} \lambda \tilde{B}). \quad (4.5)
\]
The following theorem is useful to calculate fuzzy sets $\tilde{R} \circ \tilde{B}$ and $\tilde{A} \circ \tilde{R}$ in the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and the degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ (Definition 1 and (4.4)).

**Theorem 4.3.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0,1]$. Then, the following (i) – (ii) holds:

(i) $(\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} - R_{\alpha}$;

(ii) $(\tilde{A} \circ \tilde{R})_{\alpha} = \tilde{A}_{\alpha} + R_{\alpha}$,

where $-R_{\alpha} = \{-x \mid x \in R_{\alpha}\}$.

We consider an example in the two-dimensional case to illustrate the meaning of the degree of order $\preceq_{\tilde{R}}$.

**Example 4.1** (Degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$). We consider the case when $n = 2$ and the acute closed convex cone $R_{\alpha} = \mathbb{R}_{+}^{2}$ for all $\alpha \in [0,1]$ in Example 3.1. First, for fuzzy sets $\tilde{A}$ and $\tilde{B}$ given by (3.6) and (3.7), we have $\tilde{A}_{\alpha} = [-2 + \alpha, -\alpha] \times [\alpha, 2 - \alpha]$ and $\tilde{B}_{\alpha} = [\alpha, 2 - \alpha] \times [-2 + \alpha, -\alpha]$. From Theorem 4.3, $(\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} - R_{\alpha} = (\infty, 2 - \alpha] \times (-\infty, -\alpha]$ and $(\tilde{A} \circ \tilde{R})_{\alpha} = \tilde{A}_{\alpha} + R_{\alpha} = (-2 + \alpha, \infty) \times (\alpha, \infty)$. Clearly, $\tilde{A}_{\alpha} \cap (\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} \cap (\tilde{A} \circ \tilde{R})_{\alpha} = \emptyset$ for all $\alpha \in (0,1]$, and so $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$. Similarly we can check $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = 0$. Next, we take polyhedral cone-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ by

$$
\mu_{\tilde{A}}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1|\}, 0\}, \quad (4.6)
$$

$$
\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2/2, (3x^1 + \sqrt{3}x^2 + 6)/4, (-3x^1 + \sqrt{3}x^2 + 6)/4\}, 0\} \quad (4.7)
$$

for $(x^1, x^2) \in \mathbb{R}^2$. Then similarly we can easily check $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = \min\{\text{Sub}(\tilde{B} \subseteq \tilde{R} \circ \tilde{A}), \text{Sub}(\tilde{A} \subseteq \tilde{B} \circ \tilde{R})\} = \min\{7/9, 1/3\} = 1/3$.

**5. Approximation for numerical calculation**

Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. In this section, using discrete cases, we discuss a method to approximate the degree of fuzzy order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ numerically. For simplicity, we deal with the case when $n = 2$ and fuzzy sets $\tilde{A}, \tilde{B}$ such that membership functions $\mu_{\tilde{A}}$, $\mu_{\tilde{B}}$, $\mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}$ and $\mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}$ are continuous. In the discrete case, the subbenthed degree ([1, p.28]) is defined by (4.1) with scalar cardinality

$$
|\tilde{A}| := \sum_x \mu_{\tilde{A}}(x), \quad (5.1)
$$

where the sum is taken over some finite set. Then, the degree of order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ in the discrete case is given by

$$
\min \left\{ \frac{\sum_x \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}(x)}{\sum_x \mu_{\tilde{A}}(x)}, \frac{\sum_x \mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}(x)}{\sum_x \mu_{\tilde{B}}(x)} \right\}. \quad (5.2)
$$

We approximate (4.4), using (5.2) which are easy to calculate numerically.
Let a region $C := [-c, c]^2$ such that $\tilde{A}_0 \cup \tilde{B}_0 \subseteq C = [-c, c]^2$ with $c > 0$. For $m = 1, 2, \cdots$, put a mesh $C^{(m)}$ of $C$ by $C^{(m)} := \{(x^1, x^2) \mid x^1 = ic/m, x^2 = jc/m, i, j = -m, -(m-1), \cdots, -1, 0, 1, \cdots, m-1, m\}$. Let $I_A := \int_{\mathbb{R}^2} \mu_A(x^1, x^2) \, dx^1 \, dx^2$ and define

$$I_A^{(m)} := \sum_{(x^1, x^2) \in C^{(m)}} \mu_A(x^1, x^2) \left(\frac{c}{m}\right)^2 = \sum_{i=-m}^{m-1} \sum_{j=-m}^{m} \mu_A(x^1_i^{(m)}, x^2_j^{(m)}) \left(\frac{c}{m}\right)^2$$

for $m = 1, 2, \cdots$, where $(x^1_i^{(m)}, x^2_j^{(m)}) = (ic/m, jc/m) \in C^{(m)}$. Then, the integrand $I_A^{(m)} = \int_{C \subseteq \mathbb{R}^2} \mu_A(x^1, x^2) \, dx^1 \, dx^2$ is approximated:

$$I_A^{(m)} \rightarrow I_A = \int_{\mathbb{R}^2} \mu_A(x^1, x^2) \, dx^1 \, dx^2$$

as $m \rightarrow \infty$. Here, we put an error $\epsilon_A^{(m)} := |I_A - I_A^{(m)}|$ for $m = 1, 2, \cdots$. For fuzzy sets $\tilde{B}, \tilde{A} \cap (\tilde{R} \tilde{B})$ and $\tilde{B} \cap (\tilde{A} \tilde{R})$, we also define integrands, discrete approximations and errors similarly. Then we obtain the following estimation of errors. The degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ in the discrete case on $C^{(m)}$ is given by

$$D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) := \min \left\{ \frac{\sum_{x \in C^{(m)}} \mu_{A \cap (\tilde{R} \tilde{B})}(x)}{\sum_{x \in C^{(m)}} \mu_{\tilde{A}}(x)}, \frac{\sum_{x \in C^{(m)}} \mu_{B \cap (\tilde{A} \tilde{R})}(x)}{\sum_{x \in C^{(m)}} \mu_{\tilde{B}}(x)} \right\}$$

for $m = 1, 2, \cdots$.

**Theorem 5.1.** For $m = 1, 2, \cdots$, it holds that

$$|D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) - D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})| \leq \max \left\{ \frac{\epsilon_A^{(m)} + \epsilon_{A \cap (\tilde{R} \tilde{B})}(x)}{I_A - \epsilon_A^{(m)}}, \frac{\epsilon_B^{(m)} + \epsilon_{B \cap (\tilde{A} \tilde{R})}(x)}{I_B - \epsilon_B^{(m)}} \right\}$$

if $0 < \epsilon_A^{(m)} < I_A$ and $0 < \epsilon_B^{(m)} < I_B$.

Theorem 5.1 implies the convergence $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \rightarrow D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ as $m \rightarrow \infty$ and also gives an estimation of errors. Finally, we give an example to approximate the degree of order on $\mathcal{F}(\mathbb{R}^2)$ by Theorem 5.1.

**Example 5.1** (Approximation of the degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$). We consider the case when $n = 2$ and the acute closed convex cone $R_\alpha = \mathbb{R}_+^2$ for all $\alpha \in [0, 1]$. Take a helmet-type fuzzy set $\tilde{A}$ and a polyhedron-type fuzzy set $\tilde{B}$ by

$$\mu_{\tilde{A}}(x^1, x^2) = \max\{1 - 0.25(x^1 + 1)^2 - 0.25(x^2 - 1)^2, 0\},$$

$$\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2, (3x^1 + \sqrt{3}x^2 + 6)/2, (-3x^1 + \sqrt{3}x^2 + 6)/2, 1\}, 0\}$$

for $(x^1, x^2) \in \mathbb{R}^2$. Then, we can approximate $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \approx \min\{0.0903, 0.2331\} = 0.0903$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) \approx \min\{0.5890, 0.5714\} = 0.5714$. 

Table 5.1. Approximation of degree of order $\preceq_{\overline{R}}$ in Example 5.1 ($n = 2$).

<table>
<thead>
<tr>
<th>$m$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(\tilde{A} \preceq_{\overline{R}}^{(m)} \tilde{B})$</td>
<td>.08102</td>
<td>.08781</td>
<td>.08905</td>
<td>.08962</td>
<td>.08985</td>
<td>.08999</td>
<td>.09008</td>
<td>.09014</td>
</tr>
<tr>
<td>$D(\tilde{B} \preceq_{\overline{R}}^{(m)} \tilde{A})$</td>
<td>.56905</td>
<td>.57100</td>
<td>.57066</td>
<td>.57139</td>
<td>.57134</td>
<td>.57125</td>
<td>.57136</td>
<td>.57136</td>
</tr>
</tbody>
</table>

References


