Degree of Order regarding Multidimensional Fuzzy Sets

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1. Introduction and notations

Kurano et al. [4] studied a pseudo order regarding fuzzy sets on \( \mathbb{R}^n \) on the basis of a set-relation in \( \mathbb{R}^n \) studied by Kuroiwa et al. [2] and Kuroiwa [3] for multi-criteria crisp set-valued optimizations in mathematical programming. First, we introduce a fuzzy relation, which is fuzzy partial ordering, induced by closed convex cones. Next, a pseudo order regarding fuzzy sets on \( \mathbb{R}^n \) is given by inclusions defined from the fuzzy relation, and it is also a reasonable multi-dimensional extension of the fuzzy max order regarding fuzzy numbers. For incomparable fuzzy sets on \( \mathbb{R}^n \), we present a degree of order, using a subsethood degree. This method is flexible and can be applied to various types of fuzzy decision-making and optimization problems in multi-criteria. For example, we can apply the method to problems where components on \( \mathbb{R}^n \) are related to each other in multi-criteria.

In the rest of this section, we give some notations and introduce some results regarding vector ordering on \( \mathbb{R}^n \) by convex cones. Let \( \mathbb{R} \) be the set of all real numbers and let \( \mathbb{R}^n \) be an \( n \)-dimensional Euclidean space, where \( n \) is a positive integer. We write fuzzy sets on \( \mathbb{R}^n \) and their membership functions by \( \tilde{A} \) and \( \mu_{\tilde{A}} : \mathbb{R}^n \rightarrow [0,1] \) respectively. The \( \alpha \)-cut \((\alpha \in [0,1])\) of the fuzzy set \( \tilde{A} \) on \( \mathbb{R}^n \) is defined as

\[
\tilde{A}_\alpha := \{ x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) \geq \alpha \} \quad (\alpha > 0) \quad \text{and} \quad \tilde{A}_0 := \text{cl}\{ x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) > 0 \},
\]

where \( \text{cl} \) denotes the closure of the set. A fuzzy set \( \tilde{A} \) is called convex if the \( \alpha \)-cut \( \tilde{A}_\alpha \) is a convex set for all \( \alpha \in [0,1] \). Let \( \mathcal{F}(\mathbb{R}^n) \) be the set of all convex fuzzy sets \( \tilde{A} \) whose membership functions \( \mu_{\tilde{A}} : \mathbb{R}^n \rightarrow [0,1] \) are upper-semicontinuous and normal \((\sup_{x \in \mathbb{R}^n} \mu_{\tilde{A}}(x) = 1)\) and have a compact \( 0 \)-cut. When one-dimensional case \((n = 1)\), the fuzzy sets are called fuzzy numbers and \( \mathcal{F}(\mathbb{R}) \) denotes the set of all fuzzy numbers. In this paper, we deal with fuzzy sets on \( \mathbb{R}^n \) as a multi-dimensional extension of fuzzy numbers. Let \( \mathcal{C}(\mathbb{R}^n) \) be the set of all non-empty compact convex subsets of \( \mathbb{R}^n \).

The definitions of addition and scalar multiplication on \( \mathcal{F}(\mathbb{R}^n) \) are as follows: For fuzzy sets \( \tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n) \) and a scalar \( \lambda \geq 0 \), the sum \( \tilde{A} + \tilde{B} \) and scalar multiplication \( \lambda \tilde{A} \) are given by applying Zaheh's extension principle:

\[
\mu_{\tilde{A}+\tilde{B}}(x) := \sup_{y,z \in \mathbb{R}^n : y+z=x} \min \{ \mu_{\tilde{A}}(y), \mu_{\tilde{B}}(z) \},
\]

(1.1)

\[
\mu_{\lambda \tilde{A}}(x) := \begin{cases} 
\mu_{\tilde{A}}(x/\lambda) & \text{if } \lambda > 0 \\
1_{\{0\}}(x) & \text{if } \lambda = 0
\end{cases}
\]

(1.2)

for \( x \in \mathbb{R}^n \), where \( 1_{\{\cdot\}}(\cdot) \) is an indicator and \( \{0\} \) denotes the crisp set of zero in \( \mathbb{R}^n \). By using set operations \( A + B := \{ x + y \mid x \in A, y \in B \} \) and \( \lambda A := \{ \lambda x \mid x \in A \} \) for
Let $A, B \in \mathbb{C}^n$, the following holds immediately:

\[(\tilde{A} + \tilde{B})_\alpha := \tilde{A}_\alpha + \tilde{B}_\alpha \quad \text{and} \quad (\lambda \tilde{A})_\alpha = \lambda \tilde{A}_\alpha \quad (\alpha \in [0, 1]).\]  

(1.3)

Let $K$ be a non-empty convex cone of $\mathbb{R}^n$, i.e. $x + y \in K$ and $\lambda x \in K$ hold for all $\lambda \geq 0$ and all $x, y \in K$. Using a convex cone $K$, we can define a pseudo order $\preceq_K$ on $\mathbb{R}^n$ by

(K.1) $x \preceq_K y$ means that $y - x \in K$.

Let $\mathbb{R}_+ = \{x = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n | x^i \geq 0 (i = 1, 2, \cdots, n)\}$ be the subset of entrywise nonnegative elements in $\mathbb{R}^n$. When $K = \mathbb{R}_+$, the order $x \preceq_{\mathbb{R}_+} y$ means that $x^i \leq y^i$ for all $i = 1, 2, \cdots, n$, where $x = (x^1, x^2, \cdots, x^n)$ and $y = (y^1, y^2, \cdots, y^n) \in \mathbb{R}^n$.

2. Fuzzy partial ordering by convex cones

In this section, we discuss a fuzzy relation induced by closed convex cones. Let $R_\alpha (\alpha \in [0, 1])$ be non-empty closed convex cones on $\mathbb{R}^n$ such that $\bigcap_{\alpha \in (0, 1)} R_{\alpha'} = R_\alpha$ for $\alpha \in (0, 1]$. To avoid meaningless ordering, we assume $R_\alpha (\alpha \in (0, 1])$ are acute, i.e. there exists $a \in \mathbb{R}^n$ satisfying $a \cdot x > 0$ for all $x \in R_\alpha$ with $x \neq 0$, where $\cdot$ means the inner product of vectors on $\mathbb{R}^n$. We also put the support set by $R_0 := \text{cl}(\bigcup_{\alpha \in (0, 1]} R_\alpha)$. Define a fuzzy relation $\tilde{R}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by closed convex cones $R_\alpha$ as follows.

\[\mu_{\tilde{R}}(x, y) := \sup\{\alpha \in [0, 1] | y - x \in R_\alpha\} \quad ((x, y) \in \mathbb{R}^n \times \mathbb{R}^n),\]  

(2.1)

where $\sup \emptyset := 0$. Then the $\alpha$-cut $\tilde{R}_\alpha$ is

\[\tilde{R}_\alpha = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | \mu_{\tilde{R}}(x, y) \geq \alpha\} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | y - x \in R_\alpha\}\]  

(2.2)

for $\alpha \in (0, 1]$. The fuzzy relation $\tilde{R}$ has the following properties.

**Theorem 2.1.** $\tilde{R}$ is a fuzzy partial ordering ([1]), i.e., it satisfies the following (i) – (iii):

(i) $\mu_{\tilde{R}}(x, x) = 1$ for all $x \in \mathbb{R}^n$.

(ii) $\mu_{\tilde{R}}(x, z) \geq \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{R}}(y, z)\}$ for all $x, y, z \in \mathbb{R}^n$.

(iii) If $\mu_{\tilde{R}}(x, y) > 0$ and $\mu_{\tilde{R}}(y, x) > 0$, then $x = y$.

In Theorem 2.1, the property (i), (ii) and (iii) means reflexivity, transitivity and antisymmetry respectively.
3. Ordering of fuzzy quantities by the fuzzy relation $\tilde{R}$

In this section, we introduce an order for fuzzy sets on $\mathbb{R}^n$, using the fuzzy relation $\tilde{R}$ from Section 2. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, using the sup-min composition operation with $\tilde{R}$ ([1]), we define fuzzy sets $\tilde{A} \circ \tilde{R}, \tilde{R} \circ \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ by

\begin{align*}
\mu_{\tilde{A}\tilde{B}}(x) &:= \sup_{y \in \mathbb{R}^n} \min\{\mu_{\tilde{A}}(x, y), \mu_{\tilde{B}}(y)\} \quad (x \in \mathbb{R}^n), \\
\mu_{\tilde{A}\tilde{B}}(y) &:= \sup_{x \in \mathbb{R}^n} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x, y)\} \quad (y \in \mathbb{R}^n).
\end{align*}

**Definition 1.** Consider fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The order $\tilde{A} \leq_{\tilde{R}} \tilde{B}$ means that

$$\tilde{A} \subseteq \tilde{R} \circ \tilde{B} \quad \text{and} \quad \tilde{B} \subseteq \tilde{A} \circ \tilde{R}.$$  

Further, the equivalence $\tilde{A} \sim_{\tilde{R}} \tilde{B}$ means that $\tilde{A} \leq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \leq_{\tilde{R}} \tilde{A}$. We also write $\tilde{R} >_{\tilde{R}} \tilde{B}$ when $\tilde{A} \leq_{\tilde{R}} \tilde{B}$ and $\tilde{A} \not<_{\tilde{R}} \tilde{B}$.

**Lemma 3.1.** The order $\leq_{\tilde{R}}$ is a pseudo order: For fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n)$, the following (i) and (ii) hold.

(i) $\tilde{A} \leq_{\tilde{R}} \tilde{A}$.

(ii) If $\tilde{A} \leq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \leq_{\tilde{R}} \tilde{C}$, then $\tilde{A} \leq_{\tilde{R}} \tilde{C}$.

In Lemma 3.1, the property (i) and (ii) means reflexivity and transitivity respectively. In the one-dimensional case ($n = 1$), the order $\leq_{\tilde{R}}$ coincides with the fuzzy max order, and then $\sim_{\tilde{R}}$ is replaced with $=$. The following example gives the order $\leq_{\tilde{R}}$ induced from natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$ and shows that the order is not a total order.

**Example 3.1** (The order $\leq_{\tilde{R}}$ corresponding to natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$). We consider a case when $n = 2$ and we put an acute closed convex cone $R_\alpha = \mathbb{R}_+^2 = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^i \geq 0 (i = 1, 2)\}$ for all $\alpha \in [0, 1]$. Then the corresponding fuzzy relation is

$$\mu_{\tilde{R}}((x^1, x^2), (y^1, y^2)) = \begin{cases} 1 & \text{if } y^1 \geq x^1 \text{ and } y^2 \geq x^2 \\ 0 & \text{otherwise.} \end{cases}$$

(3.3)

For $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, the definition of the order $\tilde{A} \leq_{\tilde{R}} \tilde{B}$ is reduced to the following conditions (3.4) and (3.5):

$$\mu_{\tilde{A}}(x^1, x^2) \leq \sup_{(y^1, y^2) : y^1 \geq x^1, y^2 \geq x^2} \mu_{\tilde{B}}(y^1, y^2) \quad \text{for all } (x^1, x^2) \in \mathbb{R}^2 \times \mathbb{R}^2;$$

(3.4)

$$\mu_{\tilde{A}}(y^1, y^2) \leq \sup_{(x^1, x^2) : x^1 \geq y^1, x^2 \geq y^2} \mu_{\tilde{B}}(x^1, x^2) \quad \text{for all } (y^1, y^2) \in \mathbb{R}^2 \times \mathbb{R}^2.$$  

(3.5)

Take pyramid-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined by

$$\mu_{\tilde{A}}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1\}, 0\},$$

(3.6)

$$\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{1 - |x^1 - 1|, 1 - |x^2 + 1\}, 0\}.$$  

(3.7)

for $(x^1, x^2) \in \mathbb{R}^2$. Then we can easily check that neither $\tilde{A} \leq_{\tilde{R}} \tilde{B}$ nor $\tilde{B} \leq_{\tilde{R}} \tilde{A}$ hold.
4. Degree of the fuzzy order $\preceq_{\overline{R}}$

In this section, by using a subsethood degree, we present a method to evaluate the degree of satisfaction of the fuzzy order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ for all fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The fuzzy order $\preceq_{\overline{R}}$ is not a total order (Example 3.1), however we can apply this method to fuzzy sets which are incomparable by the order $\preceq_{\overline{R}}$. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, the subsethood degree is defined by ([5],[1])

$$\text{Sub}(\tilde{A} \subseteq \tilde{B}) := \frac{|\tilde{A} \cap \tilde{B}|}{|\tilde{A}|}$$  \hspace{1cm} (4.1)

if $|\tilde{A}| > 0$, where

$$|\tilde{A}| := \int_{\mathbb{R}^n} \mu_{\tilde{A}}(x) \, dx \quad (\tilde{A} \in \mathcal{F}(\mathbb{R}^n)).$$  \hspace{1cm} (4.2)

For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, in spirit of Definition 1 we define the degree of order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ by

$$D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) := \min\{\text{Sub}(\tilde{A} \subseteq \tilde{R} \circ \tilde{B}), \text{Sub}(\tilde{B} \subseteq \tilde{A} \circ \tilde{R})\}.$$  \hspace{1cm} (4.3)

Then, the degree of order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ is written as

$$D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = \min \left\{ \frac{|\tilde{A} \cap (\tilde{R} \circ \tilde{B})|}{|\tilde{A}|}, \frac{|\tilde{B} \cap (\tilde{A} \circ \tilde{R})|}{|\tilde{B}|} \right\} = \min \left\{ \frac{\int_{\mathbb{R}^n} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R} \circ \tilde{B}}(x)\} \, dx}{\int_{\mathbb{R}^n} \mu_{\tilde{A}}(x) \, dx}, \frac{\int_{\mathbb{R}^n} \min\{\mu_{\tilde{B}}(x), \mu_{\tilde{A} \circ \tilde{R}}(x)\} \, dx}{\int_{\mathbb{R}^n} \mu_{\tilde{B}}(x) \, dx} \right\}.$$  \hspace{1cm} (4.4)

The following lemma implies a correspondence between the order $\preceq_{\overline{R}}$ in Definition 1 and the degree of order $\preceq_{\overline{R}}$ in (4.4).

**Theorem 4.1.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The order $\tilde{A} \preceq_{\overline{R}} \tilde{B}$ holds if and only if $D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = 1$.

The degree of order $\preceq_{\overline{R}}$ has the following properties.

**Lemma 4.1.** Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda \geq 0$. Then, the following (i) - (ii) holds:

(i) $D(\tilde{A} \preceq_{\overline{R}} \tilde{A}) = 1$.

(ii) If $D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = 1$ and $D(\tilde{B} \preceq_{\overline{R}} \tilde{C}) = 1$, then $D(\tilde{A} \preceq_{\overline{R}} \tilde{C}) = 1$.

The following results are related to Theorem 3.1.

**Theorem 4.2.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, $z \in \mathbb{R}^n$ and $\lambda > 0$. Then

$$D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = D(\tilde{A} + \{z\} \preceq_{\overline{R}} \tilde{B} + \{z\}) = D(\lambda \tilde{A} \preceq_{\overline{R}} \lambda \tilde{B}).$$  \hspace{1cm} (4.5)
The following theorem is useful to calculate fuzzy sets $\tilde{R} \circ \tilde{B}$ and $\tilde{A} \circ \tilde{R}$ in the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and the degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ (Definition 1 and (4.4)).

**Theorem 4.3.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0,1]$. Then, the following (i) – (ii) holds:

(i) $(\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} - R_{\alpha}$;

(ii) $(\tilde{A} \circ \tilde{R})_{\alpha} = \tilde{A}_{\alpha} + R_{\alpha}$,

where $-R_{\alpha} = \{-x \mid x \in R_{\alpha}\}$.

We consider an example in the two-dimensional case to illustrate the meaning of the degree of order $\preceq_{\tilde{R}}$.

**Example 4.1** (Degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$). We consider the case when $n = 2$ and the acute closed convex cone $R_{\alpha} = \mathbb{R}^2_{+}$ for all $\alpha \in [0,1]$ in Example 3.1. First, for fuzzy sets $\tilde{A}$ and $\tilde{B}$ given by (3.6) and (3.7), we have $\tilde{A}_{\alpha} = [-2 + \alpha, -\alpha] \times [\alpha, 2 - \alpha]$, and $\tilde{B}_{\alpha} = [\alpha, 2 - \alpha] \times [-2 + \alpha, -\alpha]$. From Theorem 4.3, $(\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} - R_{\alpha} = (-\infty, 2 - \alpha] \times (-\infty, -\alpha]$ and $(\tilde{A} \circ \tilde{R})_{\alpha} = \tilde{A}_{\alpha} + R_{\alpha} = [-2 + \alpha, \infty) \times [\alpha, \infty)$. Clearly, $\tilde{A}_{\alpha} \cap (\tilde{R} \circ \tilde{B})_{\alpha} = \emptyset$ for all $\alpha \in (0,1]$, and so $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$. Similarly we can check $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = 0$. Next, we take polyhedral cone-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ by

\[\mu_{\tilde{A}}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1]\}, 0\}, \quad (4.6)\]

\[\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2/2, (3x^1 + \sqrt{3}x^2 + 6)/4, (-3x^1 + \sqrt{3}x^2 + 6)/4\}, 0\} \quad (4.7)\]

for $(x^1, x^2) \in \mathbb{R}^2$. Then similarly we can easily check $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = \min\{\text{Sub}(\tilde{B} \subseteq \tilde{R} \circ \tilde{A}), \text{Sub}(\tilde{A} \subseteq \tilde{B} \circ \tilde{R})\} = \min\{7/9, 1/3\} = 1/3$.

5. **Approximation for numerical calculation**

Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. In this section, using discrete cases, we discuss a method to approximate the degree of fuzzy order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ numerically. For simplicity, we deal with the case when $n = 2$ and fuzzy sets $\tilde{A}, \tilde{B}$ such that membership functions $\mu_{\tilde{A}}, \mu_{\tilde{B}}, \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}$ and $\mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}$ are continuous. In the discrete case, the subsethood degree ([1, p.28]) is defined by (4.1) with scalar cardinality

\[|\tilde{A}| := \sum_x \mu_{\tilde{A}}(x), \quad (5.1)\]

where the sum is taken over some finite set. Then, the degree of order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ in the discrete case is given by

\[\min\left\{\frac{\sum_x \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}(x)}{\sum_x \mu_{\tilde{A}}(x)}, \frac{\sum_x \mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}(x)}{\sum_x \mu_{\tilde{B}}(x)}\right\} \quad (5.2)\]

We approximate (4.4), using (5.2) which are easy to calculate numerically.
Let a region $C := [-c, c]^2$ such that $	ilde{A}_0 \cup \tilde{B}_0 \subseteq C = [-c, c]^2$ with $c > 0$. For $m = 1, 2, \cdots$, put a mesh $C^{(m)}$ of $C$ by $C^{(m)} := \{(x^1, x^2) | x^1 = ic/m, x^2 = jc/m, i, j = -m, -(m-1), \cdots, -1, 0, 1, \cdots, m-1, m\}$. Let $I_A := \int_{\mathbb{R}^2} \mu_A(x^1, x^2) \, dx^1 dx^2$ and define

\[ I^{(m)}_A := \sum_{(x^1, x^2) \in C^{(m)}} \mu_A(x^1, x^2) \left( \frac{c}{m} \right)^2. \]  

(5.3)

for $m = 1, 2, \cdots$, where $(x^1, x^2) = (ic/m, jc/m) \in C^{(m)}$. Then, the integrand $I_A := \int_{\mathbb{R}^2} \mu_A(x^1, x^2) \, dx^1 dx^2$ is approximated:

\[ I^{(m)}_A \rightarrow I_A = \int_{\mathbb{R}^2} \mu_A(x^1, x^2) \, dx^1 dx^2 \]

as $m \rightarrow \infty$. Here, we put an error $\varepsilon^{(m)}_A := |I_A - I^{(m)}_A|$ for $m = 1, 2, \cdots$. For fuzzy sets $\tilde{B}$, $\tilde{A} \cap (\tilde{R} \circ \tilde{B})$ and $\tilde{B} \cap (\tilde{A} \circ \tilde{R})$, we also define integrands, discrete approximations and errors similarly. Then we obtain the following estimation of errors. The degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ in the discrete case on $C^{(m)}$ is given by

\[ D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) := \min \left\{ \frac{\sum_{x \in C^{(m)}} \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}(x)}{\sum_{x \in C^{(m)}} \mu_{\tilde{A}}(x)}, \frac{\sum_{x \in C^{(m)}} \mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}(x)}{\sum_{x \in C^{(m)}} \mu_{\tilde{B}}(x)} \right\} 

(5.4)

for $m = 1, 2, \cdots$.

**Theorem 5.1.** For $m = 1, 2, \cdots$, it holds that

\[ |D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) - D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})| \leq \max \left\{ \frac{\varepsilon^{(m)}_A + \varepsilon^{(m)}_{A \cap (\tilde{R} \circ \tilde{B})}}{I_A - \varepsilon^{(m)}_A}, \frac{\varepsilon^{(m)}_B + \varepsilon^{(m)}_{B \cap (\tilde{A} \circ \tilde{R})}}{I_B - \varepsilon^{(m)}_B} \right\} \]

(5.5)

if $0 < \varepsilon^{(m)}_A < I_A$ and $0 < \varepsilon^{(m)}_B < I_B$.

Theorem 5.1 implies the convergence $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \rightarrow D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ as $m \rightarrow \infty$ and also gives an estimation of errors. Finally, we give an example to approximate the degree of order on $\mathcal{F}(\mathbb{R}^2)$ by Theorem 5.1.

**Example 5.1** (Approximation of the degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$). We consider the case when $n = 2$ and the acute closed convex cone $R_\alpha = \mathbb{R}^2_+$ for all $\alpha \in [0, 1]$. Take a helmet-type fuzzy set $\tilde{A}$ and a polyhedron-type fuzzy set $\tilde{B}$ by

\[ \mu_\tilde{A}(x^1, x^2) = \max\{1 - 0.25(x^1 + 1)^2 - 0.25(x^2 - 1)^2, 0\}, \]

(5.6)

\[ \mu_\tilde{B}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2, (3x^1 + \sqrt{3}x^2 + 6)/2, (-3x^1 + \sqrt{3}x^2 + 6)/2, 1\}, 0\} \]

(5.7)

for $(x^1, x^2) \in \mathbb{R}^2$. Then, we can approximate $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \approx \min\{0.0903, 0.2331\} = 0.0903$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) \approx \min\{0.5890, 0.5714\} = 0.5714$. 

Table 5.1. Approximation of degree of order $\preceq_{\overline{R}}$ in Example 5.1 ($n = 2$).

<table>
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<th>$m$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
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<td>0.08102</td>
<td>0.08781</td>
<td>0.08905</td>
<td>0.08962</td>
<td>0.08985</td>
<td>0.08999</td>
<td>0.09008</td>
<td>0.09014</td>
</tr>
<tr>
<td>$D(\tilde{B} \preceq_{\overline{R}}^{(m)} \tilde{A})$</td>
<td>0.56905</td>
<td>0.57100</td>
<td>0.57066</td>
<td>0.57139</td>
<td>0.57134</td>
<td>0.57125</td>
<td>0.57136</td>
<td>0.57136</td>
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References


