Degree of Order regarding Multidimensional Fuzzy Sets

1. Introduction and notations

Kurano et al. [4] studied a pseudo order regarding fuzzy sets on $\mathbb{R}^n$ on the basis of a set-relation in $\mathbb{R}^n$ studied by Kuroiwa et al. [2] and Kuroiwa [3] for multi-criteria crisp set-valued optimizations in mathematical programming. First, we introduce a fuzzy relation, which is fuzzy partial ordering, induced by closed convex cones. Next, a pseudo order regarding fuzzy sets on $\mathbb{R}^n$ is given by inclusions defined from the fuzzy relation, and it is also a reasonable multi-dimensional extension of the fuzzy max order regarding fuzzy numbers. For incomparable fuzzy sets on $\mathbb{R}^n$, we present a degree of order, using a subsethood degree. This method is flexible and can be applied to various types of fuzzy decision-making and optimization problems in multi-criteria. For example, we can apply the method to problems where components on $\mathbb{R}^n$ are related to each other in multi-criteria.

In the rest of this section, we give some notations and introduce some results regarding vector ordering on $\mathbb{R}^n$ by convex cones. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^n$ be an $n$-dimensional Euclidean space, where $n$ is a positive integer. We write fuzzy sets on $\mathbb{R}^n$ and their membership functions by $\tilde{A}$ and $\mu_{\tilde{A}} : \mathbb{R}^n \rightarrow [0,1]$ respectively. The $\alpha$-cut ($\alpha \in [0,1]$) of the fuzzy set $\tilde{A}$ on $\mathbb{R}^n$ is defined as

$$\tilde{A}_\alpha := \{x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \tilde{A}_0 := \text{cl}\{x \in \mathbb{R}^n \mid \mu_{\tilde{A}}(x) > 0\},$$

where cl denotes the closure of the set. A fuzzy set $\tilde{A}$ is called convex if the $\alpha$-cut $\tilde{A}_\alpha$ is a convex set for all $\alpha \in [0,1]$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all convex fuzzy sets $\tilde{A}$ whose membership functions $\mu_{\tilde{A}} : \mathbb{R}^n \rightarrow [0,1]$ are upper-semicontinuous and normal ($\sup_{x \in \mathbb{R}^n} \mu_{\tilde{A}}(x) = 1$) and have a compact 0-cut. When one-dimensional case ($n = 1$), the fuzzy sets are called fuzzy numbers and $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. In this paper, we deal with fuzzy sets on $\mathbb{R}^n$ as a multi-dimensional extension of fuzzy numbers. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all non-empty compact convex subsets of $\mathbb{R}^n$.

The definitions of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are as follows: For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and a scalar $\lambda \geq 0$, the sum $\tilde{A} + \tilde{B}$ and scalar multiplication $\lambda \tilde{A}$ are given by applying Zaheh’s extension principle:

$$\mu_{\tilde{A} + \tilde{B}}(x) := \sup_{y,z \in \mathbb{R}^n : y+z=x} \min\{\mu_{\tilde{A}}(y), \mu_{\tilde{B}}(z)\}, \quad (1.1)$$

$$\mu_{\lambda \tilde{A}}(x) := \begin{cases} \mu_{\tilde{A}}(x/\lambda) & \text{if } \lambda > 0 \\ 1_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (1.2)$$

for $x \in \mathbb{R}^n$, where $1_{\{\cdot\}}(\cdot)$ is an indicator and $\{0\}$ denotes the crisp set of zero in $\mathbb{R}^n$. By using set operations $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for
$A, B \in C(\mathbb{R}^n)$, the following holds immediately:

$$(\tilde{A} + \tilde{B})_\alpha := \tilde{A}_\alpha + \tilde{B}_\alpha \quad \text{and} \quad (\lambda \tilde{A})_\alpha = \lambda \tilde{A}_\alpha \quad (\alpha \in [0, 1]).$$

(1.3)

Let $K$ be a non-empty convex cone of $\mathbb{R}^n$, i.e. $x + y \in K$ and $\lambda x \in K$ hold for all $\lambda \geq 0$ and all $x, y \in K$. Using a convex cone $K$, we can define a pseudo order $\preceq_K$ on $\mathbb{R}^n$ by

(K.1) $x \preceq_K y$ means that $y - x \in K$.

Let $\mathbb{R}_+^n = \{x = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n \mid x^i \geq 0 (i = 1, 2, \cdots, n)\}$ be the subset of entrywise nonnegative elements in $\mathbb{R}^n$. When $K = \mathbb{R}_+^n$, the order $x \preceq_{\mathbb{R}_+^n} y$ means that $x^i \leq y^i$ for all $i = 1, 2, \cdots, n$, where $x = (x^1, x^2, \cdots, x^n)$ and $y = (y^1, y^2, \cdots, y^n) \in \mathbb{R}^n$.

2. Fuzzy partial ordering by convex cones

In this section, we discuss a fuzzy relation induced by closed convex cones. Let $R_\alpha (\alpha \in [0, 1])$ be non-empty closed convex cones on $\mathbb{R}^n$ such that $\bigcap_{\alpha' \in (0, \alpha)} R_{\alpha'} = R_\alpha$ for $\alpha \in (0, 1]$. To avoid meaningless ordering, we assume $R_\alpha (\alpha \in (0, 1])$ are acute, i.e. there exists $a \in \mathbb{R}^n$ satisfying $a \cdot x > 0$ for all $x \in R_\alpha$ with $x \neq 0$, where $\cdot$ means the inner product of vectors on $\mathbb{R}^n$. We also put the support set by $R_0 := \text{cl}(\bigcup_{\alpha \in [0, 1]} R_\alpha)$. Define a fuzzy relation $\tilde{R}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by closed convex cones $R_\alpha$ as follows.

$$\mu_{\tilde{R}}(x, y) := \sup\{\alpha \in [0, 1] \mid y - x \in R_\alpha\} \quad ((x, y) \in \mathbb{R}^n \times \mathbb{R}^n),$$

(2.1)

where $\sup \emptyset := 0$. Then the $\alpha$-cut $\tilde{R}_\alpha$ is

$$\tilde{R}_\alpha = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mu_{\tilde{R}}(x, y) \geq \alpha\} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y - x \in R_\alpha\}$$

(2.2)

for $\alpha \in (0, 1]$. The fuzzy relation $\tilde{R}$ has the following properties.

**Theorem 2.1.** $\tilde{R}$ is a fuzzy partial ordering ([1]), i.e., it satisfies the following (i) – (iii):

(i) $\mu_{\tilde{R}}(x, x) = 1$ for all $x \in \mathbb{R}^n$.

(ii) $\mu_{\tilde{R}}(x, \bar{z}) \geq \min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{R}}(y, \bar{z})\}$ for all $x, y, \bar{z} \in \mathbb{R}^n$.

(iii) If $\mu_{\tilde{R}}(x, y) > 0$ and $\mu_{\tilde{R}}(y, x) > 0$, then $x = y$.

In Theorem 2.1, the property (i), (ii) and (iii) means reflexivity, transitivity and antisymmetry respectively.
3. Ordering of fuzzy quantities by the fuzzy relation $\tilde{R}$

In this section, we introduce an order for fuzzy sets on $\mathbb{R}^n$, using the fuzzy relation $\tilde{R}$ from Section 2. For fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, using the sup-min composition operation with $\tilde{R}$ ([1]), we define fuzzy sets $\tilde{A} \circ \tilde{R}, \tilde{R} \circ \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ by

$$
\mu_{\tilde{R} \circ \tilde{B}}(x) := \sup_{y \in \mathbb{R}^n} \min\{\mu_\tilde{R}(x, y), \mu_\tilde{B}(y)\} \quad (x \in \mathbb{R}^n),
$$

(3.1)

$$
\mu_{\tilde{A} \circ \tilde{R}}(y) := \sup_{x \in \mathbb{R}^n} \min\{\mu_\tilde{A}(x), \mu_\tilde{R}(x, y)\} \quad (y \in \mathbb{R}^n).
$$

(3.2)

**Definition 1.** Consider fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. The order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ means that $\tilde{A} \subseteq \tilde{R} \circ \tilde{B}$ and $\tilde{B} \subseteq \tilde{A} \circ \tilde{R}$.

Further, the equivalence $\tilde{A} \sim_{\tilde{R}} \tilde{B}$ means that $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{A}$. We also write $\tilde{A} \prec_{\tilde{R}} \tilde{B}$ when $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{A} \not\preceq_{\tilde{R}} \tilde{B}$.

**Lemma 3.1.** The order $\preceq_{\tilde{R}}$ is a pseudo order: For fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n)$, the following (i) and (ii) hold.

(i) If $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{C}$, then $\tilde{A} \preceq_{\tilde{R}} \tilde{C}$.

(ii) If $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and $\tilde{B} \preceq_{\tilde{R}} \tilde{C}$, then $\tilde{A} \preceq_{\tilde{R}} \tilde{C}$.

In Lemma 3.1, the property (i) and (ii) means reflexivity and transitivity respectively. In the one-dimensional case $\alpha = 1$, the order $\preceq_{\tilde{R}}$ coincides with the fuzzy max order, and then $\sim_{\tilde{R}}$ is replaced with $\preceq_{\tilde{R}}$. The following example gives the order $\preceq_{\tilde{R}}$ induced from natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$ and shows that the order $\preceq_{\tilde{R}}$ is not a total order.

**Example 3.1** (The order $\preceq_{\tilde{R}}$ corresponding to natural fuzzy relations on $\mathbb{R}^2 \times \mathbb{R}^2$). We consider a case when $n = 2$ and we put an acute closed convex cone $R_\alpha = \mathbb{R}^2_+ = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^i \geq 0(i = 1, 2)\}$ for all $\alpha \in [0, 1]$. Then the corresponding fuzzy relation is

$$
\mu_\tilde{R}(x^1, x^2), (y^1, y^2)) = \begin{cases} 
1 & \text{if } y^1 \geq x^1 \text{ and } y^2 \geq x^2 \\
0 & \text{otherwise}.
\end{cases}
$$

(3.3)

For $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$, the definition of the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ is reduced to the following conditions (3.4) and (3.5):

$$
\mu_\tilde{A}(x^1, x^2) \leq \sup_{(y^1, y^2) : y^1 \geq x^1, y^2 \geq x^2} \mu_\tilde{B}(y^1, y^2) \quad \text{for all } (x^1, x^2) \in \mathbb{R}^2 \times \mathbb{R}^2;
$$

(3.4)

$$
\mu_\tilde{B}(y^1, y^2) \leq \sup_{(x^1, x^2) : x^1 \geq y^1, x^2 \geq y^2} \mu_\tilde{A}(x^1, x^2) \quad \text{for all } (y^1, y^2) \in \mathbb{R}^2 \times \mathbb{R}^2.
$$

(3.5)

Take pyramid-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined by

$$
\mu_\tilde{A}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1|\}, 0\},
$$

(3.6)

$$
\mu_\tilde{B}(x^1, x^2) = \max\{\min\{1 - |x^1 - 1|, 1 - |x^2 + 1|\}, 0\}
$$

(3.7)

for $(x^1, x^2) \in \mathbb{R}^2$. Then we can easily check that neither $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ nor $\tilde{B} \preceq_{\tilde{R}} \tilde{A}$ hold.
4. Degree of the fuzzy order \( \preceq_{\overline{R}} \)

In this section, by using a subsethood degree, we present a method to evaluate the degree of satisfaction of the fuzzy order \( \tilde{A} \preceq_{\overline{R}} \tilde{B} \) for all fuzzy sets \( \tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n) \). The fuzzy order \( \preceq_{\overline{R}} \) is not a total order (Example 3.1), however we can apply this method to fuzzy sets which are incomparable by the order \( \preceq_{\overline{R}} \). For fuzzy sets \( \tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n) \), the subsethood degree is defined by ([5],[1])

\[
\text{Sub}(\tilde{A} \subseteq \tilde{B}) := \frac{|\tilde{A} \cap \tilde{B}|}{|\tilde{A}|} \tag{4.1}
\]

if \( |\tilde{A}| > 0 \), where

\[
|\tilde{A}| := \int_{\mathbb{R}^n} \mu_{\overline{A}}(x) \, dx \quad (\tilde{A} \in \mathcal{F}(\mathbb{R}^n)). \tag{4.2}
\]

For fuzzy sets \( \tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n) \), in spirit of Definition 1 we define the degree of order \( \tilde{A} \preceq_{\overline{R}} \tilde{B} \) by

\[
D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) := \min\{\text{Sub}(\tilde{A} \subseteq \tilde{R} \circ \tilde{B}), \text{sub}(\tilde{B} \subseteq \tilde{A} \circ \tilde{R})\}. \tag{4.3}
\]

Then, the degree of order \( \tilde{A} \preceq_{\overline{R}} \tilde{B} \) is written as

\[
D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = \min \left\{ \frac{|\tilde{A} \cap (\tilde{R} \circ \tilde{B})|}{|\tilde{A}|}, \frac{|\tilde{B} \cap (\tilde{A} \circ \tilde{R})|}{|\tilde{B}|} \right\} = \min \left\{ \int_{\mathbb{R}^n} \min\{\mu_{\overline{A}}(x), \mu_{\tilde{R} \circ \overline{B}}(x)\} \, dx, \int_{\mathbb{R}^n} \min\{\mu_{\overline{B}}(x), \mu_{\overline{A} \circ \overline{R}}(x)\} \, dx \right\}. \tag{4.4}
\]

The following lemma implies a correspondence between the order \( \preceq_{\overline{R}} \) in Definition 1 and the degree of order \( \preceq_{\overline{R}} \) in (4.4).

**Theorem 4.1.** Let \( \tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n) \). The order \( \tilde{A} \preceq_{\overline{R}} \tilde{B} \) holds if and only if \( D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = 1 \).

The degree of order \( \preceq_{\overline{R}} \) has the following properties.

**Lemma 4.1.** Let \( \tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{F}(\mathbb{R}^n) \) and \( \lambda \geq 0 \). Then, the following (i) – (ii) holds:

(i) \( D(\tilde{A} \preceq_{\overline{R}} \tilde{A}) = 1 \).

(ii) If \( D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = 1 \) and \( D(\tilde{B} \preceq_{\overline{R}} \tilde{C}) = 1 \), then \( D(\tilde{A} \preceq_{\overline{R}} \tilde{C}) = 1 \).

The following results are related to Theorem 3.1.

**Theorem 4.2.** Let \( \tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n) \), \( z \in \mathbb{R}^n \) and \( \lambda > 0 \). Then

\[
D(\tilde{A} \preceq_{\overline{R}} \tilde{B}) = D(\tilde{A} + \{z\} \preceq_{\overline{R}} \tilde{B} + \{z\}) = D(\lambda \tilde{A} \preceq_{\overline{R}} \lambda \tilde{B}). \tag{4.5}
\]
The following theorem is useful to calculate fuzzy sets $\tilde{R} \circ \tilde{B}$ and $\tilde{A} \circ \tilde{R}$ in the order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ and the degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ (Definition 1 and (4.4)).

**Theorem 4.3.** Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$ and $\alpha \in (0, 1]$. Then, the following (i) – (ii) hold:

(i) $(\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} - R_{\alpha}$;

(ii) $(\tilde{A} \circ \tilde{R})_{\alpha} = \tilde{A}_{\alpha} + R_{\alpha}$,

where $-R_{\alpha} = \{-x \mid x \in R_{\alpha}\}$.

We consider an example in the two-dimensional case to illustrate the meaning of the degree of order $\preceq_{\tilde{R}}$.

**Example 4.1 (Degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$).** We consider the case when $n = 2$ and the acute closed convex cone $R_{\alpha} = \mathbb{R}_+^2$ for all $\alpha \in [0, 1]$ in Example 3.1. First, for fuzzy sets $\tilde{A}$ and $\tilde{B}$ given by (3.6) and (3.7), we have $\tilde{A}_{\alpha} = [-2 + \alpha, -\alpha] \times [\alpha, 2 - \alpha]$ and $\tilde{B}_{\alpha} = [\alpha, 2 - \alpha] \times [-2 + \alpha, -\alpha]$. From Theorem 4.3, $(\tilde{R} \circ \tilde{B})_{\alpha} = \tilde{B}_{\alpha} - R_{\alpha} = (-\infty, 2 - \alpha] \times (-\infty, -\alpha]$ and $(\tilde{A} \circ \tilde{R})_{\alpha} = \tilde{A}_{\alpha} + R_{\alpha} = [-2 + \alpha, \infty] \times [\alpha, \infty)$. Clearly, $\tilde{A}_{\alpha} \cap (\tilde{R} \circ \tilde{B})_{\alpha} = \emptyset$ for all $\alpha \in (0, 1]$, and so $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$. Similarly we can check $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = 0$. Next, we take polyhedral cone-type fuzzy sets $\tilde{A}$ and $\tilde{B}$ by

\[
\mu_{\tilde{A}}(x^1, x^2) = \max\{\min\{1 - |x^1 + 1|, 1 - |x^2 - 1|\}, 0\},
\]

\[
\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2/2, (3x^1 + \sqrt{3}x^2 + 6)/4, (-3x^1 + \sqrt{3}x^2 + 6)/4\}, 0\}
\]

for $(x^1, x^2) \in \mathbb{R}^2$. Then similarly we can easily check $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) = 0$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) = \min\{|\text{Sub}(\tilde{B} \subseteq \tilde{R} \circ \tilde{A}), \text{Sub}(\tilde{A} \subseteq \tilde{B} \circ \tilde{R})\} = \min\{7/9, 1/3\} = 1/3$.

### 5. Approximation for numerical calculation

Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R}^n)$. In this section, using discrete cases, we discuss a method to approximate the degree of fuzzy order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ numerically. For simplicity, we deal with the case when $n = 2$ and fuzzy sets $\tilde{A}, \tilde{B}$ such that membership functions $\mu_{\tilde{A}}, \mu_{\tilde{B}}, \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}$ and $\mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}$ are continuous. In the discrete case, the subsethood degree ([1, p.28]) is defined by (4.1) with scalar cardinality

\[
|\tilde{A}| := \sum_x \mu_{\tilde{A}}(x),
\]

where the sum is taken over some finite set. Then, the degree of order $\tilde{A} \preceq_{\tilde{R}} \tilde{B}$ in the discrete case is given by

\[
\min\left\{ \frac{\sum_x \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}(x)}{\sum_x \mu_{\tilde{A}}(x)}, \frac{\sum_x \mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}(x)}{\sum_x \mu_{\tilde{B}}(x)} \right\}.
\]

We approximate (4.4), using (5.2) which are easy to calculate numerically.
Let a region $C := [-c, c]^2$ such that $\tilde{A}_0 \cup \tilde{B}_0 \subseteq C = [-c, c]^2$ with $c > 0$. For $m = 1, 2, \cdots$, put a mesh $C^{(m)}$ of $C$ by $C^{(m)} := \{(x^1, x^2) \mid x^1 = ic/m, x^2 = jc/m, i, j = -m, -(m-1), \cdots, -1, 0, 1, \cdots, m-1, m\}$. Let $I_{\tilde{A}} := \int_{\mathbb{R}^2} \mu_{\tilde{A}}(x^1, x^2) \, dx^1 \, dx^2$ and define

$$I^{(m)}_{\tilde{A}} := \sum_{(x^1, x^2) \in C^{(m)}} \mu_{\tilde{A}}(x^1, x^2) \left(\frac{c}{m}\right)^2 = \sum_{i=-m}^{m} \sum_{j=-m}^{m} \mu_{\tilde{A}}(x^1_{i,m}, x^2_{j,m}) \left(\frac{c}{m}\right)^2$$

(5.3)

for $m = 1, 2, \cdots$, where $(x^1_{i,m}, x^2_{j,m}) = (ic/m, jc/m) \in C^{(m)}$. Then, the integrand

$I_{\tilde{A}} := \int_{\mathbb{R}^2} \mu_{\tilde{A}}(x^1, x^2) \, dx^1 \, dx^2$ is approximated:

$$I^{(m)}_{\tilde{A}} \to I_{\tilde{A}} := \int_{C} \mu_{\tilde{A}}(x^1, x^2) \, dx^1 \, dx^2 = \int_{\mathbb{R}^2} \mu_{\tilde{A}}(x^1, x^2) \, dx^1 \, dx^2$$

as $m \to \infty$. Here, we put an error $\varepsilon^{(m)}_{\tilde{A}} := |I_{\tilde{A}} - I^{(m)}_{\tilde{A}}|$ for $m = 1, 2, \cdots$. For fuzzy sets $\tilde{B}$, $\tilde{A} \cap (\tilde{R} \circ \tilde{B})$ and $\tilde{B} \cap (\tilde{A} \circ \tilde{R})$, we also define integrands, discrete approximations and errors similarly. Then we obtain the following estimation of errors. The degree of order $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ in the discrete case on $C^{(m)}$ is given by

$$D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) := \min\left\{\frac{\sum_{x \in C^{(m)}} \mu_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}(x)}{\sum_{x \in C^{(m)}} \mu_{\tilde{A}}(x)}, \frac{\sum_{x \in C^{(m)}} \mu_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}(x)}{\sum_{x \in C^{(m)}} \mu_{\tilde{B}}(x)}\right\}$$

(5.4)

for $m = 1, 2, \cdots$.

**Theorem 5.1.** For $m = 1, 2, \cdots$, it holds that

$$|D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) - D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})| \leq \max\left\{\frac{\varepsilon^{(m)}_{\tilde{A}} + \varepsilon^{(m)}_{\tilde{A} \cap (\tilde{R} \circ \tilde{B})}}{I_{\tilde{A}} - \varepsilon^{(m)}_{\tilde{A}}}, \frac{\varepsilon^{(m)}_{\tilde{B}} + \varepsilon^{(m)}_{\tilde{B} \cap (\tilde{A} \circ \tilde{R})}}{I_{\tilde{B}} - \varepsilon^{(m)}_{\tilde{B}}}\right\}$$

(5.5)

if $0 < \varepsilon^{(m)}_{\tilde{A}} < I_{\tilde{A}}$ and $0 < \varepsilon^{(m)}_{\tilde{B}} < I_{\tilde{B}}$.

Theorem 5.1 implies the convergence $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \to D(\tilde{A} \preceq_{\tilde{R}} \tilde{B})$ as $m \to \infty$ and also gives an estimation of errors. Finally, we give an example to approximate the degree of order on $\mathcal{F}(\mathbb{R}^2)$ by Theorem 5.1.

**Example 5.1 (Approximation of the degree of order $\preceq_{\tilde{R}}$ on $\mathcal{F}(\mathbb{R}^2)$).** We consider the case when $n = 2$ and the acute closed convex cone $R_\alpha = \mathbb{R}_+^2$ for all $\alpha \in [0, 1]$. Take a helmet-type fuzzy set $\tilde{A}$ and a polyhedron-type fuzzy set $\tilde{B}$ by

$$\mu_{\tilde{A}}(x^1, x^2) = \max\{1 - 0.25(x^1 + 1)^2 - 0.25(x^2 - 1)^2, 0\}, \quad (5.6)$$

$$\mu_{\tilde{B}}(x^1, x^2) = \max\{\min\{-\sqrt{3}x^2, 3x^1 + \sqrt{3}x^2 + 6\}/2, (-3x^1 + \sqrt{3}x^2 + 6)/2, 1\}, 0\} \quad (5.7)$$

for $(x^1, x^2) \in \mathbb{R}^2$. Then, we can approximate $D(\tilde{A} \preceq_{\tilde{R}} \tilde{B}) \approx \min\{0.0903, 0.2331\} = 0.0903$ and $D(\tilde{B} \preceq_{\tilde{R}} \tilde{A}) \approx \min\{0.5890, 0.5714\} = 0.5714$. 


Table 5.1. Approximation of degree of order $\preceq_{R}$ in Example 5.1 ($n = 2$).

<table>
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<th>10</th>
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<td>.08999</td>
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<td>.09014</td>
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<td>.57066</td>
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<td>.57134</td>
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References


