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AHP による区間ウェイト制約を導入した DEA

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Abstract:
AHP is proposed to give the importance grade with respect to many items. However, a decision maker tends to give the inconsistent information about the importance grade of input and output items. Then a comparison matrix obtained by a decision maker has inconsistent elements. Therefore to deal with a decision maker's inconsistency, interval AHP, where the importance grade of the item is given as an interval, is proposed. In this paper we also assume that a decision maker's inconsistency is represented as an interval. Its center is obtained by eigenvector method and its radius is obtained by interval regression analysis using the obtained centers. To choose the crisp importance grades and the crisp efficinency in the decision maker's judgement, we use DEA, which is an evaluation method from the optimistic viewpoint with respect to many input and output items. The weight in DEA and the importance grade through AHP are similar and we normalize data in order to make the weight in DEA itself represent the importance grade in AHP.

keywords:DEA, AHP, Interval importance grades

1 Introduction

The efficiency is considered as the ratio of weighted sum of output data to that of input data. It is natural to take the importance grade of an item as its weight. However, it is not usually easy for a decision maker to give the determined importance grade directly. Therefore, AHP (Analytic Hierarchical Process) is proposed to determine the importance grades of each item [1]. AHP is a method to deal with the importance grades with respect to many items. In conventional AHP, the crisp importance grade of each item can be obtained by solving eigenvector problem with a comparison matrix whose elements are given by a decision maker by comparing all possible pairs of items. Based on the idea that a human judgement is inconsistent, the estimated weights should contains uncertainty. Thus, the model that gives the importance grade as an interval to reflect the inconsistency of a comparison matrix is proposed [2].

We take another way to obtain the interval importance grades based on eigenvector method and interval regression analysis. Assuming that the estimated weight is an interval denoted by its center and its radius, two problems for finding the center and the radius are formulated. The centers are obtained by eigenvector method in the same way as conventional AHP. Using the obtained centers, the radius is obtained by interval regression analysis where each radius is minimized subject to the constraint conditions that the estimated intervals include the elements of the given comparison matrix [3]. When a decision maker gives comparison matrices for input and output items, the interval importance grades of input and output items are obtained respectively. The obtained interval importance grades can be considered as the acceptable importance grades for a decision maker.

To give the crisp efficiency, we choose the most optimistic importance grades for the analyzed object in the interval by DEA (Data Envelopment Analysis) [4][5]. DEA is a well-known method to evaluate DMUs (Decision Making Units) from the optimistic viewpoint. The weights in DEA and the importance grades through AHP are similar then DEA is used to choose the most optimistic importance grades of input and output items in the decision maker's acceptable ranges. In order to make the weight in DEA represent the importance grade through AHP itself, we normalize all data based on DMU_o. The efficiencies obtained from the normalized data and the original data are equal. The study with respect to combination of AHP and DEA was done in [6], where the interval importance grades are obtained through interval AHP with an interval comparison matrix and they are introduced to DEA as the weight constraints.

Our proposed approaches to obtain the interval importance grades and to introduce them to DEA are different from the study [6] in the sense of that our aim is to choose the importance grades in a possible ranges which are estimated from a decision maker's judgement.

2 Interval AHP

When there are n items $I_1, ..., I_n$, a decision maker compares a pair of items for all possible pairs then we can obtain a comparison matrix A as follows.

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & 1 \end{pmatrix}$$

where the element of matrix A, a_{ij}, shows the importance grade of I_i obtained by comparing with I_j, the orthogonal elements are equal to 1, that is $a_{ii} = 1$ and the reciprocal property is satisfied, that is $a_{ij} = 1/a_{ji}$.

The more the number of compared items become, the more difficult it is to give consistent comparison values, since a decision maker compare only two items at one time. The obtained comparison matrix has inconsistent elements each other. Therefore, it is more suitable to give the items interval importance grades and partial order of the items is obtained by them. Then, we estimate the importance grade of item i, as an interval denoted as W_i, that is determined by its center w_i^c and its radius d_i as follows.

$$W_i = [{}^L w_i, {}^U w_i] = [w_i^c - d_i, w_i^c + d_i]$$

where ${}^L w_i$ and ${}^U w_i$ are the upper and the lower bounds of the interval. In order to determine interval importance grades, we have two problems where one is to obtain the center and the other is to obtain the radius. The center is obtained by eigenvector method with the obtained comparison matrix A. The eigenvector problem is formulated as follows.

$$Aw = \lambda w \tag{1}$$

where λ is the eigenvalue, w is the eigenvector and they are the decision variables of this probelm. Solving (1), the eigenvector $(w_1^c, \ldots, w_n^c)$ for the principal eigenvalue λ_{max} is obtained as the center of the interval importance grades of each item $(I_1, \ldots, I_n)$. The center w_i^{c*} is normalized to be $\sum_{i=1}^n w_i^{c*} = 1$.

The radius is obtained based on interval regression analysis, which is to find the estimated intervals to include the original data. In our problem, a_{ij} is approximated as an interval ratio such that the following relation holds.

$$a_{ij} \in \frac{W_i}{W_j} = \left[\frac{w_i^{c*} - d_i}{w_j^{c*} + d_j}, \frac{w_i^{c*} + d_i}{w_j^{c*} - d_j}\right] \tag{2}$$

where W_i and W_j are the estimated inteval importance grades and W_i/W_j is defined as the maximum range.

The interval importance grades are determined in consideration of the inconsistency contained in a comparison matrix. With using the obtained centers w_i^{c*} by (1), the radius should be minimized subject to the constraint conditions that the relation (2) for all elements should be satisfied.

$$\begin{aligned} & \min \quad \lambda \\ \text{s.t.} \quad & \frac{w_i^{c*} - d_i}{w_j^{c*} + d_j} \le a_{ij} \le \frac{w_i^{c*} + d_i}{w_j^{c*} - d_j}, \\ & \quad i = 1, \ldots, n-1, \; j = i+1, \ldots, n \\ & d_i \le \lambda, \quad i = 1, \ldots, n \end{aligned} \tag{3}$$

The first constraint condition shows the inclusion relation (2). Instead of minimizing the sum of radii, we minimize the maximun of them. This can be reduced to LP problem. The radius of the interval importance grades reflect some inconsistency in the given matrix. In other words, the obtained importance grades can be regarded as the possible ranges estimated from the given data. The interval importance grade shows the acceptable range for a decision maker.

3 Choice of the optimistic weights and efficiency by DEA

3.1 DEA with the normalized data

In DEA the maximum ratio of output data to input data is assumed as the efficiency and it is calculated from the optimistic viewpoint for each DMU. The basic DEA model is formulated as following LP problem.

$$\begin{aligned} \theta_o^{E*} = \max_{\boldsymbol{u}} & \; \boldsymbol{u}^t \boldsymbol{y}_o \\ \text{s.t.} \quad \boldsymbol{v}^t \boldsymbol{x}_o & = 1 \\ -\boldsymbol{v}^t X + \boldsymbol{u}^t Y & \le 0 \\ \boldsymbol{u} & \ge 0 \\ \boldsymbol{v} & \ge 0 \end{aligned} \tag{4}$$

where the decision variables are the weight vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $X \in \Re^{m \times n}$ and $Y \in \Re^{k \times n}$ are input and output matrices consisting of all input and output vectors that are all positive and the number of DMUs is n. (4) gives the optimistic weights, $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ for DMU_o and the efficiency is obtaind by them. In case that the optimal value of the objective function is equal to 1, the optimal weights are not determined identically.

In conventional DEA (4), it is difficult to discuss the importance grades of input and output items by comparing their weights, because they depend on the scales of the original data. The efficiency is obtained as the ratio of the hypothetical output to the hypothetical input, where the products of data and weights are summed up. It can be said that the product of data and weight represents the importance grade in evaluation more exactly than the weights only. Then we normalize the given input and output data based on DMU_o so that the input and output weights represent the importance grades of the items.

The normalized input and output denoted as $\hat{x}_{jp}$ and $\hat{y}_{jr}$, $(j=1,\ldots,n)$ are obtained as follows.

$$\begin{aligned}\hat{x}_{jp} &= x_{jp}/x_{op}, \quad p=1,\ldots,m\\ \hat{y}_{jr} &= y_{jr}/y_{or}, \quad r=1,\ldots,k\end{aligned}$$

The problem to obtain the efficiency with the normalized input and output are formulated as follows.

$$\begin{aligned}\theta_o^{E*} = \ & \max_{\boldsymbol{u}} \boldsymbol{u}^t\hat{\boldsymbol{y}}_o\\ \text{s.t.}\ & \boldsymbol{v}^t\hat{\boldsymbol{x}}_o = 1\\ & -\boldsymbol{v}^t\hat{X} + \boldsymbol{u}^t\hat{Y} \leq \mathbf{0}\\ & \boldsymbol{u} \geq \mathbf{0}\\ & \boldsymbol{v} \geq \mathbf{0}\end{aligned} \tag{5}$$

where $\hat{X}, \hat{Y}$ are the all normalized data and denoted as follows.

$$\begin{array}{c} \quad\ 1 \qquad\quad o \qquad\quad n \\ \hat{X} = \begin{pmatrix} \hat{x}_{11} & \cdots & 1 & \cdots & \hat{x}_{1n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \hat{x}_{m1} & \cdots & 1 & \cdots & \hat{x}_{mn}\end{pmatrix}\end{array}$$

$$\hat{Y} = \begin{pmatrix} \hat{y}_{11} & \cdots & 1 & \cdots & \hat{y}_{1n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \hat{y}_{k1} & \cdots & 1 & \cdots & \hat{y}_{kn}\end{pmatrix}$$

The efficiency from the normalized input and output is equal to that from the original data by conventional DEA. This fact can be verified by simple calculation.

Assuming the optimal solutions of (5) as $\boldsymbol{u}^* = ({u_1}^*,\ldots,{u_k}^*)^t$ and $\boldsymbol{v}^* = ({v_1}^*,\ldots,{v_m}^*)^t$, the following relation can be easily found.

$$\begin{aligned}\theta_o^{E*} &= {u_1}^* + \cdots + {u_k}^*\\ {v_1}^* &+ \cdots + {v_m}^* = 1\end{aligned}$$

The above two equations follow that the obtained weight represents the importance grade itself. Then we can use DEA with the normalized data to choose the optimistic weight in the interval importance grade obtained by a decision maker through interval AHP.

3.2 Optimistic importance grades in interval importance grades

The importance grades of output and input items obtained by comparison matrices given by a decision maker are calculated as the following intervals through interval AHP.

$$\begin{aligned}W_p^{in} &= [{}^L w_p^{in}, {}^U w_p^{in}], \quad p=1,\cdots,m\\ W_r^{out} &= [{}^L w_r^{out}, {}^U w_r^{out}], \quad r=1,\cdots,k\end{aligned}$$

The centers of the interval importance grades of input and output items through AHP sum up to one. On the other hand in DEA with the normalized data, input and output weights sum up to one and the efficiency respectively. We obtain the optimistic weights and efficiency through DEA by considering the interval importance grades through interval AHP as the weight constraints in DEA. By DEA, we can determine the optimistic weights for eachc DMU in the possible ranges. The input weights are constrained by the obtained interval importance grades directly and we need to modify the output weights so that the sum of them should be one because the obtained importance grades sum up to one. The constraint conditions for the input and output weights are as follows.

$$\begin{aligned}{}^L w_r^{out} &\leq \frac{u_r}{\sum_{r=1}^k u_r} \leq {}^U w_r^{out}, \quad r=1,\cdots,k\\ {}^L w_p^{in} &\leq v_p \leq {}^U w_p^{in}, \quad p=1,\cdots,m\end{aligned} \tag{6}$$

where u_r and v_p are the variables in DEA and ${}^L w_r^{out}$, ${}^U w_r^{out}$, ${}^L w_p^{in}$ and ${}^U w_p^{in}$ are the bounds of the interval importance grades of input p and output r. The problem to choose the most optimistic weights for DMU_o in a decision maker's judgement is formulated as follows by adding (6) to (5) as the constraint conditions.

$$\begin{aligned}\theta_o^{E*} &= \max_{\boldsymbol{u}} \textstyle\sum_{r=1}^k u_r\\ \text{s.t.}\ & \textstyle\sum_{p=1}^m v_p = 1\\ & -\boldsymbol{v}^t\hat{X} + \boldsymbol{u}^t\hat{Y} \leq \mathbf{0}\\ & u_r \geq \left(\textstyle\sum_{r=1}^k u_r\right) {}^L w_r^{out}, \quad r=1,\ldots,k\\ & u_r \leq \left(\textstyle\sum_{r=1}^k u_r\right) {}^U w_r^{out}, \quad r=1,\ldots,k\\ & v_p \geq {}^L w_p^{in}, \quad p=1,\ldots,m\\ & v_p \leq {}^U w_p^{in}, \quad p=1,\ldots,m\\ & \boldsymbol{u} \geq \mathbf{0}\end{aligned} \tag{7}$$

By data normalization, the interval importance grades are used as the weight constraints naturally, because the weight itself represents the importance grade. In case that the efficiency is equal to one, the weights are not determined identically, even though any weights that give the efficiency are in the intervals obtained by a decision maker.

4 Numericl example

We use one input and four outputs data shown in Table 1 as an example where A,...,J are denoted as DMUs.

	x_1	y_1	y_2	y_3	y_4
A	1	1	8	1	3
B	1	2	3	4	4
C	1	2	6	6	1
D	1	3	3	5	5
E	1	3	7	4	2
F	1	4	2	3	1
G	1	4	5	5	3
H	1	5	2	1	6
I	1	6	2	7	1
J	1	7	1	2	5

Table 1: Data with 1-input and 4-output

The comparison matrix given by a decision maker is shown in Table 2. By the eigenvector problem (1), the centers of the importance grades of output items are obtained as follows.

$$(w_1^{c*}, w_2^{c*}, w_3^{c*}, w_4^{c*}) = (0.080, 0.583, 0.051, 0.286)$$

The radius is obtained by (3) and the interval importance grades are also shown in Table 2.

With using the normalized data, the efficiencies obtained by the proposed model (7) and by conventional DEA (5) are shown in Table 3. The efficiency through DEA is determined only from the most optimistic viewpoint for each DMU without considering any decision maker's judgement. In the proposed model, the efficiency can be obtained from the most optimistic viewpoint for each DMU in a decision maker's acceptable importance grades. Therefore, the efficiencies in the proposed model are smaller than those in conventional DEA.

We pick out B and C to remark the result in view of the chosen weights in Table 4 and Figure 1. In Table 4, the chosen output weights and the efficiencies by the proposed model are shown. The sums of the obtained weights are normalized to one. All the weights show the optimistic ones in the obtained interval importance grades. In Figure 1, where the lines show the interval importance grades through interval AHP and $\times$ and $\circ$ show the chosen weights that give the efficiencies of B and C respectively. The decision maker's inconsistent information about the importance grades are represented as the intervals and in the interval each item's weight is chosen based on DEA where DMU_o is evaluated from the optimistic viewpoint.

	y_1	y_2	y_3	y_4	importance grades
y_1	1	1/5	5	1/3	[0.012,0.149]
y_2	5	1	7	3	[0.515,0.652]
y_3	1/5	1/7	1	1/7	[0.028,0.074]
y_4	3	1/3	7	1	[0.217,0.354]

Table 2: Comparison matrix and importance grades of the output items

	proposed model	DEA
A	1.000	1.000
B	0.601	0.850
C	0.596	1.000
D	0.627	1.000
E	0.952	1.000
F	0.344	0.706
G	0.878	1.000
H	0.440	1.000
I	0.347	1.000
J	0.231	1.000

Table 3: Efficiencies

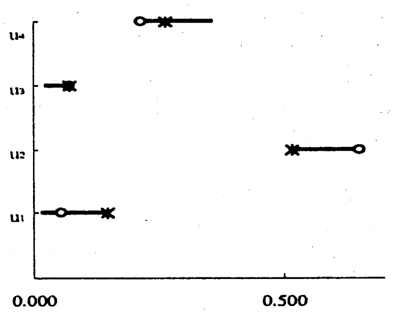

Figure 1: Output weights of B and C

5 Concluding remarks

In this paper, we dealt with a decision maker's inconsistent information about the importance grade of each item as an interval through interval AHP and chose the most optimistic one for DMU_o in the interval by DEA. A decision maker gives comparison matrices for input and output items respectivly based on his/her judgement. From the comparison matrix that contains inconsistent elements each other due to a decision maker's judgement, the interval importance grade of each item

	output weights				
	[0.012,0.0149]	[0.515,0.652]	[0.028,0.074]	[0.217,0.354]	efficiencies
B	0.149	0.515	0.074	0.262	0.601
C	0.057	0.652	0.074	0.217	0.596

Table 4: Chosen output weights for B and C

is obtained by AHP and interval regression analysis. The interval importance grade shows the acceptable range for the decision maker. To make the input and output weights in DEA represent the importance grades of input and output items through AHP, we formulated DEA with the normalized data. The efficiencies are the same as those by conventional DEA and the obtained item's weight itself represents its importance grade. Then, we used DEA to choose the most optimistic importance grade by considering the interval importance grades through interval AHP as the weight constraints in DEA directly.

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