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A Simple Derivation of Josephson Formulae in Superconductivity*

Izumi OJIMA
RIMS, Kyoto University

Abstract

A simple and general derivation of Josephson formulae for the tunneling currents is presented on the basis of Sewell's general formulation of superconductivity in use of off-diagonal long range order (ODLRO).

1 Introduction

Why *superconductivity* and *Josephson effect*?: Because superconductivity provides a very important prototype of

· *SSB* (=spontaneous symmetry breaking)

and

· *collective phenomena* caused by SSB (e.g., supercurrent, Meissner effect, Josephson effect, magnetic flux quantization, etc.).

Among these collective phenomena, Josephson effect [1] exhibits the *physical roles of order parameters* arising from SSB in relation to *phase coexistence.* To formulate this effect in a concise way, we first note the key roles played by the *Cooper pair* condensates [2] in superconductivity. They yield the non-trivial order parameter which can be formulated as a "variable at infinity" based on cluster property in thermodynamic *pure phase* (= factor state with GNS representation having trivial centre) [3, 4]. According to Sewell's "macroscopic quantum theoretical approach", we adopt his general characterization of superconducting BCS states $\langle\cdot\rangle_{BCS}$ [5] in the form of *off-diagonal long range order* (ODLRO):

$$|\langle\psi(X_1+\frac{\xi_1}{2})\psi(X_1-\frac{\xi_1}{2})\psi^\dagger(X_2-\frac{\xi_2}{2})\psi^\dagger(X_2+\frac{\xi_2}{2})\rangle_{BCS}-\Psi(X_1,\xi_1)\Psi^*(X_2,\xi_2)| \underset{|\vec{X}_1-\vec{X}_2|-\infty}{\longrightarrow} 0.$$

Here $\psi(x)$ is the second quantized non-relativistic electron field obeying CAR: $\{\psi(x),\ \psi^\dagger(y)\}=\delta^3(x-y)$, and $\Psi(X,\xi)=\langle\psi(X+\frac{\xi}{2})\psi(X-\frac{\xi}{2})\rangle_{BCS}$

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denotes the "macroscopic wave function" which is non-vanishing at spatial infinity $|\vec{X}| \to \infty$. Starting from this assumption, he gave a general proof of Meissner effect ($\vec{B} = 0$ inside of superconductor) [5]. Our aim here is two-fold:

i) In the context of superconductivity: To understand its essential features in a model-independent way;

ii) In general context: To extract useful hints for understanding symmetry breaking, collective phenomena and phase coexistence, etc.

Obtained result is: A simple and general derivation of (*dc* and *ac*) Josephson formulae for tunneling currents due to phase difference between two superconductors separated by a thin barrier of insulator (=Josephson junction).

From this we can conclude that Josephson tunneling current is a direct consequence of ODLRO and of non-invariance of energy in each side of junction under gauge transformation due to phase difference, which should be contrasted with the traditional derivations such as those based on the perturbative treatment of the tunneling Hamiltonian, or, based on the phenomenological Ginzburg-Landau equation.

2 Simple Derivation of Josephson Formulae

A completely model-independent approach is desirable, but we need here the standard BCS Hamiltonian [6, 3] arising from electron-phonon coupling:

$$H_{BCS}(\Lambda; w) = \int_\Lambda dx [\frac{1}{2m} \vec{\nabla}\psi^\dagger(x) \cdot \vec{\nabla}\psi(x) - \mu\psi^\dagger(x)\psi(x)]$$
$$+ \frac{1}{|\Lambda|} \int_\Lambda dx \int_\Lambda dy \int_\Lambda dz \int_\Lambda du \; w(x,y,z,u) \times \times \psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z),$$

where m is the mass of an electron and μ is the chemical potential of electrons in bulk superconductor occupying a macroscopically extended spatial region Λ with volume $|\Lambda|$.

To see the essence, we here use a simple picture of weakly coupled superconductors placed in two spatial regions Λ_1, Λ_2 ($\subset \Lambda$) separated by a Josephson junction regarded as a *phase boundary* $W \equiv \partial\Lambda_1 = \partial\Lambda_2$. In view of the wide applicability of BCS model (at least in non high T_c cases), we suppose that the differences in properties of superconductors in Λ_1 and in Λ_2 should be absorbed in differences in potentials $w_{\Lambda_1}(x,y,z,u) \equiv w(x,y,z,u)$ for $x,y,z,u \in \Lambda_1$; $w_{\Lambda_2}(x,y,z,u) \equiv w(x,y,z,u)$ for $x,y,z,u \in \Lambda_2$. Up to this freedom, dynamics of superconductors are described universally by H_{BCS} and the differences of realized thermodynamic phases are reduced to the choice of states $\langle\cdot\rangle$.

Remark 1 *To give a precise meaning to "thermodynamic phases", we need the thermodynamic limit with volume* $\to \infty$. *Then, the sizes of regions* Λ_1, Λ_2 *are macroscopically finite and microscopically "infinite". Then, the location of junction* $W = \partial\Lambda_1$ *goes to "spatial infinity" far away from* Λ_1 *with outside region* Λ_2 *lying beyond it. It is possible to give a precise meaning to this heuristic expression in a non-standard analytic formulation [7].*

The next cruicial point is: How to define tunneling current? Because of the non-local terms in H_{BCS} we have *no locally conserved* electric current. So we need to clarify in which sense the electric current is conserved. Defining the electric charge $Q(\Lambda)$ in Λ by

$$Q(\Lambda) = -|e| \int_\Lambda dx\ \psi^\dagger(x)\psi(x),$$

we see that it generates global $U(1)$-gauge transformation:

$$\begin{aligned} [iQ(\Lambda),\ \psi(x)] &= i|e|\psi(x), \\ [iQ(\Lambda),\ \psi^\dagger(x)] &= -i|e|\psi^\dagger(x), \end{aligned}$$

where $e = -|e|$: unit of electric charge. Then the meaning of *conservation* is found in the expression:

$$[H_{BCS}(\Lambda; w),\ Q(\Lambda)] = 0 \text{ for } \forall\Lambda.$$

De nition2 *Tunneling current* J *between* Λ_1 *and* Λ_2 *is defined by*

$$J = \frac{d}{dt}\langle Q(\Lambda_1)\rangle_{BCS} = \langle [iH_{BCS}(\Lambda),\ Q(\Lambda_1)]\rangle_{BCS}.$$

Remark 3 i) J *cannot be non-vanishing without outside region* Λ_2.

ii) *Its heuristic meaning is seen in* $J \propto dN/dt = \partial H/\partial\theta$ *in the number-phase picture of Ginzburg-Landau theory [8].*

Decomposing $H_{BCS}(\Lambda; w)$ into regions Λ_1 and Λ_2:

$$H_{BCS}(\Lambda; w) = H_{BCS}(\Lambda_1; \frac{|\Lambda_1|}{|\Lambda|}w) + H_{BCS}(\Lambda_2; \frac{|\Lambda_2|}{|\Lambda|}w) + H_{12},$$

and combining it with the local (anti-)commutativity due to CAR, we see that J can be reduced to

$$J = \langle [iH_{12},\ Q(\Lambda_1)]\rangle_{BCS} = -\langle [iQ(\Lambda_1),\ H_{12}]\rangle_{BCS}$$

with H_{12} given by

$$H_{12} = \frac{1}{|\Lambda|} \sum_{\{i,j,k,l\}=\{1,2\}} \int_{\Lambda_i} dx \int_{\Lambda_j} dy \int_{\Lambda_k} dz \int_{\Lambda_l} du\ w(x,y,z,u)\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z).$$

Then we have

$$
\begin{aligned}
J &= -\langle [iQ(\Lambda_1),\ H_{12}]\rangle_{BCS} \\
&= i|e|\frac{1}{|\Lambda|}\sum_{\{i,j,k,l\}=\{1,2\}}\int_{\Lambda_i}dx\int_{\Lambda_j}dy\int_{\Lambda_k}dz\int_{\Lambda_l}du \\
&\times w(x,y,z,u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\langle\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z)\rangle_{BCS}.
\end{aligned}
$$

In thermodynamic limit $\Lambda, \Lambda_1, \Lambda_2 \longrightarrow \infty$ with $|\Lambda_1|/|\Lambda_2|$ fixed, the junction at the boundary W goes infinitely far away, and hence, the terms coming from Λ_2 beyond W are replaced by their expectation values in $\langle\cdot\rangle_{BCS}$ owing to cluster property:

$$
\begin{aligned}
&|\langle\psi^\#(x_1^{(1)})\cdots\psi^\#(x_{k_1}^{(1)})\psi^\#(x_1^{(2)})\cdots\psi^\#(x_{k_2}^{(2)})\rangle_{BCS} \\
&-\langle\psi^\#(x_1^{(1)})\cdots\psi^\#(x_{k_1}^{(1)})\rangle_{BCS}\langle\psi^\#(x_1^{(2)})\cdots\psi^\#(x_{k_2}^{(2)})\rangle_{BCS}| \\
&\underset{\Lambda_1,\Lambda_2\longrightarrow\infty,\ |\Lambda_1|/|\Lambda_2|:fixed}{\longrightarrow} 0,
\end{aligned}
$$

where $x_1^{(1)},\cdots,x_{k_1}^{(1)}\in\Lambda_1$, $x_1^{(2)},\cdots,x_{k_2}^{(2)}\in\Lambda_2$ and $\psi^\#=\psi$ or $\psi^\dagger$.

Here we recall that under the assumption of spatial homogeneity at infinity the expectation values of odd powers of fermionic operators ψ, $\psi^\dagger$ vanish (i.e., Bose-Fermi superselection rule holds) [9]. Then we see that, in the integrand $\chi_{\Lambda_i}(x)\chi_{\Lambda_j}(y)\chi_{\Lambda_k}(z)\chi_{\Lambda_l}(u)\times(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\langle\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z)\rangle_{BCS}$ ($\chi_\Lambda\equiv$ indicator function of set Λ), surviving terms come from the cases with $i=j=1, k=l=2$ or $i=j=2, k=l=1$:

$$
\begin{aligned}
&\sum_{\{i,j,k,l\}=\{1,2\}}\chi(x,y,z,u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\langle\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z)\rangle_{BCS} \\
&\underset{\Lambda_1,\Lambda_2\longrightarrow\infty,\ |\Lambda_1|/|\Lambda_2|:fixed}{\longrightarrow}\ \sum_{i=j\neq k=l}\chi(x,y,z,u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1}) \\
&\qquad\times\langle\psi^\dagger(x)\psi^\dagger(y)\rangle_{BCS}\langle\psi(u)\psi(z)\rangle_{BCS}\ ,
\end{aligned}
$$

where $\chi(x,y,z,u)\equiv\chi_{\Lambda_i}(x)\ \chi_{\Lambda_j}(y)\chi_{\Lambda_k}(z)\chi_{\Lambda_l}(u)$.

Assuming "*almost* spatial homogeneity" in Λ_1 and Λ_2 w.r.t. "macroscopic wave function" $\Psi(X,\xi)$ in ODLRO in the sense of

$$
\begin{aligned}
\langle\psi(x)\psi(y)\rangle_{BCS} &\simeq \Psi(\frac{x+y}{2}, x-y) \\
&= \begin{cases} |\Psi_1|e^{i\theta_1}\ (x,y\in\Lambda_1), \\ |\Psi_2|e^{i\theta_2}\ (x,y\in\Lambda_2), \end{cases}
\end{aligned}
$$

we obtain the desired formula for *dc*-Josephson current by picking out phase factors from the integrands:

$$
J\simeq \text{constant}\times(e^{i(\theta_1-\theta_2)}-e^{-i(\theta_1-\theta_2)})\propto\sin(\Delta\theta),
$$

where $\Delta\theta \equiv \theta_1 - \theta_2$ is the phase difference of Cooper pairs across the junction.

Using this, we can easily derive the *ac*-Josephson formula in situations with voltage gap V across the junction, simply by replacing the above $\Delta\theta$ with $\Delta\theta + 2eVt$. This is due to the following reason: In spite of lost local gauge invariance in H_{BCS} in spatial directions due to non-local interactions, local gauge invariant coupling is still meaningful in temporal direction. By this temporal gauge freedom, situation with voltage gap V is realized by a time-dependent local gauge transformation:

$$A^{\mu} = (\phi,\ \vec{A} = 0) \longrightarrow (\phi + \frac{\partial}{\partial t}(Vt) = V,\ \vec{A} = 0);$$
$$\psi(x,t) \longrightarrow e^{ieVt}\psi(x,t),$$

which results in $\Delta\theta \longrightarrow \Delta\theta + 2eVt$. Thus we have $J_{ac} \propto \sin(\Delta\theta + 2eVt)$ for *ac*-Josephson current.

In the above, we need *(almost) spatial homogeneity* to extract phase difference. Although precise coefficients cannot be determined by such qualitative discussion, we can extract in a similar way the term in energy density coming from $\Delta\theta$ at the boundary located infinitely far away:

$$\frac{\langle H_{12}\rangle_{BCS}}{|\Lambda|} \underset{\Lambda_1,\Lambda_2 \longrightarrow \infty,\ |\Lambda_1|/|\Lambda_2|:fixed}{\longrightarrow} \text{constant} \times \cos(\Delta\theta).$$

If the coefficient of $\cos(\Delta\theta)$ is of negative sign, this guarantees self-consistency of the postulate that phase of Cooper pair condensates should be spatially homogeneous in favour of $\Delta\theta = 0$ in the absence of a constraint maintaining phase difference at the barrier.

To verify this consistency in a more satisfactory way, it seems necessary to confront a challenging problem of how one can justify the notion of a *point-like order parameter* $\Psi(x)$ of Cooper pairs appearing in Ginzburg-Landau approach. It is crucial also for discussing Type II superconductivity involving spatial inhomogeneity and local gauge invariance problem.

3 Discussion: Infinities and Infinitesimals

Here some comments are in order on the use of non-standard analysis [7]: It can be useful in describing situations with infinitely large regions Λ_1, Λ_2 separated by a boundary W at infinity. What is important is that it allows to treat simultaneously theories with finite and infinite volumes *without disconnecting* the two approaches.

Before introducing the distinctions among finite, ∞ and $1/\infty$ (level of *internal objects*), it looks as if we were in finite volume theory. Once such distinctions are introduced (interpretation in a *non-standard model*) by regarding $|\Lambda_1|$ and $|\Lambda_2|$ as *infinite numbers* (whose ratio is kept finite), infinite volume theory is seen to be contained in the former, extracted by the procedure of taking *standard parts* of quantities with all infinitesimals such as $1/|\Lambda_1|$ thrown away.

With only one thermodynamic phase, the procedure of extracting standard parts may be equivalent to the usual formulation in thermodynamic limit. In the present situation with *two* infinitely large regions Λ_1 and Λ_2, we have still "another world" in Λ_2 *beyond the infinitely distant* boundary W of infinitely extended Λ_1. This is difficult to be treated in the usual formulation; it can be described without difficulty in such a framework that all the infinities and infinitesimals are fully legitimate quantities. Moreover, all the limiting or approximate relations appearing above are replaced by simple algebraic equivalence relations modulo infinitesimals, in which one of its conceptual advantages can be found.

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