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Algebraic Quantum Field Theory

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To get matters in perspective, I must begin by mentioning two other branches of quantum field theory. Renormalized perturbation theory has the task of making numerical computations of scattering cross sections, these being the quantities that form the backbone of experimental high energy physics. Success and failure lie close together. Summing the first few terms of the perturbation series for quantum electrodynamics gives results in extraordinary agreement with experiment. Applications to the Salam–Weinberg theory meet with less success whilst the large effective coupling constant in strong interactions precludes the use of the perturbative methods. Furthermore, the perturbative expansion gives one no idea as to whether there is an underlying field theory whose scattering theory is governed by the perturbative expansion.

Constructive field theory sets itself the goal of constructing interacting models based on the ideas of renormalization theory. Again, success and failure lie close together. It proved possible to construct a whole family of interacting models in two spacetime dimensions such as the $P(\phi)_2$ models, the polynomial models. Two models, ϕ_3^4 and Y_3, the quartic interaction and the Yukawa coupling were constructed in three spacetime dimensions but the methods did not lead to any theories in the physical four dimensional spacetime. Instead it is believed that attempts to construct ϕ_4^4 or quantum electrodynamics in this way actually lead to free field models.

Algebraic quantum field theory was innovative both mathematically and physically. The fields $f \mapsto \phi(f) := \int f(x)\phi(x)dx$ as unbounded operator-valued distributions in Hilbert space were replaced by the net $\mathcal{O} \mapsto \mathrm{F}(\mathcal{O})$ of algebras of bounded operators that they generate. Here $\mathrm{F}(\mathcal{O})$ is to be regarded as the algebra of bounded operators generated by the $\phi(f)$ with supp$f \subset \mathcal{O}$. This allows one to use the well developed theory of bounded operators on Hilbert space. We also implicitly claim that spacetime enters only through the assignement of algebras to regions $\mathcal{O}$ in spacetime, where it is usual to restrict $\mathcal{O}$ to be a double cone, that is the intersection of a backward light cone in one point with a forward light cone in another. This changes the way that we look at spacetime.

More important was the recognition that the fundamental object was not $\mathcal{O} \mapsto \mathtt{F}(\mathcal{O})$ but a smaller net $\mathcal{O} \mapsto \mathtt{A}(\mathcal{O})$, $\mathtt{A}(\mathcal{O}) \subset \mathtt{F}(\mathcal{O})$. $\mathtt{A}(\mathcal{O})$ is to be thought of as generated by the observable polynomials in the fields whose test functions f have supports in $\mathcal{O}$ or, alternatively, in terms of its physical interpretation as being generated by the observables that can be measured within $\mathcal{O}$.

Algebraic quantum field theory proceeds axiomatically, postulating certain basic 'laws' of physics: local commutativity, positivity of the energy, duality, the Reeh–Schlieder property, local normality, additivity and the split property. These laws have a physical interpretation and on the basis of these laws, or some subset of them, conclusions are drawn about the behaviour of the system that themselves allow a physical interpretation. This is complemented by studying simple models where these laws can be verified or their independence demonstrated.

As an illustration let me spell out the law of local commutativity.

$$A_1A_2 = A_2A_1, \quad A_1 \in \mathtt{A}(\mathcal{O}_1),\ A_2 \in \mathtt{A}(\mathcal{O}_2),\ \mathcal{O}_1 \perp \mathcal{O}_2.$$

Here $\mathcal{O}_1 \perp \mathcal{O}_2$ means that the two regions in question are causally disjoint, or, as one usually says in Minkowski space, spacelike separated. This law allows a simple physical interpretation. One knows from elementary quantum mechanics that you cannot make simultaneous measurements of quantities that do not commute with one another. In the relativistic setting, this means that measurements made in $\mathcal{O}_i$ affect the results of measurements made in the causal future of $\mathcal{O}_i$. When $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally disjoint, neither intersect the causal future of the other. Thus the measurements do not interfer with one another and the observables should commute leading to the above law of local commutativity.

In general, a field net $\mathtt{F}$ does not satisfy this law of local commutativity. Indeed, I have stressed the distinction between the field net and the observable net and this is a manifestation of the appearance of non–observable quantities in the field net. However, a simple generalization of local commutativity suffices to describe the spacelike commutation properties of the fields. A field net has what is called a $\mathtt{Z}_2$–grading. That is it is a direct sum of two pieces:

$$\mathrm{F}(\mathcal{O}) = \mathrm{F}_+(\mathcal{O}) \oplus \mathrm{F}_-(\mathcal{O}).$$

so that $F \in \mathrm{F}$ can be written uniquely as a sum of its Bose and Fermi parts: $F_+ + F_-$. Given $F_1 \in \mathrm{F}(\mathcal{O}_1)$ and $F_2 \in \mathrm{F}(\mathcal{O}_2)$ with $\mathcal{O}_1 \perp \mathcal{O}_2$, we have

$$F_{1+}F_{2+} = F_{2+}F_{1+} \; F_{1+}F_{1-} = F_{1-}F_{1+} \; F_{1-}F_{2-} = -F_{2-}F_{1-}.$$

These are referred to as Bose–Fermi commutation relations. Algebraic quantum field theory has succeeded in understanding why this simple generalization is sufficient.

Let me now explain a simple but important mathematical construction. By a *state* of A we mean a positive normalized linear functional, i.e. $A \mapsto \omega(A) \in \mathrm{C}$ is linear, $\omega(A^*A) \geq 0$ and $\omega(I) = 1$. Then the GNS construction associates with ω a representation π_ω of A on a Hilbert space $\mathcal{H}_\omega$ with a cyclic vector Ω_ω such that

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega), \quad A \in \mathrm{A}.$$

Ω_ω is cyclic when $\pi_\omega(\mathrm{A})\Omega_\omega$ is dense in $\mathcal{H}_\omega$.

Thus we can pass from any state ω to the more familiar Hilbert space picture in which the algebra is represented concretely by bounded linear operators and the state by a vector Ω_ω. Nevertheless, there is an important difference between this mathematical idea of state on A and the physical idea of the state of a physical system. In fact, only a small fraction of the states on A allow a reasonable interpretation as physical states. If, in the above construction, ω is physically relevant then the other states given by density matrices on $\mathcal{H}_\omega$,

$$\omega_\rho(A) := \mathrm{Tr}(\rho\pi_\omega(A)),$$

are physically relevant and π_ω is physically relevant. These other states are the normal states of the representation π_ω and include the special case of a vector state defined by a unit vector Φ

$$\omega_\Phi(A) := (\Phi, \pi_\omega(A)\Phi).$$

This is seen by taking ρ to be the projection onto the one dimensional subspace spanned by Φ.

As states of particular physical relevance we have, in the realm of statistical physics, the thermal equilibrium states characterized by an inverse temperature β and a chemical potential μ. In the realm of many body physics, we have the ground states and in elementary particle physics, the vacuum state. On the other hand, not all states of relevance to elementary particle can be normal states of the vacuum representation because, among such states, there are states with non–zero baryon or lepton numbers which must belong to different superselection sectors.

I investigated this phenomenon of superselection sectors in joint work with S. Doplicher and R. Haag. Our intuition was that states of relevance to elementary particle physics should tend rapidly to the vacuum state for measurements which tend spacelike to infinity. This is the theoretical counterpoint to the experimental efforts to achieve a high vacuum by pumping out the system and by using lots of concrete to shield from the effects of cosmic rays. We decided to select as physically relevant to elementary particle physics those representations π which satisfy

$$\pi|\mathcal{O}^{\perp} \simeq \pi^0|\mathcal{O}^{\perp}, \qquad (\mathcal{S})$$

or, in more detail, if given $\mathcal{O} \in \mathcal{K}$, there is a unitary $V_{\mathcal{O}}$ such that $V_{\mathcal{O}}\pi(A) = \pi^0(A)V_{\mathcal{O}}$, $A \in \mathrm{A}(\mathcal{O}_1)$ and $\mathcal{O}_1$ and $\mathcal{O}$ are causally disjoint. A superselection sector is now defined as an equivalence class of an irreducible representation satisfying the selection criterion. Using the term charge generically to denote a parameter distinguishing a superselction sector from the vacuum sector, we were able to show that there was a law of charge composition of the form

$$\pi \otimes \pi' = \pi^1 \oplus \pi^2 \oplus \cdots \oplus \pi^n,$$

where all representations involved are irreducible but not necessarily inequivalent. This is also referred to as a fusion rule.

Then there is a law of charge conjugation. Given an irreducible representation π satisfying the selection criterion, there is another irreducible representation $\bar{\pi}$ satisfying the selection criterion and unique up to equivalence such that $\pi \otimes \bar{\pi}$ contains π^0. If the 1–particle states of a particle are vector states of π, then the 1–particle states of the antiparticle are vector states of $\bar{\pi}$. This gives one the correct definition of antiparticle since particle and antiparticle can annihilate each other to produce photons and photon states lie in the vacuum sector.

Finally, to every sector there is a statistics parameter $\lambda \in \pm\frac{1}{\mathrm{N}}$. $\lambda = \frac{1}{d}$ means para–Bose statistics of order d, $d = 1$ being ordinary Bose statistics. $\lambda = -\frac{1}{d}$ means para-Fermi statistics of order d, $d = -1$ being ordinary Fermi statistics.

To illustrate the role of parastatistics, we first imagine a world without electromagnetic interactions. Then a proton cannot be distinguished from a neutron but must be treated as the same elementary particle, the nucleon. But the nucleon is then a para-Fermion of order two. A second example is the quark which is treated as a para-Fermion of order three. The quark does not manifest itself as a particle in the sense of scattering theory but appears as a constituent of other elementary particles such as the proton in the scaling limit.

An alternative description of superselection structure is given by the following result of Doplicher and myself. There is a canonical net of field algebras $\mathcal{O} \mapsto \mathtt{F}(\mathcal{O})$, the original observable net appearing as the fixed-point net of the action of a compact group G of automorphisms of $\mathtt{F}$: $\mathtt{A}(\mathcal{O}) = \mathtt{F}(\mathcal{O})^G$. G, the gauge group, is the group of all automorphisms of $\mathtt{F}$ leaving $\mathtt{A}$ pointwise invariant. The representation π of $\mathtt{A}$ on the vacuum Hilbert space of $\mathtt{F}$ has the form

$$\pi = \oplus_{i \in \hat{G}} d_i \pi_i,$$

where i runs over the equivalence classses of continuous unitary representations of G and π_i over the equivalence classes of irreducible representations of $\mathtt{A}$ which satisfy the selection criterion. $d_i = \frac{1}{|\lambda_i|}$ is just the dimension of the corresponding irreducible representation of G. The superselection structure is described in terms of the representation theory of G with one exception. The distinction between Bosonic and Fermionics parts corresponds to singling out an element k of the centre of G whose square is the identity. The Bose part of $\mathtt{F}$ is the part invariant under k, the Fermi part changes sign. k is represented by 1 in the representation of G corresponding to a para–Bose sector and by -1 in that corresponding to a para–Fermi sector.

The selection criterion denoted by $(\mathcal{S})$ above is too restrictive to cover the cases of physical interest. The importance of the above work is therefore that it points the way as to how to obtain interesting results from a criterion of this sort. At this stage Buchholz and Fredenhagen made an important contribution. They showed that if a sector described massive particles as evinced by the presence of an isolated mass hyperboloid in the energy–momentum spectrum of the sector, then the corresponding irreducible representation satisfies the following weaker form of the selection criterion

$$\pi|\mathcal{C}^\perp \simeq \pi_0|\mathcal{C} \qquad (\mathcal{C}).$$

Here $\mathcal{C}$ denotes a spacelike cone, that is a cone based on a double cone with a vertex spacelike to the double cone. Using their criterion $(\mathcal{C})$, Buchholz and Fredenhagen were able to reproduce the results of the above analysis in space dimensions ≥ 3.

In deriving the criterion $(\mathcal{C})$, Buchholz and Fredenhagen assume the absence of massless particles. But there are massless particles in nature. In particular, the photon has mass zero and the corresponding field, the electromagnetic field satisfies Gauss's law according to which the total charge inside a sphere is the flux of the electric field through the sphere. This implies that when the electric charge associated with a sector is non–zero, the electric field always extends to spacelike infinity, being non–zero possibly just within some spacelike cone. This contradicts $(\mathcal{S})$ but it also contradicts $(\mathcal{C})$ since $(\mathcal{C})$ is supposed to hold for *any* choice of spacelike cone $(\mathcal{C})$.

At this point, I would like to mention work that has been going on for a number of years to find a new selection criterion that is sufficiently general to include quantum electrodynamics and hence the photon. This work has been done in collaboration with Buchholz, Doplicher, Morchio and Strocchi. We propose a new selection criterion $(\mathcal{N})$ whereby states are not localized on the whole algebra but only on a suitable large subalgebra. The subalgebra is not invariant under Lorentz transformations and therefore involves singling out a Lorentz frame. In the case of quantum electrodynamics, the algebra is supposed to be generated by the 0–component of the electric current and the magnetic field since these quantities remain localized in contrast to the electric field. We have a simple model exihibiting sectors satisfying $(\mathcal{N})$ but not $(\mathcal{S})$ or $(\mathcal{C})$. The key question is of course whether quantum electrodynamics satisfies $(\mathcal{N})$ and this question is presently under investigation using renormalized perturbation theory. More than this, matters have reached a decisive stage. We need to know that Feynmann integrals corresponding to diagrams with one external zero mass photon vanish off–shell. A negative result would force us to revise our ideas. A positive result would provide non–trivial evidence in favour of our hypothesis since the specific form of the interaction enters into the computations.