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THERMOELASTICITY SYSTEM IN SHAPE MEMORY PROBLEMS

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Abstract

We are concerned with a nonlinear 3–D thermoelastic system, which arises as a model of dynamical processes in shape memory materials. The system has the form of a fourth order viscoelasticity equation coupled with the temperature equation. For such a system we have proved global in time existence for arbitrary data, as well as the uniqueness of the solution. In addition, we have studied stability of solutions with respect to distributed body forces and heat sources.

1 Thermoelasticity system

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain with a smooth boundary $\partial\Omega$. We consider the following problem $(\mathbf{P})$:

$$\mathbf{u}_{tt} - \nu\mathbf{Q}\mathbf{u}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{u} = \boldsymbol{\nabla} \cdot F_{/\epsilon}(\epsilon, \theta) + \mathbf{b}, \tag{1.1}$$

$$c(\epsilon, \theta)\theta_t - k\Delta\theta = \theta F_{/\theta\epsilon}(\epsilon, \theta) : \epsilon_t + \nu(\mathbf{A}\epsilon_t) : \epsilon_t + g \quad \text{in} \quad Q_T, \tag{1.2}$$

with initial

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_t(0, \mathbf{x}) = \mathbf{u}_1(\mathbf{x}), \tag{1.3}$$

$$\theta(0, \mathbf{x}) = \theta_0(\mathbf{x}) \ \text{ in } \ \Omega, \tag{1.4}$$

and boundary conditions

$$\mathbf{u} = 0, \quad \mathbf{Q}\mathbf{u} = 0, \tag{1.5}$$

$$\boldsymbol{\nabla}\theta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{1.6}$$

where

$$c(\boldsymbol{\epsilon},\theta) = c_v - \theta F_{/\theta\theta}(\boldsymbol{\epsilon},\theta). \tag{1.7}$$

Here Ω refers to the region in a reference configuration.
The subscript t denotes differentiation with respect to time, $\nabla\cdot$ is the divergence with respect to $\mathbf{x}$, $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, $F_{/\epsilon}(\boldsymbol{\epsilon},\theta)$ and $F_{/\theta}(\boldsymbol{\epsilon},\theta)$ denote derivatives of $F(\boldsymbol{\epsilon},\theta)$ with respect to $\boldsymbol{\epsilon},\theta$.

The meaning of the quantities in problem (P) is as follows: $\mathbf{u} : Q_T \to \mathbb{R}^n$ is the displacement vector, $\theta : Q_T \to \mathbb{R}_+$ is the absolute temperature, $\boldsymbol{\epsilon} = (\epsilon_{ij})$ with $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i/j} + u_{j/i})$, is the linearized strain tensor, $\boldsymbol{\epsilon}_t = \boldsymbol{\epsilon}(\mathbf{u}_t)$ is the strain rate tensor, $F(\boldsymbol{\epsilon},\theta)$ is the elastic energy, $c(\boldsymbol{\epsilon},\theta)$ is the specific heat coefficient.
The positive constants c_v, k, ν and κ correspond to thermal specific heat, heat conductivity, viscosity and interface energy.
The vector $\mathbf{b}$ is a distributed external force and g is a distributed heat source.
The linear map

$$\mathbf{u} \mapsto \mathbf{A}\boldsymbol{\epsilon}(\mathbf{u}) = \lambda \operatorname{trace} \boldsymbol{\epsilon}(\mathbf{u})\, \mathbf{I} + 2\mu\boldsymbol{\epsilon}(\mathbf{u}), \tag{1.8}$$

where $\lambda, \mu > 0$ are Lamé constants and $\mathbf{I} = (\delta_{ij})$ is the unit matrix, represents Hooke's law for the homogeneous isotropic material. Here $\mathbf{A} = (A_{ijkl})$ with

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

is the fourth order elasticity tensor satisfying the following symmetry conditions:

$$A_{ijkl} = A_{jikl} = A_{klij}.$$

The second order differential operator $\mathbf{Q}$ defined by

$$\mathbf{u} \mapsto \mathbf{Q}\mathbf{u} = \nabla \cdot (\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u})), \tag{1.9}$$

is known as operator of linearized elasticity. By (1.8) it admits the representation

$$\mathbf{Q}\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}). \tag{1.10}$$

In the divergence $\nabla\cdot$ we use the convention of the contraction over the last index, i.e.

$$\nabla \cdot (\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u})) = \partial_j(\, A_{ijkl}\epsilon_{kl}(\mathbf{u})\,) = A_{ijkl}\partial_j\epsilon_{kl}(\mathbf{u}) = \mathbf{A}\nabla\boldsymbol{\epsilon}(\mathbf{u}).$$

Moreover, throughout the summation convention is used, and the following notation: for vectors $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$ and tensors $\mathbf{B} = (B_{ij})$, $\mathbf{C} = (C_{ij})$, $\mathbf{A} = (A_{ijkl})$ we write $\mathbf{a}\cdot\mathbf{b} = a_ib_i$, $\mathbf{B} : \mathbf{C} = B_{ij}C_{ij}$, $\mathbf{aB} = a_iB_{ij}$, $\mathbf{Ba} = B_{ij}a_j$, $\mathbf{BA} = B_{ij}A_{ijkl}$, etc.

To problem (P) corresponds the free energy functional of the Ginzburg–Landau form

$$f(\boldsymbol{\epsilon}, \nabla\boldsymbol{\epsilon}, \theta) = -c_v\theta\log\theta + F(\boldsymbol{\epsilon}(\mathbf{u}),\theta) + \frac{\kappa}{8} \mid \mathbf{Q}\mathbf{u} \mid^2, \tag{1.11}$$

with the subsequent terms representing thermal energy, elastic energy and interfacial energy.
The main characteristic feature of (1.11) as a model of shape memory materials is the nonlinearity in $\boldsymbol{\epsilon}$: $F(\boldsymbol{\epsilon},\theta)$ is a multiple–well in $\boldsymbol{\epsilon}$ with the shape changing qualitatively with θ. The second characteristic feature is the presence of higher order term with

coefficient κ, which accounts for interaction effects on phase interfaces. Terms of this type are known in the so called multiscale approach to modelling of phase transitions. The particular form of κ–term in (1.11) can be interpreted as a resultant of mechanical forces acting on a layer element of interface.

A typical example of the elastic energy is the Falk–Konopka model [10] in the form of sixth order polynomial in terms of ϵ_{ij}:

$$F(\epsilon,\theta) = \sum_{i=1}^{3} F_i^2(\theta) J_i^2(\epsilon) + \sum_{i=1}^{5} F_i^4(\theta) J_i^4(\epsilon) + \sum_{i=1}^{2} F_i^6(\theta) J_i^6(\epsilon), \tag{1.12}$$

where $J_i^k(\epsilon)$, $i = 1,\ldots,i^k$, are k-th order crystallographical invariants, that is appropriate combinations of the strain tensor components ϵ_{ij} and $F_i^k(\theta)$ are corresponding temperature-dependent coefficients.

The form (1.12) represents a generalization of the well known 1–D Landau–Devonshire energy [8],[9]

$$F(\epsilon,\theta) = \alpha_1(\theta-\theta_c)\epsilon^2 - \alpha_2\epsilon^4 + \alpha_3\epsilon^6, \tag{1.13}$$

where $\alpha_i > 0$ are constant parameters, and $\theta_c > 0$ is a critical temperature.

2 Motivation and known results

The equations (1.1),(1.2) express balance laws of linear momentum and energy (under assumption of constant material density)

$$\mathbf{u}_{tt} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{b}, \tag{2.1}$$

$$e_t + \nabla \cdot \mathbf{q} - \boldsymbol{\sigma} : \epsilon_t = g, \tag{2.2}$$

with constitutive equations for stress tensor $\boldsymbol{\sigma}$ and energy flux $\mathbf{q}$ accounting for interfacial effects:

$$\boldsymbol{\sigma} = \frac{\delta f}{\delta \epsilon} + \boldsymbol{\sigma}^v, \qquad \boldsymbol{\sigma}^v = \nu \mathbf{A}\epsilon_t, \tag{2.3}$$

$$\mathbf{q} = -k\nabla\theta + \mathbf{p}, \qquad \mathbf{p} = -\epsilon_t f_{/\mathbf{D}\epsilon}. \tag{2.4}$$

Here $\delta f/\delta\epsilon$ denotes the first variation of the functional f, and the internal energy e is given by Gibbs relation

$$e = f + \theta s, \qquad s = -f_{/\theta}, \tag{2.5}$$

where s is the entropy. In case of free energy (1.11) expressions (2.3), (2.4) become

$$\boldsymbol{\sigma} = F_{/\epsilon} - \frac{\kappa}{4}\mathbf{A}\epsilon(\mathbf{Qu}) + \nu\mathbf{A}\epsilon_t, \tag{2.6}$$

$$\mathbf{q} = -k\nabla\theta - \frac{\kappa}{4}\epsilon_t(\mathbf{AQu}). \tag{2.7}$$

The stress tensor $\boldsymbol{\sigma}$ contains three contributions: elastic stress, interfacial stress and viscous stress according to Hooke's–like law.

The energy flux is composed of the usuall heat flux expressed by Fourier law, and nonequilibrium phase interface flux $\mathbf{p}$. The latter, called interstitial work flux corresponding to working of phase interfaces, appears in Dunn–Serrin [6] thermodynamical theory of higher grade thermoelastic materials.

In the literature there are known several 3–D free energy models describing phase transitions in crystals, in particular model of Falk–Konopka [10], model of Ericksen [7], which has been analysed numerically in [4], [14], 2–D gradient–type model of Barsch–Krumhansl [2], and 2–D model for noncrystalline shape memory material [23].

In 1–D case the dynamical model describing martensitic phase transitions of the shear type has been developed by Falk [8],[9]. There the shear strain plays the role of the order parameter, that is quantity characterizing different phases. The corresponding free energy has the form of Ginzburg–Landau functional, dependent on strain, absolute temperature and strain gradient.

The resulting 1–D nonlinear thermoelasticity system has been the subject of intensive mathematical studies, see e.g. references in the monograph of Brokate and Sprekels [3], and [1].

A 3–D dynamical model corresponding to Falk–Konopka energy has been derived in [17]. In a special case it constitutes an analog of 1–D Falk's dynamical model. The governing constitutive laws (2.3),(2.4) (imposed by entropy principle) are characteristic for thermoelastic materials of higher grade.

A conceptually different 3–D evolution model is due to Frémond [11]. The free energy depends on volumetric proportions of phases, strain tensor, absolute temperature and gradient of strain tensor trace. For the existing mathematical analysis of this model we refer e.g. to [3].

We mention also the model due to Fried–Gurtin [12], with energy dependent on strain tensor, multicomponent order parameter and its gradient.

3 Assumptions and main results

We assume that

(D) the boundary $\partial\Omega$ is of class C^2.

Further assumptions concern the elastic energy:

(FE–1) Structure: $F(\epsilon,\theta)$ is of class C^3 on $S^2\times[0,\infty)$, where S^2 denotes the set of symmetric tensors of second order in $\mathbb{R}^n$. We assume the splitting

$$F(\epsilon,\theta)=F_1(\epsilon,\theta)+F_2(\epsilon),$$

where $F_1(\epsilon,\theta)$ is linear in θ over certain interval $[0,\theta_1)$ and satisfies (FE–2) for large values of θ.

(FE–2) Growth conditions: Let ϵ_1 and θ_1 be certain constants. There exists a constant Λ such that for $|\epsilon|\geq\epsilon_1$ and $\theta\geq\theta_1$ the following conditions are satisfied:

$$\begin{aligned}
&|F_{1/\epsilon\epsilon}(\epsilon,\theta)|\leq\Lambda+\Lambda\theta^r|\epsilon|^{q-1}, && |F_{2/\epsilon\epsilon}(\epsilon)|\leq\Lambda+\Lambda|\epsilon|^{\bar q-1},\\
&|F_{1/\epsilon\theta}(\epsilon,\theta)|\leq\Lambda+\Lambda\theta^{r-1}|\epsilon|^{q}, && |F_{1/\theta\theta}(\epsilon,\theta)|\leq\Lambda+\Lambda\theta^{r-2}|\epsilon|^{q+1},
\end{aligned}$$

where

$$0<r<\frac{2}{p_n},\qquad 1<q\leq q_n\left(\frac{1}{p_n}-\frac{r}{2}\right),\qquad 1<\bar q\leq\frac{q_n}{p_n},$$

$p_n = n + 2$, and q_n is the Sobolev exponent for which the imbedding of $W_2^1(\Omega)$ into $L_{q_n}(\Omega)$ is continuous, that is $q_n = 2n/(n-2)$ for $n \geq 3$ and q_n is any finite number for $n = 2$.

We note that the above conditions imply the following growth of $F(\epsilon, \theta)$:

$$|F_1(\epsilon, \theta)| \leq \Lambda + \Lambda\theta^r |\epsilon|^{q+1}, \qquad |F_2(\epsilon)| \leq \Lambda + \Lambda |\epsilon|^{\bar{q}+1}.$$

The next two assumptions are physically natural.

(FE–3) Concavity with respect to θ (thermal stability):

$$F_{1/\theta\theta}(\epsilon, \theta) \leq 0 \quad \text{for} \quad (\epsilon, \theta) \in S^2 \times [0, \infty).$$

This implies the lower bound for the specific heat coefficient

$$0 < c_v \leq c(\epsilon, \theta) \quad \text{for} \quad (\epsilon, \theta) \in S^2 \times [0, \infty).$$

(FE–4) Lower bound for the internal energy:

$$-\Lambda \leq (F_1(\epsilon, \theta) - \theta F_{1/\theta}(\epsilon, \theta)) + F_2(\epsilon) \quad \text{for} \quad (\epsilon, \theta) \in S^2 \times [0, \infty).$$

The most restrictive is the assumption on θ–growth exponent $r < 1/2$ and the assumption on ϵ–growth exponent $\bar{q} \leq 6/5$ in 3–D case.
In 2–D case the latter assumption is not active, since q and $\bar{q}$ are then any large numbers. Hence our assumptions admit the form of sixth order polynomial (1.12) only in 2–D case. In 3–D case they require the growth with respect to ϵ close to quadratic. The temperature dependence is restricted to quadratic terms $F_i^2(\theta)$ (as in 1–D model (1.13)). The growth condition on r is needed both in 2– and 3–D case.

We are looking for the solution in the anisotropic Sobolev space

$$V(p) = \{\, (\mathbf{u}, \theta) \in \mathbf{W}_p^{4,2}(Q_T) \times W_p^{2,1}(Q_T) \,\},$$

with a parameter p related to the L_p–integrability. The assumptions on the initial data and the source terms correspond to this space.

(BV–p) Let $\delta > 0$, $p > 1$, $p_1 = p + \delta$. The initial conditions satisfy the inclusions

$$\mathbf{u}_0 \in \mathbf{W}_p^{4-2/p}(\Omega), \quad \mathbf{u}_1 \in \mathbf{W}_p^{2-2/p}(\Omega),$$

$$0 \leq \theta_0 \in W_{p_1}^{2-2/p_1}(\Omega),$$

and the compatibility relations. The source terms satisfy

$$\mathbf{b} \in \mathbf{L}_p(Q_T), \quad g \in L_{p_1}(Q_T), \quad g \geq 0 \ \text{a.e.}$$

The first main result concerns the existence of solutions to problem (P).

Theorem 3.1 *Existence.*
Under assumptions (D), (FE–1) – (FE–4), (BV–p) and the condition

$$0 < \sqrt{\kappa} < \nu$$

there exists for $p \geq p_n$ a solution $(\mathbf{u}, \theta) \in V(p)$ to problem (P) for any $T > 0$. Moreover, the following a priori estimates hold,

$$\| \mathbf{u} \|_{\mathbf{W}_p^{4,2}(Q_T)} \leq \Lambda, \quad \| \theta \|_{W_p^{2,1}(Q_T)} \leq \Lambda, \tag{3.1}$$

with a constant Λ dependent on the data of the problem, Ω and time T.

The condition between κ and ν is needed for parabolic decomposition of elasticity equation (1.1).
This theorem has several consequences concerning regularity of the solution:

Corollary 3.1 *For a solution to problem (P) the following holds: the functions* $\mathbf{u}$, $\nabla \mathbf{u}$, $\nabla^2 \mathbf{u}$, $\mathbf{u}_t$, θ *are continuous in Q_T, and*

$$|\mathbf{u}|,\ |\nabla \mathbf{u}|,\ |\nabla^2 \mathbf{u}|,\ |\mathbf{u}_t| \ \leq \Lambda, \qquad 0 \leq \theta \leq \Lambda \quad \textit{in } Q_T,$$

$$\| \nabla^3 \mathbf{u} \|_{\mathbf{L}_p(Q_T)},\ \| \nabla \mathbf{u}_t \|_{\mathbf{L}_p(Q_T)},\ \| \nabla \theta \|_{\mathbf{L}_p(Q_T)} \leq \ \Lambda \qquad \textit{for } p_n \leq p < \infty,$$

$$c_v \leq c(\boldsymbol{\epsilon}, \theta) \leq c_{max} = c_{max}(\Lambda).$$

The second result concerns the uniqueness of the solution. The proof requires an additional regularity, which holds, provided $p > p_n$. Moreover, stronger assumptions on $F(\boldsymbol{\epsilon}, \theta)$ and g have to be imposed.

(FE–5) The function $F_1(\boldsymbol{\epsilon}, \theta)$ is of class C^4 on the set $S^2 \times [0, \infty)$, and the heat source satisfies

$$g \in L_\infty(Q_T) \quad \text{and} \quad g \geq 0 \ \ a.e.$$

Theorem 3.2 *Uniqueness.*
Let the assumptions of Theorem 3.1 and (FE–5) be satisfied, and $p > p_n$. Then the solution to the problem (P) is unique for any $T > 0$.

We note that in case of uniqueness the solution has additional continuity properties:

Corollary 3.2 *For a solution to problem (P) the following holds in case $p > p_n$:*
$\nabla^3 \mathbf{u}, \nabla \mathbf{u}_t, \nabla \theta$ *are continuous in Q_T, and satisfy the bounds*

$$|\nabla^3 \mathbf{u}|,\ |\nabla \mathbf{u}_t|,\ |\nabla \theta| \ \leq \ \Lambda.$$

Then, in particular, the stress tensor $\boldsymbol{\sigma}$ is continuous in Q_T.

The third main result establishes the stability of solution.

Theorem 3.3 *Stability.*
Under the assumptions of Theorem 3.2 the solutions (u^i, θ^i) corresponding to the right–hand sides $(\mathbf{b}^i, g^i)$, $i = 1, 2$, satisfy the inequality

$$||(\mathbf{u}^2 - \mathbf{u}^1, \theta^2 - \theta^1)||_{V(p)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}) \tag{3.2}$$

for any finite $p > p_n$ and $T > 0$, where Λ is a constant dependent on the data of the problem, Ω and time T.

4 Existence theorem – outline of the proof

The proof is based on the parabolic decomposition of equation (1.1) and the application of the Leray–Schauder fixed point theorem. Here we present the main steps, the details are given in [19].
Further on Λ denotes a constant, depending only on the data of the problem, domain and time horizon.
Step 1. Parabolic decomposition. Chosing numbers α, β so that

$$\alpha + \beta = \nu, \qquad \alpha\beta = \frac{\kappa}{4}, \tag{4.1}$$

(1.1) with initial conditions (1.3) and boundary conditions (1.5) decomposes into the following systems of BVP's for a vector field $\mathbf{w}$:

$$\begin{aligned} \mathbf{w}_t - \beta\mathbf{Q}\mathbf{w} &= \boldsymbol{\nabla} \cdot F_{/\epsilon}(\epsilon, \theta) + \mathbf{b}, \quad \text{in } Q_T, \\ \mathbf{w}(0, \mathbf{x}) &= \mathbf{u}_1(\mathbf{x}) - \alpha\mathbf{Q}\mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{w} &= 0 \quad \text{on } S_T, \end{aligned} \tag{4.2}$$

and the displacement $\mathbf{u}$:

$$\begin{aligned} \mathbf{u}_t - \alpha\mathbf{Q}\mathbf{u} &= \mathbf{w}, \quad \text{in } Q_T, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } S_T, \end{aligned} \tag{4.3}$$

The condition between parameters κ and ν assures that $\Re\alpha, \Re\beta > 0$. System (4.2),(4.3) is coupled with the BVP (1.2), (1.4),(1.6) for θ.
Step 2. The solution map. We use the Leray–Schauder theorem in the following formulation [5]:

Theorem 4.1 *Let $\mathcal{B}$ be a Banach space. Assume that $T\ :\ [0,1] \times \mathcal{B} \longrightarrow \mathcal{B}$ is a map with the following properties:*

(i) *For any fixed $\lambda \in [0,1]$ the map $T(\lambda, \cdot)\ :\ \mathcal{B} \longrightarrow \mathcal{B}$ is completely continuous.*

(ii) *For every bounded subset $\mathcal{C}$ of $\mathcal{B}$, the family of maps $T(\cdot, \chi)\ :\ [0,1] \longrightarrow \mathcal{B}$, $\chi \in \mathcal{C}$, is uniformly equicontinuous.*

(iii) *There is a bounded subset $\mathcal{C}$ of $\mathcal{B}$, such that any fixed point in $\mathcal{B}$ of $T(\lambda, \cdot)$, $0 \leq \lambda \leq 1$, is contained in $\mathcal{C}$.*

(iv) *$T(0, \cdot)$ has precisely one fixed point in $\mathcal{B}$.*

Then $T(1, \cdot)$ has at least one fixed point in $\mathcal{B}$.

To define the corresponding operator we extend the definition of $F_1(\epsilon, \theta)$ to all values of θ in $\mathbb{R}$ in such a way that it is of class C^3, and that

$$F_{1/\theta\theta}(\epsilon, \theta) \geq 0 \quad \text{for} \quad (\epsilon, \theta) \in S^2 \times (-\infty, 0).$$

We note that with such extension the specific heat coefficient $c(\epsilon,\theta)$ remains bounded from below for all $(\epsilon,\theta)\in S^2\times\mathbb{R}$. We define now the map T_λ from $V(p)$ into $V(p)$,

$$T_\lambda\ :\ (\bar{\mathbf{u}},\bar{\theta})\longrightarrow(\mathbf{u},\theta),\qquad \lambda\in[0,1],$$

by means of the following three problems: BVP for $\mathbf{w}$

$$\begin{aligned} \mathbf{w}_t-\beta\mathbf{Q}\mathbf{w}&=\lambda[\boldsymbol{\nabla}\cdot F_{/\epsilon}(\bar{\epsilon},\bar{\theta})+\mathbf{b}]\quad\text{in } Q_T,\\ \mathbf{w}(0,\mathbf{x})&=\lambda[\mathbf{u}_1(\mathbf{x})-\alpha\mathbf{Q}\mathbf{u}_0(\mathbf{x})]\quad\text{in } \Omega,\\ \mathbf{w}&=0\quad\text{on } S_T, \end{aligned}\tag{4.4}$$

where $\bar{\epsilon}=\epsilon(\bar{\mathbf{u}})$, BVP for $\mathbf{u}$

$$\begin{aligned} \mathbf{u}_t-\alpha\mathbf{Q}\mathbf{u}&=\mathbf{w}\quad\text{in } Q_T,\\ \mathbf{u}(0,\mathbf{x})&=\lambda\mathbf{u}_0(\mathbf{x})\quad\text{in } \Omega,\\ \mathbf{u}&=0\quad\text{on } S_T, \end{aligned}\tag{4.5}$$

and BVP for θ

$$\begin{aligned} c_\lambda(\epsilon,\bar{\theta})\theta_t-k\Delta\theta&=\lambda[\bar{\theta}F_{/\theta\epsilon}(\epsilon,\bar{\theta}):\epsilon_t+\nu(\mathbf{A}\epsilon_t):\epsilon_t+g]\quad\text{in } Q_T,\\ \theta(0,\mathbf{x})&=\lambda\theta_0(\mathbf{x})\quad\text{in } \Omega,\\ \boldsymbol{\nabla}\theta\cdot\mathbf{n}&=0\quad\text{on } S_T, \end{aligned}\tag{4.6}$$

where

$$c_\lambda(\epsilon,\bar{\theta})=c_v-\lambda\bar{\theta}F_{/\theta\theta}(\epsilon,\bar{\theta}).$$

Clearly a fixed point of T_1 in V(p) is equivalent to a solution $(\mathbf{u},\theta)$ in V(p) of the decomposed system and thus constitutes a solution to problem (P) in V(p).
In further steps of the proof we shall verify assumptions of Theorem 4.1.
Step 3. First we show that T_λ is well defined in $V(p)$, i.e. the image $T_\lambda(V(p))$ belongs to $V(p)$. Here we use the fact that the systems (4.2),(4.3) are parabolic in the general Solonnikov sense [19], and therefore the following result applies [22],[21].

Lemma 4.1 *For a domain with $\partial\Omega\in C^2$, the solution of the system*

$$\begin{aligned} \mathbf{u}_t-\mathbf{Q}\mathbf{u}&=\mathbf{f}\quad\textit{in } Q_T,\\ \mathbf{u}(0,\mathbf{x})&=\mathbf{u}_0\quad\textit{in } \Omega,\\ \mathbf{u}&=0\quad\textit{on } S_T, \end{aligned}$$

satisfies the following inequality for $1<p<\infty$,

$$\|\mathbf{u}\|_{\mathbf{W}_p^{2,1}(Q_T)}\le\Lambda\{\|\mathbf{f}\|_{\mathbf{L}_p(Q_T)}+\|\mathbf{u}_0\|_{\mathbf{W}_p^{2-2/p}(\Omega)}\}.$$

The application of Lemma 4.1 to BVP (4.4) implies $\mathbf{w}\in\mathbf{W}_p^{2,1}(Q_T)$, $\mathbf{u}\in\mathbf{W}_p^{4,2}(Q_T)$ and the corresponding bounds by $(\bar{\mathbf{u}},\bar{\theta})$ in $V(p)$–norm. Furthermore, from the classical parabolic theory [15], $\theta\in W_p^{2,1}(Q_T)$ and is bounded by the same norm. Therefore $T_\lambda(\bar{\mathbf{u}},\bar{\theta})\in V(p)$.

Step 4. Equicontinuity of T_λ. A direct comparison of solutions $(\mathbf{w}^i, \mathbf{u}^i, \theta^i)$, $i = 1, 2$, corresponding to $\lambda^i \in [0,1]$, using Lemma 4.1, shows that

$$\|\mathbf{w}^1 - \mathbf{w}^2\|_{\mathbf{W}_p^{2,1}(Q_T)}, \ \|\mathbf{u}^1 - \mathbf{u}^2\|_{\mathbf{W}_p^{4,2}(Q_T)} \le \Lambda \mid \lambda^1 - \lambda^2 \mid . \tag{4.7}$$

By the classical parabolic theory, using (4.7), we get estimate

$$\|\theta^1 - \theta^2\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda \mid \lambda^1 - \lambda^2 \mid . \tag{4.8}$$

Thus assumption (ii) of the Leray–Schauder theorem is satisfied.
Step 5. Uniqueness of the fixed point of T_λ for $\lambda = 0$.
By the regularity of the problem, for $\lambda = 0$ the system (4.4)–(4.6) has the unique solution $(\mathbf{w}, \mathbf{u}, \theta) = (0,0,0)$. Therefore $V(p) \ni (\mathbf{u}, \theta) = (0,0)$ is the unique fixed point of $T_0(\cdot)$.

The essential part of the proof is the verification of assumption (iii) of the Leray–Schauder theorem.
Step 6. A priori bound for a fixed point. Without loss of generality we set $\lambda = 1$ and assume that $(\mathbf{u}, \theta) \in V(p)$ is a fixed point of T_1.
Step 6.1 First we show that temperature is nonnegative. With this we prove energy estimates, and then improve them reccursively.

Lemma 4.2 *If $(\mathbf{u}, \theta)$ is a fixed point of T_1, that is it constitutes a solution to problem (P) in $V(p)$, then $\theta \ge 0$ in Q_T.*

Proof. Let $(\mathbf{u}, \theta) \in V(p)$ for $p = n + 2$, and $g \in L_p(Q_T)$. We consider the parabolic problem for η:

$$\begin{aligned} \eta_t - \nabla \cdot (\bar{k} \nabla \eta) + \mathbf{d} \cdot \nabla \eta - a\eta &= f \quad \text{in } Q_T, \\ \eta(0, \mathbf{x}) &= \theta_0(\mathbf{x}) \quad \text{in } \Omega, \\ \nabla \eta \cdot \mathbf{n} &= 0 \quad \text{on } S_T, \end{aligned} \tag{4.9}$$

where

$$\bar{k} = \frac{k}{c(\epsilon, \theta)}, \qquad \mathbf{d} = \nabla \bar{k},$$

$$a = \frac{F_{/\theta\epsilon}(\epsilon, \theta) : \epsilon_t}{c(\epsilon, \theta)}, \qquad f = \frac{\nu(\mathbf{A}\epsilon_t) : \epsilon_t + g}{c(\epsilon, \theta)}.$$

By assumption and the imbedding theorems, θ, ϵ, $\nabla\epsilon$ are continuous in Q_T, ϵ_t ,$\nabla\theta$ are bounded in $\mathbf{L}_p(\Omega)$ for any $t \in [0, T]$, the coefficient $\bar{k}$ is bounded from below and from above by positive constants, and the right–hand side f is nonnegative.

With this we check that the assumptions of the existence theorem [15], Thm III 5.1, are satisfied. Therefore there exists the unique generalized solution to problem (4.9), with $\eta \in V_2^{1,1/2}(Q_T)$. Next we apply the modified version [19] of the stability result [15], Thm III 4.5, to the solutions of the problem (4.9). To this purpose we take smooth functions a^m, d_i^m, $f^m \ge 0$, $\theta_0^m \ge 0$ converging to a, d_i, f, θ_0 in appropriate norms. By maximum principle [15], Thm I 2.2, the classical solution η^m to problem (4.9) with smooth data satisfies $\eta^m \ge 0$ in Q_T. By stability theorem

$$\eta^m \to \eta \qquad \text{strongly in} \quad V_2^{1,0}(Q_T).$$

For the definition of spaces $V_2^{1,1/2}(Q_T)$, $V_2^{1,0}(Q_T)$ see [15]. With this we conclude the following: For any nonnegative smooth function ϕ and any $t \in (0,T)$

$$0 \leq \int_\Omega \phi\eta^m \, dx = \int_\Omega \phi(\eta^m - \eta) \, dx + \int_\Omega \phi\eta \, dx,$$

where the first integral on the right–hand side converges to 0. Therefore,

$$\int_\Omega \phi\eta \, dx \geq 0$$

and consequently $\eta \geq 0$ a.e. in Q_T. It is now enough to observe that η coincides with θ, since (4.9) is equivalent to (1.2),(1.4),(1.6). □

Step 6.2 Energy estimates. We multiply equation (1.1) by $\mathbf{u}_t$, equation (1.2) by 1, and integrate over Q_t performing integration by parts and using boundary conditions. Then in view of the nonnegativity of temperature and boundedness from below of the internal energy, using Gronwall's inequality we arive at

Lemma 4.3 *A fixed point of T_1 satisfies, for almost any $t \in (0,T)$,*

$$\int_\Omega [\frac{1}{2}|\mathbf{u}_t|^2 + c_v\theta + (F_1(\epsilon,\theta) - \theta F_{1/\theta}(\epsilon,\theta)) + F_2(\epsilon) + \frac{\kappa}{8}|\mathbf{Qu}|^2] \, dx \leq \Lambda, \tag{4.10}$$

with a constant Λ dependent only on the initial data, the sources $\mathbf{b}, g$ and time T.

Step 6.3 Improvement of estimates. The energy estimates allow to obtain more refined bounds for the fixed point. The essential tool is the following ellipticity property of the operator $\mathbf{Q}$, see Nečas [16] p. 260:

Lemma 4.4 *Assume $\partial\Omega \in C^2$. Then for the solution of the problem*

$$\mathbf{Qu} = \mathbf{f} \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

it holds

$$\mathbf{f} \in L_2(\Omega) \Longrightarrow \mathbf{u} \in \mathbf{W}_2^2(\Omega).$$

Due to this property, and (4.10), we obtain the following chain of implications

$$\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{W}_2^2(\Omega))} \leq \Lambda \;\Rightarrow\; \|\epsilon\|_{L_\infty(0,T;\mathbf{W}_2^1(\Omega))} \leq \Lambda \;\Rightarrow\; \|\epsilon\|_{L_\infty(0,T;\mathbf{L}_{q_n}(\Omega))} \leq \Lambda. \tag{4.11}$$

The further procedure consists in obtaining more estimates from the decomposed system. To this purpose we use the regularity property of parabolic systems given by the following generalization [19] of Friedman–Nečas Lemma [13]:

Lemma 4.5 *Let $\mathbf{u}$ be a solution of*

$$\begin{aligned} \mathbf{u}_t - \mathbf{Qu} &= \mathbf{f} + \nabla \cdot \boldsymbol{\sigma} \quad \text{in } Q_T, \\ \mathbf{u}(0,\mathbf{x}) &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } S_T. \end{aligned}$$

If $\mathbf{f}, \boldsymbol{\sigma} \in \mathbf{L}_p(Q_T)$ for $1 < p < \infty$, then

$$\| \nabla\mathbf{u} \|_{\mathbf{L}_p(Q_T)} \leq \Lambda(p,T,\Omega)(\| \mathbf{f} \|_{\mathbf{L}_p(Q_T)} + \| \boldsymbol{\sigma} \|_{\mathbf{L}_p(Q_T)}).$$

Using growth conditions (FE–2) and Lemma 4.5 we get

$$\|F_{/\epsilon}\|_{\mathbf{L}_{pn}(Q_T)} \le \Lambda + \Lambda\|\theta\|^{2/p_n}_{L_2(Q_T)} \quad \Rightarrow \quad \|\nabla \mathbf{w}\|_{\mathbf{L}_{pn}(Q_T)} \le \Lambda + \Lambda\|\theta\|^{2/p_n}_{L_2(Q_T)}. \tag{4.12}$$

Consequently, by Lemma 4.1,

$$\|\nabla \mathbf{u}\|_{\mathbf{W}^{2,1}_{pn}(Q_T)} \le \Lambda + \Lambda\|\theta\|^{2/p_n}_{L_2(Q_T)}. \tag{4.13}$$

With these estimates we are able to prove the temperature bounds. To this end we multiply the temperature equation by θ and integrate over Q_t using boundary conditions. Due to growth conditions and Gronwall's inequality we get

Lemma 4.6 *There exists a constant* Λ, *such that*

$$\|\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta\|_{\mathbf{L}_2(Q_T)} \le \Lambda.$$

By this lemma it follows from (4.13) by imbedding theorems that

$$\|\epsilon\|_{\mathbf{W}^{2,1}_{pn}(Q_T)} \le \Lambda \quad \Rightarrow \quad |\epsilon| \le \Lambda \quad \text{in } Q_T, \quad \|\nabla\epsilon\|_{\mathbf{L}_p(Q_T)} \le \Lambda \quad \text{for} \quad p \ge p_n. \tag{4.14}$$

Continuity of ϵ, together with (FE–1), (FE–2) implies

$$c_v \le c(\epsilon, \theta) \le c_{max}(\Lambda) \quad \text{in } \ Q_T. \tag{4.15}$$

We apply now the classical parabolic theory to BVP for θ. Estimating the right–hand side in $L_{pn/2}(Q_T)$ norm we conclude

$$\|\theta\|_{W^{2,1}_{pn/2}(Q_T)} \le \Lambda \quad \Rightarrow \quad \|\nabla\theta\|_{\mathbf{L}_{pn}(Q_T)}, \ \|\theta\|_{L_p(Q_T)} \le \Lambda \quad \text{for } p \ge \frac{1}{2}p_n. \tag{4.16}$$

Next we perform an iterative improvement of a priori bounds. Going back to the decomposed system we conclude the following:

$$\|\nabla \cdot F_{/\epsilon}\|_{\mathbf{L}_{pn}(Q_T)} \le \Lambda \quad \Rightarrow \quad \mathbf{w} \in \mathbf{W}^{2,1}_{pn}(Q_T) \quad \Rightarrow \quad \mathbf{u} \in \mathbf{W}^{4,2}_{pn}(Q_T). \tag{4.17}$$

Hence, by the imbedding,

$$\|\epsilon_t\|_{\mathbf{L}_p(Q_T)} \le \Lambda \quad \text{for} \quad p \ge p_n. \tag{4.18}$$

Cosidering again the temperature equation gives the following:

$$\|\theta\|_{W^{2,1}_p(Q_T)} \le \Lambda \quad \text{for } p \ge p_n \quad \Rightarrow \quad |\theta| \le \Lambda \quad \text{in } Q_T, \ \|\nabla\theta\|_{\mathbf{L}_p(Q_T)} \le \Lambda \quad \text{for } p \ge p_n.$$

In the last iteration, using the above bounds, we get

$$\|\nabla \cdot F_{/\epsilon}\|_{\mathbf{L}_p(Q_T)} \le \Lambda \quad \text{for } p \ge p_n,$$

hence

$$\|\mathbf{w}\|_{\mathbf{W}^{2,1}_p(Q_T)} \le \Lambda \quad \Rightarrow \quad \|\mathbf{u}\|_{\mathbf{W}^{4,2}_p(Q_T)} \le \Lambda \quad \text{for } p \ge p_n.$$

This completes the derivation of a priori bounds for a fixed point.

Step 7. Complete continuity. Let $(\bar{\mathbf{u}}^n, \bar{\theta}^n)$ be a bounded sequence in $V(p)$, such that

$$(\bar{\mathbf{u}}^n, \bar{\theta}^n) \rightharpoonup (\bar{\mathbf{u}}, \bar{\theta}) \quad \text{weakly in} \quad V(p) \quad \text{for} \quad n \to \infty.$$

We shall show that for the images of T_λ

$$(\mathbf{u}^n, \theta^n) = T_\lambda(\bar{\mathbf{u}}^n, \bar{\theta}^n) \tag{4.19}$$

it holds

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in} \quad \mathbf{W}_p^{4,2}(Q_T), \tag{4.20}$$

$$\theta^n \to \theta \quad \text{strongly in} \quad W_p^{2,1}(Q_T) \tag{4.21}$$

for $n \to \infty$, where

$$(\mathbf{u}, \theta) = T_\lambda(\bar{\mathbf{u}}, \bar{\theta}), \tag{4.22}$$

that is T_λ maps the bounded subsets into precompact subsets in $V(p)$. The arguments are based on the Aubin compactness theorem, which implies strong convergences

$$\bar{\mathbf{u}}^n \to \bar{\mathbf{u}} \quad \text{in} \quad L_p(0,T,\mathbf{W}_p^3(\Omega)), \quad \bar{\theta}^n \to \bar{\theta} \quad \text{in} \quad L_p(0,T,W_p^1(\Omega)),$$

and therefore

$$\nabla \cdot F_{/\epsilon}(\bar{\epsilon}^n, \bar{\theta}^n) \to \nabla \cdot F_{/\epsilon}(\bar{\epsilon}, \bar{\theta}) \quad \text{strongly in} \quad \mathbf{L}_p(Q_T),$$

where $\bar{\epsilon}^n = \epsilon(\bar{\mathbf{u}}^n)$, $\quad \bar{\epsilon} = \epsilon(\bar{\mathbf{u}})$. Consequently, an application of Lemma 4.1 implies that

$$\mathbf{w}^n \to \mathbf{w} \quad \text{strongly in} \quad \mathbf{W}_p^{2,1}(Q_T).$$

Thereby, the convergence (4.20) results.
To prove the convergence (4.21) we consider the BVP for the difference $\eta^n = \theta^n - \theta$. By exploiting the additional regularity of data g, θ_0 in (BV–p), which implies that $\theta^n \in W_{p_1}^{2,1}(Q_T)$, we show that the right–hand side converges to 0 in $L_p(Q_T)$ norm, and therefore $\eta^n \to 0$ strongly in $W_p^{2,1}(Q_T)$. □

5 Uniqueness theorem – outline of the proof

The proof consists in the direct comparison of two solutions by means of energy estimates and the application of Gronwall's inequality. Let $(\mathbf{u}^1, \theta^1)$ and $(\mathbf{u}^2, \theta^2)$ be the solutions corresponding to the same data. To simplify notation we set for $i = 1, 2$

$$\mathbf{v} = \mathbf{u}^2 - \mathbf{u}^1 \qquad \eta = \theta^2 - \theta^1 \qquad \epsilon^i = \epsilon(\mathbf{u}^i) \qquad \epsilon_t^i = \epsilon(\mathbf{u}_t^i),$$

$$c^i = c(\epsilon^i, \theta^i) \qquad \gamma^i = \frac{1}{c^i} \qquad F_{/\epsilon}^i = F_{/\epsilon}(\epsilon^i, \theta^i) \qquad F_{/\theta\epsilon}^i = F_{/\theta\epsilon}(\epsilon^i, \theta^i).$$

The difference $(\mathbf{v},\eta) \in V(p)$ satisfies the following BVP:

$$\mathbf{v}_{tt} - \nu \mathbf{Q}\mathbf{v}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} = \boldsymbol{\nabla}\cdot(F^2_{/\epsilon} - F^1_{/\epsilon}), \tag{5.1}$$

$$\begin{aligned}\eta_t - k\gamma^1\Delta\eta &= \gamma^2\theta^2 F^2_{/\theta\epsilon} : \boldsymbol{\epsilon}^2_t - \gamma^1\theta^1 F^1_{/\theta\epsilon} : \boldsymbol{\epsilon}^1_t \\ &\quad + \nu\gamma^2(\mathbf{A}\boldsymbol{\epsilon}^2_t) : \boldsymbol{\epsilon}^2_t - \nu\gamma^1(\mathbf{A}\boldsymbol{\epsilon}^1_t) : \boldsymbol{\epsilon}^1_t \\ &\quad + (\gamma^2 - \gamma^1)g \\ &\quad + k(\gamma^2 - \gamma^1)\Delta\theta^2 \\ &=: R_1 + R_2 + R_3 + R_4 \quad \text{in} \quad Q_T,\end{aligned} \tag{5.2}$$

$$\mathbf{v}(0,x) = 0, \quad \mathbf{v}_t(0,x) = 0, \quad \eta(0,x) = 0 \quad \text{in} \quad \Omega, \tag{5.3}$$

$$\mathbf{v} = \mathbf{Q}\mathbf{v} = 0, \quad \boldsymbol{\nabla}\eta\cdot\mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{5.4}$$

In the first step we obtain energy estimates for the mechanical part. To this end we multiply (5.1) by $\mathbf{v}_t$ and integrate over Q_t. Integration by parts, and the use of initial and boundary conditions yield for almost any $t \in (0,T)$ the identity

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2} \mid \mathbf{v}_t \mid^2 + \frac{\kappa}{8} \mid \mathbf{Q}\mathbf{v} \mid^2\right) dx + \nu \int_{Q_t} (\mathbf{A}\boldsymbol{\epsilon}(\mathbf{v}_t)) : \boldsymbol{\epsilon}(\mathbf{v}_t)\, dxdt' \\ &= -\int_{Q_t} (F^2_{/\epsilon} - F^1_{/\epsilon}) : \boldsymbol{\epsilon}(\mathbf{v}_t)\, dxdt'.\end{aligned} \tag{5.5}$$

Besides, we have the following identity,

$$\frac{1}{2}\int_\Omega |\boldsymbol{\epsilon}(\mathbf{v})|^2\, dx = \frac{1}{2}\int_{Q_t} \frac{d}{dt}|\boldsymbol{\epsilon}(\mathbf{v})|^2\, dxdt' = \int_{Q_t} \boldsymbol{\epsilon}(\mathbf{v}) : \boldsymbol{\epsilon}(\mathbf{v}_t)\, dxdt'. \tag{5.6}$$

Combining (5.5) and (5.6) and using the estimate

$$\mid F^2_{/\epsilon} - F^1_{/\epsilon} \mid \leq \Lambda(\mid \boldsymbol{\epsilon}(\mathbf{v}) \mid + \mid \eta \mid), \tag{5.7}$$

which follows from the regularity assumption (FE–5) and the uniform bounds on $\boldsymbol{\epsilon}^i, \theta^i$ in Q_T, by Young's and Gronwall's inequalities, we arrive at

$$||\mathbf{v}_t||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\boldsymbol{\epsilon}(\mathbf{v})||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\mathbf{Q}\mathbf{v}||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\boldsymbol{\epsilon}(\mathbf{v}_t)||_{\mathbf{L}_2(Q_T)} \leq \Lambda ||\eta||_{L_2(Q_T)}. \tag{5.8}$$

Hence, due to Lemma 4.4,

$$||\mathbf{v}||_{L_\infty(0,T;\mathbf{W}^2_2(\Omega))} \leq \Lambda ||\eta||_{L_2(Q_T)}. \tag{5.9}$$

The energy estimates for the thermal part are obtained from multiplying equation (5.2) by η and integrating over Q_t. In view of the bound

$$0 < \frac{k}{c_{max}} \leq k\gamma^1,$$

it follows that

$$\frac{1}{2}\int_\Omega \eta^2\, dx + \frac{k}{c_{max}}\int_{Q_t} |\boldsymbol{\nabla}\eta|^2\, dxdt' \leq -\int_{Q_t} k\eta\boldsymbol{\nabla}\eta\cdot\boldsymbol{\nabla}\gamma^1\, dxdt' + \sum_{i=1}^4 \int_{Q_t} R_i\eta\, dxdt'. \tag{5.10}$$

Further procedure consists in majorizing the right–hand side of (5.10) and the subsequent application of Gronwall's inequality. To this end we use the uniform bounds on $\boldsymbol{\epsilon}^i, \theta^i$, $\nabla \boldsymbol{\epsilon}^i$, $\boldsymbol{\epsilon}^i_t$ and $\nabla\theta$, which imply uniform bounds on c^i and $\nabla\gamma^i$ in Q_T, as well as the estimates

$$|R_1|,\ \ |R_2|,\ \ |R_3| \le \Lambda(|\eta| + |\boldsymbol{\epsilon}(\mathbf{v})| + |\boldsymbol{\epsilon}(\mathbf{v}_t)|). \tag{5.11}$$

All integrals in (5.10), except the last one, are estimated directly by applying Young's inequality and (5.8). The R_4–term is first integrated by parts, and majorized by using the bounds

$$|\gamma^2 - \gamma^1| \le \Lambda(|\eta| + |\boldsymbol{\epsilon}(\mathbf{v})|), \tag{5.12}$$
$$|\nabla(\gamma^2 - \gamma^1)| \le \Lambda(|\eta| + |\nabla\eta| + |\boldsymbol{\epsilon}(\mathbf{v})| + |\nabla\boldsymbol{\epsilon}(\mathbf{v})|), \tag{5.13}$$

and estimate (5.8). In consequence we arrive at

$$\int_\Omega \eta^2(t,\mathbf{x})\,dx + \frac{k}{2c_{max}}\int_{Q_t} |\nabla\eta|\,dxdt' \le \Lambda\int_{Q_t}\eta^2\,dxdt'. \tag{5.14}$$

Hence, via Gronwall's inequality, $\eta = 0$ in Q_T. Simultaneously, by inequality (5.9), $\mathbf{v} = 0$ in Q_T. This completes the proof of uniqueness. □

6 Stability theorem – outline of the proof

The proof uses similar arguments as the existence theorem, utilizing:

- parabolic decomposition;
- Solonnikov theory of parabolic systems, Lemma 4.1;
- ellipticity property of operator $\mathbf{Q}$, Lemma 4.4;
- regularity property of parabolic systems, Lemma 4.5.

Here we present the main steps of the proof, the details are given in [20].

The starting point are the energy estimates for $\mathbf{v} = \mathbf{u}^2 - \mathbf{u}^1$, $\eta = \theta^2 - \theta^1$, where $(\mathbf{u}^i, \theta^i)$ are solutions to the problem (P) corresponding to $(\mathbf{b}^i, g^i)$. We use the same simplifying notation as in the uniqueness proof. Functions $(\mathbf{v}, \eta) \in V(p)$ satisfy BVP (5.1)–(5.4), where in addition the right–hand side of (5.1) contains $\mathbf{b}^2 - \mathbf{b}^1$ and R_3–term in (5.2) has the form

$$R_3 = \gamma^2 g^2 - \gamma^1 g^1.$$

By repeating the arguments of the uniqueness proof, we get the following energy inequality:

$$\begin{aligned}||\mathbf{v}_t||_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + ||\mathbf{v}||_{L_\infty(0,T;\mathbf{W}^2_2(\Omega))} + ||\eta||_{L_\infty(0,T;L_2(\Omega))} + ||\boldsymbol{\epsilon}(\mathbf{v}_t)||_{\mathbf{L}_2(Q_T)} + ||\nabla\eta||_{\mathbf{L}_2(Q_T)} \le \\ \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}),\end{aligned} \tag{6.1}$$

which leads to more refined estimates. By virtue of imbedding and parabolic estimates it follows from (6.1) that

$$||\boldsymbol{\epsilon}(\mathbf{v})||_{L_\infty(0,T;\mathbf{L}_{q_n}(\Omega))} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}), \tag{6.2}$$

$$||\eta||_{L_{2p_n/n}(Q_T)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}). \tag{6.3}$$

Let $(\mathbf{w}^1, \mathbf{u}^1), (\mathbf{w}^1, \mathbf{u}^1)$ denote the corresponding solutions of the decomposed problems (4.2),(4.3) and

$$\mathbf{z} = \mathbf{w}^2 - \mathbf{w}^1.$$

The functions $(\mathbf{z}, \mathbf{v})$ satisfy the following BVP's:

$$\begin{aligned} \mathbf{z}_t - \beta \mathbf{Q} \mathbf{z} &= \boldsymbol{\nabla} \cdot (F^2_{/\epsilon} - F^1_{/\epsilon}) + \mathbf{b}^2 - \mathbf{b}^1 \quad \text{in} \quad Q_T, \\ \mathbf{z}(0, x) &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{z} &= 0 \quad \text{on} \quad S_T, \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} \mathbf{v}_t - \alpha \mathbf{Q} \mathbf{v} &= \mathbf{z} \quad \text{in} \quad Q_T, \\ \mathbf{v}(0, x) &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{v} &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{6.5}$$

By using Lemma 4.5 and estimate (5.7) we get

$$||\boldsymbol{\nabla} \mathbf{z}||_{\mathbf{L}_p(Q_T)} \leq \Lambda(||\epsilon(\mathbf{v})||_{\mathbf{L}_p(Q_T)} + ||\eta||_{L_p(Q_T)} + ||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)}) \tag{6.6}$$

for $1 < p < \infty$. Consequently, the application of (6.2) and (6.3) yields

$$||\boldsymbol{\nabla} \mathbf{z}||_{\mathbf{L}_p(Q_T)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}) \quad \text{for} \quad p \leq \frac{2p_n}{n}. \tag{6.7}$$

We note that in case $2 \leq n \leq 4$,

$$2 \leq \frac{p_n}{2} \leq \frac{2p_n}{n} < q_n.$$

By virtue of Lemma 4.1, (6.7) implies

$$||\boldsymbol{\nabla} \mathbf{v}||_{\mathbf{W}^{2,1}_p(Q_T)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}) \quad \text{for} \quad p \leq \frac{2p_n}{n}. \tag{6.8}$$

Hence, in particular (provided $n \leq 4$),

$$||\epsilon(\mathbf{v})||_{\mathbf{W}^{2,1}_{p_n/2}(Q_T)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n/2}(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}). \tag{6.9}$$

The application of the imbedding theorem implies that

$$||\boldsymbol{\nabla} \epsilon(\mathbf{v})||_{\mathbf{L}_{p_n}(Q_T)}, \quad ||\epsilon(\mathbf{v})||_{\mathbf{L}_p(Q_T)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n/2}}(Q_T) + ||g^2 - g^1||_{L_2(Q_T)}) \tag{6.10}$$

for $p \geq \frac{p_n}{2}$. The above estimates allow us to obtain further bounds on η by applying again the classical parabolic theory. To this end let us rewrite η–equation (5.2) in the form

$$c^1 \eta - k \Delta \eta = \sum_{i=1}^{4} R_i^* \quad \text{where} \quad R_i^* = c^1 R_i. \tag{6.11}$$

We start with limiting the right–hand side in $L_2(Q_T)$–norm by

$$\Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}).$$

The terms R_1^*, R_2^*, R_3^* are estimated directly by using bounds (5.11),(6.3),(6.8). The R_4^*–term is estimated by applying Hölder inequality, and using bounds (5.12),(6.2),(6.3). Since the coefficient c^1 in (6.11) is positive and bounded, the parabolic theory gives

$$||\eta||_{W_2^{2,1}(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}). \qquad (6.12)$$

Hence, by imbedding theorem,

$$||\eta||_{L_p(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_2(Q_T)} + ||g^2 - g^1||_{L_2(Q_T)}) \quad \text{for} \quad 2 \le p \le \frac{q_n p_n}{n}. \qquad (6.13)$$

We note that in case $n > 2$,

$$\frac{q_n p_n}{n} = q_n \frac{n+2}{n} > q_n.$$

With estimates (6.10),(6.13) we can bound the right–hand side of η–equation (6.11) in $L_{p_n/2}(Q_T)$–norm by $\Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n/2}(Q_T)} + ||g^2 - g^1||_{L_{p_n/2}(Q_T)})$. To this purpose we use similar arguments as above. Consequently, we arrive at

$$||\eta||_{W_{p_n/2}^{2,1}(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n/2}(Q_T)} + ||g^2 - g^1||_{L_{p_n/2}(Q_T)}), \qquad (6.14)$$

and by the imbedding,

$$||\nabla\eta||_{\mathbf{L}_{p_n}(Q_T)},\ ||\eta||_{L_p(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n/2}(Q_T)} + ||g^2 - g^1||_{L_{p_n/2}(Q_T)}) \qquad (6.15)$$

for $p \ge \frac{p_n}{2}$. This concludes the first round of estimation. In order to obtain further improvements, we return again to the system (6.4), (6.5). Using bound

$$|\nabla \cdot (F^2_{/\epsilon} - F^1_{/\epsilon})| \le \Lambda(|\boldsymbol{\epsilon}(\mathbf{v})| + |\eta| + |\nabla\boldsymbol{\epsilon}(\mathbf{v})| + |\nabla\eta|), \qquad (6.16)$$

which holds owing to the continuity of $\boldsymbol{\epsilon}^i, \theta^i, \nabla\boldsymbol{\epsilon}^i$ and $\nabla\theta^i$, together with estimates (6.10),(6.15), Lemma 4.5 gives

$$||\mathbf{z}||_{\mathbf{W}_{p_n}^{2,1}(Q_T)},\ ||\mathbf{v}||_{\mathbf{W}_{p_n}^{4,2}(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n}(Q_T)} + ||g^2 - g^1||_{L_{p_n/2}(Q_T)}). \qquad (6.17)$$

Hence, due to imbedding,

$$||\nabla\boldsymbol{\epsilon}(\mathbf{v})||_{\mathbf{L}_p(Q_T)},\ ||\boldsymbol{\epsilon}(\mathbf{v}_t)||_{\mathbf{L}_p(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n}(Q_T)} + ||g^2 - g^1||_{L_{p_n/2}(Q_T)}) \qquad (6.18)$$

for $p \ge p_n$. Now we can return again to η–equation and bound its right–hand side in $L_p(Q_T)$–norm for any $p \ge p_n$ by $\Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n/2}(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)})$. In the analysis of the R_4^*–term by Hölder inequality we take advantage of the fact, that $\theta^i \in W_{p_1}^{2,1}(Q_T)$ for $p_1 = p + \delta, \quad \delta > 0$.

Similarly as above, the parabolic theory gives

$$||\eta||_{W_p^{2,1}(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n}(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}) \quad \text{for} \quad p \ge p_n. \qquad (6.19)$$

By the imbedding,

$$||\nabla\eta||_{\mathbf{L}_p(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_{p_n}(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}) \quad \text{for} \quad p \ge p_n. \qquad (6.20)$$

Thanks to (6.10), (6.15), (6.18), (6.20) we can estimate the right–hand side of system (6.4) in $\mathbf{L}_p(Q_T)$–norm for $p \ge p_n$. The subsequent application of Solonnikov theory allows to obtain the final estimates

$$||\mathbf{z}||_{\mathbf{W}_p^{2,1}(Q_T)},\ ||\mathbf{v}||_{\mathbf{W}_p^{4,2}(Q_T)} \le \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}), \qquad (6.21)$$

This completes the proof. □

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