OPTIMAL CONTROL FOR SEMILINEAR ABSTRACT EQUATIONS OF PARABOLIC TYPE

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1. INTRODUCTION

In the preceding paper [8], the authors studied the optimal control problems for the Keller-Segel equations. In that paper we showed the existence of optimal control and the first order necessary condition by formulating the Keller-Segel equations as a semilinear abstract equation. Many papers have already been published to study the control problems for nonlinear parabolic equations. In the books Ahmed [1] and Barbu [2], some general frameworks are given for handling the semilinear parabolic equations with monotone perturbations. In [1] the nonlinear terms are monotone functions with linear growth, and in [2] they are generalized to the multivalued maximal monotone operators determined by lower semicontinuous convex functions. Papageorgiou et al. [3] have studied some quasilinear parabolic equations of monotone type. This note is the generalization of [8] as a semilinear abstract equation of non-monotone type.

Notations. $\mathbb{R}$ denotes the sets of real numbers. Let $I$ be an interval in $\mathbb{R}$. $L^p(I;\mathcal{H})$, $1 \leq p \leq \infty$, denotes the $L^p$ space of measurable functions in $I$ with values in a Hilbert space $\mathcal{H}$. $C(I;\mathcal{H})$ denotes the space of continuous functions in $I$ with values in $\mathcal{H}$. Let $\mathcal{D}(I)$ denote the space of $C^\infty$-functions with compact support on $I$ and $\mathcal{D}'(I)$ denote the space of distributions on $I$. For simplicity, we shall use a universal constant $C$ to denote various constants which are determined in each occurrence in a specific way by $\delta, M$, and so forth. In a case when $C$ depends also on some parameter, say $\theta$, it will be denoted by $C_\theta$.

2. THE FORMULATION OF PROBLEM

Let $\mathcal{V}$ and $\mathcal{H}$ be two separable real Hilbert spaces with dense and compact embedding $\mathcal{V} \hookrightarrow \mathcal{H}$. Identifying $\mathcal{H}$ and its dual $\mathcal{H}'$ and denoting the dual space of $\mathcal{V}$ by $\mathcal{V}'$, we have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. We denote the scalar product of $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|$. The duality product between $\mathcal{V}'$ and $\mathcal{V}$ which coincides with the scalar product of $\mathcal{H}$ on $\mathcal{H} \times \mathcal{H}$ is denoted by $\langle \cdot, \cdot \rangle$, and the norms of $\mathcal{V}$ and $\mathcal{V}'$ by $||\cdot||$ and $||\cdot||_*$, respectively.

$\mathcal{U} = L^2(0,T;\mathcal{V}')$ and $\mathcal{U}_{ad}$ is closed, bounded and convex subset of $\mathcal{U}$.

We consider the following Cauchy problem

$$\begin{align*}
\frac{dY}{dt} + AY &= F(Y) + U(t), \quad 0 < t \leq T, \\
Y(0) &= Y_0
\end{align*}$$

(E)
in the space $\mathcal{V}'$. Here, $A$ is the positive definite self-adjoint operator of $\mathcal{H}$ defined by a symmetric sesquilinear form $a(Y, \overline{Y})$ on $\mathcal{V}$, $(AY, \overline{Y}) = a(Y, \overline{Y})$, which satisfies

\begin{align*}
(a.i) & & |a(Y, \overline{Y})| \leq M||Y||\|\overline{Y}\|, & \ Y, \overline{Y} \in \mathcal{V}, \\
(a.ii) & & a(Y, Y) \geq \delta||Y||^2, & \ Y \in \mathcal{V}
\end{align*}

with some $\delta$ and $M > 0$. $A$ is also a bounded operator from $\mathcal{V}$ to $\mathcal{V}'$. $F(\cdot)$ is a given continuous function from $\mathcal{V}$ to $\mathcal{V}'$ satisfying

\begin{align*}
(f.i) & & \|F(Y)\| \leq \eta\|Y\| + \phi_\eta(|Y|), & \ Y \in \mathcal{V}; \\
(f.ii) & & \|F(\overline{Y}) - F(Y)\| \leq \eta\|\overline{Y} - Y\| + (\|\overline{Y}\| + \|Y\| + 1)\psi_\eta(|\overline{Y}| + |Y|)|\overline{Y} - Y|, & \overline{Y}, Y \in \mathcal{V}.
\end{align*}

$U(\cdot) \in L^2(0, T; \mathcal{V}')$ is a given function and $Y_0 \in \mathcal{H}$ is an initial value.

We then obtain the following result (For the proof, see Ryu and Yagi [8]).

**Theorem 2.1.** Let (a.i), (a.ii), (f.i), and (f.ii) be satisfied. Then, for any $U \in L^2(0, T; \mathcal{V}')$ and $Y_0 \in \mathcal{H}$, there exists a unique weak solution

$$Y \in H^1(0, T(Y_0, U); \mathcal{V}') \cap C([0, T(Y_0, U)]; \mathcal{H}) \cap L^2(0, T(Y_0, U); \mathcal{V})$$

to (E), the number $T(Y_0, U) > 0$ is determined by the norms $\|U\|_{L^2(0, T; \mathcal{V}')}^2$ and $|Y_0|$.

In this section we are concerned with the following problem

\begin{align*}
(P) & & \text{Minimize } J(U),
\end{align*}

where the cost functional $J(U)$ is of the form

$$J(U) = \int_0^S \|DY(U) - Y_d\|^2 dt + \gamma \int_0^S \|U\|^2 dt, \ U \in \mathcal{U}_{ad}.$$

Here, $Y(U), U \in \mathcal{U}_{ad}$, is the weak solution of (E) and is assumed to exist on a fixed interval $[0, S]$. $D$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}$ and $Y_d$ is a fixed element of $L^2(0, S; \mathcal{V})$. $\gamma$ is a nonnegative constant.

**Remark.** Let $Y_0 \in \mathcal{H}$ be fixed. By Theorem 2.1, for $U \in \mathcal{U}_{ad}$, $Y(U)$ exists on the interval $[0, T(U)]$ with $T(U) > 0$ depending on $\|U\|_{L^2(0, T; \mathcal{V}')}$. Hence, $0 < S \leq \inf\{T(U); U \in \mathcal{U}_{ad}\}$.

We prove the following theorem.
Theorem 2.2. There exists an optimal control $\bar{U} \in \mathcal{U}_{ad}$ for (P) such that

$$J(\bar{U}) = \min_{U \in \mathcal{U}_{ad}} J(U).$$

Proof. The proof can be carried out in the same way as that of Theorem 2.1 (see [8, Theorem 2.1]). As it is standard (cf. [2, Chap. 5, Proposition 1.1] and [6, Chap. III, Theorem 15.1]), we will only sketch.

Let $\{U_n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence such that $\lim_{n \to \infty} J(U_n) = \min_{U \in \mathcal{U}_{ad}} J(U)$. Since $\{U_n\}$ is bounded, we can assume that $U_n \rightharpoonup \bar{U}$ weakly in $L^2(0, S; \mathcal{V})$. For simplicity, we will write $Y_n$ instead of the solution $Y(U_n)$ of (E) corresponding to $U_n$,

$$\begin{aligned}
\frac{dY_n}{dt} + AY_n &= F(Y_n) + U_n(t), \quad 0 < t \leq S, \\
Y_n(0) &= Y_0.
\end{aligned}$$

Taking the scaler product of the equation and $Y_n$, we obtain that

$$\frac{1}{2} \frac{d}{dt} |Y_n(t)|^2 + \langle AY_n(t), Y_n(t) \rangle = \langle F(Y_n(t), Y(nt) \rangle + \langle U_n(t), Y_n(t) \rangle.$$

Then, from (a.ii) and (f.i),

$$\frac{1}{2} \frac{d}{dt} |Y_n(t)|^2 + \delta ||Y_n(t)||^2 \leq \eta ||Y_n(t)||^2 + \{\phi_\eta(||Y_n(t)||) + ||U_n(t)||_*\} ||Y_n(t)||.$$ 

With some increasing, locally Lipschitz continuous function $\phi: [0, \infty) \to [0, \infty)$, it follows that

$$\frac{d}{dt} |Y_n(t)|^2 + \delta ||Y_n(t)||^2 \leq \phi(||Y_n(t)||^2) + \frac{8}{\delta} ||U_n(t)||^2, \quad 0 < t \leq S,$$

$$|Y_n(0)|^2 = |Y_0|^2.$$ 

Let $z_n(t) = |Y_n(t)|^2 - \frac{8}{\delta} \int_0^t ||U_n(s)||^2 ds, 0 \leq t \leq S$. Since $\int_0^S ||U_n(s)||^2 ds \leq C$, it follows that

$$\frac{dz_n}{dt} \leq \phi(z_n + 8C\delta^{-1}).$$

On the other hand, let $z(t)$ be a solution to the ordinary differential equation

$$\frac{dz}{dt} = \phi(z + 8C\delta^{-1}), \quad 0 \leq t \leq S,$$

$$z(0) = |Y_0|^2.$$

Then, by the theorem of comparison, $z_n(t) \leq z(t)$ for all $0 \leq t \leq S$. Hence, $|Y_n(t)|^2 \leq \|z\|_{C([0, S])} + 8C\delta^{-1}$.

The sequence $\{Y_n\}$ is thus bounded in $L^\infty(0, S; \mathcal{H})$. As a consequence, it follows from (2.1) that $\{Y_n\}$ is bounded in $L^2(0, S; \mathcal{V})$ also. Moreover, from (f.i), $\{|dY_n|/dt\}$ is
bounded in $L^2(0, S; \mathcal{V}')$. Therefore, choosing a subsequence if necessary, we can assume that

$$Y_n \rightarrow \overline{Y} \quad \text{weakly in} \quad L^2(0, S; \mathcal{V}),$$

$$\frac{dY_n}{dt} \rightarrow \frac{d\overline{Y}}{dt} \quad \text{weakly in} \quad L^2(0, S; \mathcal{V}').$$

Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, it is shown by [5, Chap. 1, Theorem 5.1] that

(2.2) \quad \quad Y_n \rightarrow \overline{Y} \quad \text{strongly in} \quad L^2(0, S, \mathcal{H}).

Let us verify that $\overline{Y}$ is a solution to (E) with the control $\overline{U}$. Let $\xi \in D(0, S)$ and $V \in \mathcal{V}$, and put $\Phi(t) = \xi(t)V$. Then,

\[
\int_0^S \langle Y'(t), \Phi(t) \rangle dt + \int_0^S \langle AY(t), \Phi(t) \rangle dt = \int_0^S \langle F(Y(t), \Phi(t) \rangle dt + \int_0^S \langle U(t), \Phi(t) \rangle dt.
\]

Let here $n$ tend to infinity. It is then observed from (f.ii) that

\[
\int_0^S |\langle F(Y_n(t) - F(\overline{Y}(t), \Phi(t)) \rangle | dt \leq \eta \int_0^S ||Y_n(t) - \overline{Y}(t)|| ||\Phi(t)|| dt
\]

\[
+ \int_0^S (||Y_n(t)|| + ||\overline{Y}(t)|| + 1)\psi_\eta(|Y_n(t)| + |\overline{Y}(t)|)||Y_n(t) - \overline{Y}(t)|| ||\Phi(t)|| dt,
\]

where $\eta > 0$ is arbitrary. From (2.2) it is seen that $\int_0^S \langle F(Y_n), \Phi(t) \rangle dt$ converges to $\int_0^S \langle F(\overline{Y}(t)), \Phi(t) \rangle dt$ as $n \rightarrow \infty$. Therefore, we obtain that

\[
\int_0^S \langle \overline{Y}'(t), \Phi(t) \rangle dt + \int_0^S \langle A\overline{Y}(t), \Phi(t) \rangle dt
\]

\[
= \int_0^S \langle F(\overline{Y}(t), \Phi(t) \rangle dt + \int_0^S \langle \overline{U}(t), \Phi(t) \rangle dt.
\]

This then shows that $\overline{Y}(t)$ satisfies the equation of (E) for almost all $t \in (0, S)$. In a similar way it is also shown that $\overline{Y}(0) = Y_0$, note from [4, Chap. XVIII, Theorem 1] that $\overline{Y} \in C([0, S]; \mathcal{H})$. Hence, $\overline{Y}$ is the unique solution to (E) with the control $\overline{U}$; that is, $\overline{Y} = Y(\overline{U})$.

Since $Y_n - Y_d$ is weakly convergent to $\overline{Y} - Y_d$ in $L^2(0, S; \mathcal{V})$, we have:

$$\min_{U \in \mathcal{U}_{ad}} J(U) \leq J(\overline{U}) \leq \lim_{n \rightarrow \infty} J(U_n) = \min_{U \in \mathcal{U}_{ad}} J(U).$$

Hence, $J(\overline{U}) = \min_{U \in \mathcal{U}_{ad}} J(U)$.

\[\square\]
3. FIRST ORDER NECESSARY CONDITION

In this section, we show the first order necessary condition for the Problem (P). We denote the scalar products in \( \mathcal{V} \) and \( \mathcal{V}' \) by \( \langle \cdot, \cdot \rangle_{\mathcal{V}} \) and \( \langle \cdot, \cdot \rangle_{\mathcal{V}'} \), respectively. In order to the necessary conditions of optimality, we need some additional assumptions:

(f.iii) The mapping \( F(\cdot): \mathcal{V} \to \mathcal{V}' \) is Fréchet differentiable and for each \( \eta > 0 \), there exists an increasing continuous functions \( \mu_{\eta}, \nu : [0, \infty) \to [0, \infty) \) such that

\[
|\langle F'(Y)Z, P \rangle| \leq \eta \||Z||P|| + (||Y|| + 1)\mu_{\eta}(|Y|)||Z||P||, \quad Y, Z, P \in \mathcal{V},
\]

\[
\nu(||Y||)||Z||P||, \quad Y, Z, P \in \mathcal{V}.
\]

(f.iv) \( F'(\cdot) \) is continuous from \( \mathcal{H} \) into \( \mathcal{L}(\mathcal{V}, \mathcal{V}') \).

**Proposition 3.1.** Let (a.i), (a.ii), (f.i), (f.ii), (f.iii), and (f.iv) be satisfied. The mapping \( Y : \mathcal{U}_{ad} \to H^{1}(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^{2}(0, S; \mathcal{V}) \) is Gâteaux differentiable with respect to \( U \). For \( V \in \mathcal{U}_{ad}, Y'(U)V = Z \) is the unique solution in \( H^{1}(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^{2}(0, S; \mathcal{V}) \) of the problem

\[
\begin{aligned}
\frac{dZ}{dt} + AZ - F'(Y)Z = V(t), & \quad 0 < t \leq S, \\
Z(0) = 0.
\end{aligned}
\]

**Proof.** Let \( U, V \in \mathcal{U}_{ad} \) and \( 0 \leq h \leq 1 \). Let \( Y_{h} \) and \( Y \) be the solutions of (E) corresponding to \( U + hV \) and \( U \), respectively.

**Step 1.** \( Y_{h} \to Y \) strongly in \( C([0, S]; \mathcal{H}) \) as \( h \to 0 \). Let \( W = Y_{h} - Y \). Obviously, \( W \) satisfies

\[
\begin{aligned}
\frac{dW}{dt} + AW - (F(Y_{h}(t)) - F(Y(t))) = hV(t), & \quad 0 < t \leq S, \\
W(0) = 0.
\end{aligned}
\]

Taking the scalar product of the equation (3.2) with \( W \), we obtain that

\[
\frac{1}{2} \frac{d}{dt} |W(t)|^{2} + \langle AW(t), W(t) \rangle = \langle F(Y_{h}(t)) - F(Y(t)), W(t) \rangle + \langle hV(t), W(t) \rangle.
\]

Using (a.ii) and (f.ii), we have

\[
\frac{1}{2} \frac{d}{dt} |W(t)|^{2} + \delta \|W(t)\|^{2}
\leq \frac{\delta}{2} \|W(t)\|^{2} + (\|Y_{h}(t)\|^{2} + \|Y(t)\|^{2} + 1)\psi_{\frac{\delta}{4}}(|Y_{h}(t)| + |Y(t)|)W(t)\|^{2}
+ 4h^{2}\delta^{-1}\|V(t)\|^{2}.
\]
Therefore,

\begin{equation}
\frac{1}{2}|W(t)|^2 + \frac{\delta}{2} \int_0^t \|W(s)\|^2 ds
\leq \int_0^t \left( \|Y_h(s)\|^2 + \|Y(s)\|^2 + 1 \right) \psi_\frac{1}{4} \left( |Y_h(s)| + |Y(s)| \right)^2 |W(s)|^2 ds
+ 4h^2 \delta^{-1} \int_0^S \|V(s)\|^2 ds.
\end{equation}

Using Gronwall's lemma, we obtain that

\[ |W(t)|^2 \leq C h^2 \|V\|^2 \int_0^S (\|Y_h(s)\|^2 + \|Y(s)\|^2 + 1) \mu |Y_h(s)| + |Y(s)| |W(s)|^2 ds \]

for all \( t \in [0, S] \). Hence, \( Y_h \to Y \) strongly in \( C([0, S]; \mathcal{H}) \) as \( h \to 0 \).

**Step 2.** \( Y_h \to Y \) strongly in \( H^1(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V}) \) as \( h \to 0 \). We rewrite the problem (3.2) in the form

\begin{equation}
\begin{aligned}
\frac{d}{dt} \frac{Y_h - Y}{h} + A \frac{Y_h - Y}{h} - \frac{F(Y_h) - F(Y)}{h} &= V(t), \quad 0 < t \leq S, \\
\frac{Y_h - Y}{h}(0) &= 0.
\end{aligned}
\end{equation}

On the other hand, we consider the linear problem (3.1). From (a.i), (a.ii), (f.i), (f.ii), and (f.iii), we can easily verify that (3.1) possesses a unique weak solution \( Z \in H^1(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V}) \) on \([0, S] \) (cf. [4, Chap. XVIII, Theorem 2]). Define \( F_h' = \int_0^1 F'(Y + \theta(Y_h - Y)) d\theta \). Then \( \overline{W} = \frac{Y_h - Y}{h} - Z \) satisfies

\begin{equation}
\begin{aligned}
\frac{d}{dt} \overline{W}(t) + A \overline{W}(t) - F_h'(\overline{W}(t)) &= (F_h' - F'_o)Z(t), \quad 0 < t \leq S, \\
\overline{W}(0) &= 0.
\end{aligned}
\end{equation}

Taking the scalar product of the equation of (3.5) with \( \overline{W} \), we obtain that

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} |\overline{W}(t)|^2 + \langle A \overline{W}(t), \overline{W}(t) \rangle &\leq \frac{\delta}{2} |\overline{W}(t)|^2 + (\|Y(t)\|^2 + \|Y_h(t) - Y(t)\|^2 + 1) \mu (|Y_h|^2 + |Y|^2) |\overline{W}(t)|^2 \\
&\quad + \frac{4}{\delta} \| (F_h' - F'_o)Z(t) \|^2;
\end{aligned}
\]

where \( \mu : [0, \infty) \to [0, \infty) \) is some increasing continuous function. Therefore,

\begin{equation}
|\overline{W}(t)|^2 + \delta \int_0^t \|\overline{W}(s)\|^2 ds 
\leq \int_0^t (\|Y(s)\|^2 + \|Y_h(s)\|^2 + 1) \mu (|Y_h|^2 + |Y|^2) |\overline{W}(s)|^2 ds \\
&\quad + \frac{8}{\delta} \| (F_h' - F'_o)Z(t) \|^2_{L^2(0, S; \mathcal{V}')}.
\end{equation}
From (f.iii), we have \( \|F'_{h}Z(t)\|_* \leq C\|Z(t)\| \), \( t \in [0, S] \). Since \( Y_h \to Y \) strongly in \( \mathcal{H} \), it follows from (f.iv) that

\[
F'_{h}Z(t) \to F'_{0}Z(t) \quad \text{strongly in } \mathcal{V}' \text{ a.e.}
\]

By the dominated convergence theorem, we have

\[
\|(F'_{h} - F'_{0})Z(t)\|_{L^2(0, S; \mathcal{V}')}^2 \to 0 \quad \text{as } h \to 0.
\]

Using Gronwall's lemma, it follows from (3.6) that \( Y_h \to Y \) is strongly convergent in \( H^1(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V}) \). \( \square \)

With the aid of this proposition, we can easily show the first order necessary condition.

**Theorem 3.2.** Let \( \overline{U} \) be an optimal control of (P) and let \( \overline{V} \in L^2(0, S; \mathcal{V}) \cap C([0, S]; \mathcal{H}) \cap H^1(0, S; \mathcal{V}') \) be the optimal state, that is \( \overline{V} \) is the solution to (E) with the control \( \overline{U}(t) \).

Then, there exists a unique solution \( P \in L^2(0, S; \mathcal{V}) \cap C([0, S]; \mathcal{H}) \cap H^1(0, S; \mathcal{V}') \) to the linear problem

\[
\begin{cases}
-\frac{dP}{dt} + AP - F'_{0}\overline{Y}^*P = D^*\Lambda(D\overline{Y} - Y_d), & 0 \leq t < S, \\
P(S) = 0
\end{cases}
\]

in \( \mathcal{V}' \), where \( \Lambda : \mathcal{V} \to \mathcal{V}' \) is a canonical isomorphism; moreover,

\[
\int_{0}^{S}\langle \Lambda P + \gamma\overline{U}, V - \overline{U} \rangle_{\mathcal{V}'} dt \geq 0 \quad \text{for all } V \in \mathcal{U}_{ad}.
\]

**Proof.** Since \( J \) is Gâteaux differentiable at \( \overline{U} \) and \( \mathcal{U}_{ad} \) is convex, it is seen that

\[
J'_{0}(\overline{U})(V - \overline{U}) \geq 0 \quad \text{for all } V \in \mathcal{U}_{ad}.
\]

On the other hand, we verify that

\[
J'_{0}(\overline{U})(V - \overline{U}) = \int_{0}^{S}\langle DY(\overline{U}) - Y_d, DZ \rangle_{\mathcal{V}} dt + \gamma \int_{0}^{S}\langle \overline{U}, V - \overline{U} \rangle_{\mathcal{V}'} dt
\]

with \( Z = Y'(\overline{U})(V - \overline{U}) \). Let \( P \) be the unique solution of (3.7) in \( H^1(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V}) \). From (a.i), (a.ii), (f.i), (f.ii), and (f.iii), we can guarantee that such a solution \( P \) exists (cf. [4, Chap. XVIII, Theorem 2]). Thus, in view of Proposition 3.1 the first integral in the right hand side of (3.8) is shown to be

\[
\int_{0}^{S}\langle DY(\overline{U}) - Y_d, DZ \rangle_{\mathcal{V}} dt = \int_{0}^{S}\langle D^*\Lambda(D\overline{U} - Y_d), Z \rangle dt
\]

\[
= \int_{0}^{S}\langle -\frac{dP}{dt} + AP - F'_{0}\overline{Y}^*P, Z \rangle dt = \int_{0}^{S}\langle P, \frac{dZ}{dt} + AZ - F'_{0}\overline{Y}Z \rangle dt
\]

\[
= \int_{0}^{S}\langle \Lambda P, V - \overline{U} \rangle_{\mathcal{V}'} dt.
\]
Hence,
\[ \int_0^S \langle \Lambda P + \gamma \overline{U}, V - \overline{U} \rangle_{\mathcal{V}} J dt \geq 0, \quad \text{for all } V \in \mathcal{U}_{ad}. \]

**Remark.** Note that our result covers that of [8, 9] when the sensitivity function \( \chi(\rho) \) is linear function of \( \rho \), \( \chi(\rho) = b \rho \) \((b \text{ being a positive constant})\). Furthermore, since all assumptions of our abstract result are satisfied when \( \chi(\rho) = \frac{b \rho}{1+\rho} \), our result is also applied in this case.

**References**