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二次元外部ラプラス問題のFEM-CSM近似解の誤差評価

An error estimate of an FEM-CSM combined method for planar exterior Laplace problems

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講演者の提案してきた外部ラプラス問題のFEM-CSM結合解法の誤差評価に関しては、これまでいくつかの機会に結果を述べた（Reference 2、Reference 3など）。吟味不足で、そこに述べた定理は正確ではなかった。この報告でこの誤りを正したい。この報告の詳細は、Reference 4に述べてある。

1. Introduction

Fix a simply connected bounded domain $\mathcal{O}$ in the plane. Assume that the boundary $C$ of $\mathcal{O}$ is sufficiently smooth. The exterior domain of $C$ is denoted by $\Omega$. Let $D_a$ be the interior of the disc with radius $a$ having the origin as its center. Fix a function $f \in L^2(\Omega)$ whose support, $\text{supp}(f)$, is bounded. Choose $a$ so large that the open disc $D_a$ may contain the union $\mathcal{O} \cup \text{supp}(f)$ in its interior. The following Poisson equation (E) is employed as a model problem.

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
  u &= 0 \quad \text{on } C, \\
  |u|_{r>a} &< \infty.
\end{aligned}
\]

(E)

The intersection of the domain $\Omega$ and the disc $D_a$ is said to be the interior domain, denoted by $\Omega_i$: $\Omega_i = \Omega \cap D_a$. Consider the Dirichlet inner product $a(u, v)$ for $u, v \in H^1(\Omega_i)$:

\[
a(u, v) = \int_{\Omega_i} \nabla u \nabla v \, d\Omega.
\]
Let $\Gamma_a$ be the boundary of the disc $D_a$. Since the trace $\gamma_a v$ on $\Gamma_a$ is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, the boundary bilinear form of Steklov type $b(u, v)$ is well defined for $u, v \in H^1(\Omega_i)$. The precise definition of $b(u, v)$ will be given in Section 3. Define a continuous symmetric bilinear form $t(u, v)$ for $u, v \in H^1(\Omega_i)$ through

$$t(u, v) = a(u, v) + b(u, v).$$

Let $F(v)$ be a continuous linear functional on $H^1(\Omega_i)$ defined through the following formula for $v \in H^1(\Omega_i)$:

$$F(v) = \int_{\Omega_i} fv \, d\Omega.$$

A function space $V$ is defined as follows:

$$V = \{ v \in H^1(\Omega_i) : v = 0 \text{ on } C \}.$$

Using these notations, the following weak formulation problem (II) is defined.

$$\begin{cases}
    t(u, v) = F(v), & v \in V, \\
    u \in V.
\end{cases}$$

We admit the equivalence between the equation (E) and the problem (II).

2. A CSM approximate problem for Laplace equation in the exterior region of a disc

Let $D_a$ be the interior of the disc with radius $a$ having the origin as its center, and let $\Gamma_a$ be the boundary of $D_a$. Let $\Omega_e = (D_a \cup \Gamma_a)^C$, which is said to be the exterior domain. We use the notation $r = r(\theta)$ for the point in the plane corresponding to the complex number $re^{i\theta}$ with $r = |r|$ where $|r|$ is the Euclidean norm of $r \in \mathbb{R}^2$. Similarly we use $a = a(\theta)$, and $\rho = \rho(\theta)$, corresponding to $ae^{i\theta}$ with $a = |a|$, and $\rho e^{i\theta}$ with $\rho = |\rho|$, respectively.

Let $f(a(\theta))$ be a continuous function on the circle $\Gamma_a$. The function $f(a(\theta))$ is a $2\pi$ periodic function of $\theta$. Denote the problem to find a harmonic function $u = u(r)$ coinciding with $f$ on $\Gamma_a$, which is bounded in $\Omega_e$, by (E$_f$).
\begin{align*}
\text{(E}_f\text{)}
\begin{cases}
-\Delta u = 0 \quad \text{in } \Omega, \\
u = f \quad \text{on } \Gamma_a, \\
\sup_{\Omega_e}|u| < \infty.
\end{cases}
\end{align*}

Fix a positive integer \( N \). Set 
\[ \theta_1 = \frac{2\pi}{N}. \]

For any \( j \in \mathbb{Z} \), denote \( j\theta_1 \) by \( \theta_j \). Fix a positive number \( \rho \) so as to satisfy \( 0 < \rho < a \). For the fixed positive integer \( N \), set the points \( \vec{\rho}_j, a_j, 0 \leq j \leq N-1 \), as follows:
\[ a_j = a(\theta_j), \quad \vec{\rho}_j = \vec{\rho}(\theta_j) \quad \text{with} \quad 0 < \rho < a. \]

The points \( \vec{\rho}_j \), and \( a_j \), are said to be the charge, and the collocation, points, respectively. The arrangement of the set of points of charge points and collocation points introduced as above is called the equi-distant equally phased arrangement of charge points and collocation points.

Now we define a CSM approximate problem \( (\text{E}_f^{(N)}) \) for the continuous problem \( (\text{E}_f) \) as follows:
\begin{align*}
\text{(E}_f^{(N)}\text{)}
\begin{cases}
u^{(N)}(r) = \sum_{j=0}^{N-1} q_j G_j(r) + q_N, \\
u^{(N)}(a_j) = f(a_j), \quad 0 \leq j \leq N-1, \\
\sum_{j=0}^{N-1} q_j = 0,
\end{cases}
\end{align*}

where
\[ G_j(r) = E(r - \vec{\rho}_j) - E(r), \quad E(r) = -\frac{1}{2\pi} \log r. \]

Let
\[ N(\gamma) = \frac{\log 2}{-\log \gamma} \quad \text{with} \quad \gamma = \frac{\rho}{a}. \]

Theorem B (Cf. Katsurada-Okamoto[1].) Fix a positive number \( b, 0 < b < a \). Let \( u(r) \) be harmonic in a domain containing the exterior domain of the disc with radius \( b \) having the origin as its center. Suppose that \( N \geq N(\gamma) \). Let \( u^{(N)}(r) \) be the solution of the problem \( (\text{E}_f^{(N)}) \) with the data \( f(a(\theta)) = u(a(\theta)) \). Then there exist constants \( B > 0 \) and \( \beta \in (0, 1), \)
dependent on parameters $a$, $b$ and $\rho$, independent of $u$ (with the property above) and $N$, such that the following two estimates are valid:

\[
\max_{\mathbf{r} \in \mathring{\Omega}_e} |u(\mathbf{r}) - u^{(N)}(\mathbf{r})| \leq B \cdot \beta^N \cdot \max_{|\mathbf{r}|=b} |u(\mathbf{r})|,
\]

\[
\max_{\mathbf{r} \in \mathring{\Omega}_e} \left| \nabla u(\mathbf{r}) - \nabla u^{(N)}(\mathbf{r}) \right|_{\mathbb{R}^2} \leq B \cdot \beta^N \cdot \max_{|\mathbf{r}|=b} |u(\mathbf{r})|.
\]

3. Boundary bilinear form of Steklov type for exterior Laplace problems and its CSM-approximation form

Let $f(\theta)$ be a complex valued continuous $2\pi$-periodic function of $\theta$. For $n \in \mathbb{Z}$, a continuous Fourier coefficient $f_n$ of the function $f(\theta)$ is defined through

\[
f_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) e^{-in\theta} d\theta.
\]

For functions $u(a(\theta))$ and $v(a(\theta))$ of $H^{1/2}(\Gamma_a)$, let us introduce the boundary bilinear form of Steklov type for exterior Laplace problem through the following formula (1):

\[
(1) \quad b(u, v) = 2\pi \sum_{n=-\infty}^{\infty} |n| f_n \overline{g_n},
\]

where $f_n$ and $g_n$ are continuous Fourier coefficients of $u(a(\theta))$ and $v(a(\theta))$, respectively.

The CSM approximate form for $b(u, v)$, which is denoted by $b^{(N)}(u, v)$, is represented through the following formula (2):

\[
(2) \quad b^{(N)}(u, v) = -\frac{2\pi a}{N} \sum_{j=0}^{N-1} \frac{\partial u^{(N)}(a_j)}{\partial r} v^{(N)}(a_j),
\]

where $u^{(N)}(\mathbf{r})$ and $v^{(N)}(\mathbf{r})$ are CSM-approximate solutions of the problem $(\mathbb{E}_{f}^{(N)})$ with $f = u(a(\theta))$, and $f = v(a(\theta))$, respectively.

4. An FEM-CSM combined method for exterior Laplace problems

We say that the function $v(a(\theta))$ is an equi-distant piecewise linear continuous $2\pi$-periodic function with $N$ nodal points if it is expressed in the
following form:

$$v(a(\theta)) = \frac{\theta_{j+1} - \theta}{\theta_1} v(a(\theta_j)) + \frac{\theta - \theta_j}{\theta_1} v(a(\theta_{j+1})), $$

$$ \theta_j \leq \theta \leq \theta_{j+1}, \quad 0 \leq j \leq N - 1.$$ 

And we use the notation, $a(v) = a(v, v)^{1/2}$, for $v \in V$.

A family of finite dimensional subspaces of $V$, $\{V_N : N = N_0, N_0 + 1, \ldots \}$ is supposed to have the following properties:

$(V_N - 1)$ \quad $V_N \subset C(\overline{\Omega_i})$.

$(V_N - 2)$ \quad \begin{cases} 
\text{For any } v \in V_N, \ v(a(\theta)) \text{ is an equi–distant} \\
\text{piecewise linear continuous } 2\pi–\text{periodic} \\
\text{function with } N \text{ nodal points.}
\end{cases}$

$(V_N - 3)$ \quad \begin{cases} 
\text{There is a constant } C \text{ independent of } N \\
\text{such that for any } v \in V \cap H^2(\Omega_i) \\
\min_{v_N \in V_N} a(v - v_N) \leq \frac{C}{N} ||v||_{H^2(\Omega_i)}.
\end{cases}$

For $u, v \in H^1(\Omega_i) \cap C(\overline{\Omega_i})$, we define bilinear forms $t^{(N)}(u, v)$ as follows.

$$t^{(N)}(u, v) = a(u, v) + b^{(N)}(u, v).$$

Now our approximate problem $(\Pi^{(N)})$ is stated as follows.

$(\Pi^{(N)})$ \quad \begin{cases} 
t^{(N)}(u_N, v) = F(v), \ v \in V_N, \\
u_N \in V_N.
\end{cases}$

**Theorem 1** Let $u$ be the solution of the problem $(\Pi)$, and let $u_N$ be the solution of the problem $(\Pi^{(N)})$. Suppose that $\text{supp}(f)$ is contained in a disc $D_b$ with the radius $b(< a)$ having the origin as its center. Let the function $D(\xi)$ of $\xi \in (0, 1)$ be defined through

$$D(\xi) = \frac{4\xi}{(1 - \xi)^3}.$$
Let \( N \geq N(\gamma) \). Then there is a constant \( C \) such that
\[
\|u - u_N\|_{H^1(\Omega_i)} \leq C \left\{ B\beta^N + \frac{1 + D\left(\frac{b}{a}\right)}{N} \right\} \|f\|_{L^2(\Omega_i)},
\]
where the constants \( B \) and \( \beta \in (0, 1) \) are described in Theorem B for the set of parameters \( \{a, b, \rho\} \). In the above, the constant \( C \) is independent of the inhomogeneous data \( f \) and \( N \).

The proof of Theorem 1 is written in [4].

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References