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BOUNDARY ELEMENT APPROXIMATION OF MINIMAL SURFACES AND CONFORMAL MAPPINGS

Takuya Tsuchiya (土屋 卓也)
Department of Mathematical Sciences, Faculty of Science
Ehime Unviersity (愛媛大学理学部数理科学科)
tsuchiya@math.sci.ehime-u.ac.jp

Kazuki Yoshida (吉田 和樹)
Graduate School of Science and Engineering, Doctor Course
Ehime University (愛媛大学大学院理工学研究科博士課程)
kazu@math.sci.ehime-u.ac.jp

Abstract. In this paper, boundary element approximation of minimal surfaces and conformal mappings defined on the unit disk is considered. Since minimal surfaces are characterized as stationary points of the Dirichlet integral in certain subsets of a functional space, we approximate the Dirichlet integral using the boundary element method and define the boundary element minimal surfaces as stationary points of the discretized Dirichlet integral. The boundary element conformal mappings are defined by the same way. Convergence of the boundary element minimal surfaces to the exact solutions is proved. A numerical example is given.

Key words. conformal mappings, minimal surfaces, boundary elements, variational principle, Dirichlet's integral

AMS subject classifications. 30C30, 65E05, 65N30

1 Introduction

Let $B \subset \mathbb{R}^2$ be the unit disk, and $\gamma \subset \mathbb{R}^n$ $(n \geq 2)$ a closed Jordan curve. The **Plateau problem** is to find a map $x = (x^1, \cdots, x^n) \in C(\overline{B}; \mathbb{R}^n) \cap H^1(B; \mathbb{R}^n)$ such that

$$\Delta x = (\Delta x^1, \cdots, \Delta x^n) = 0 \text{ in } B, \tag{1.1}$$

$$(x_{u_1}, x_{u_2}) = |x_{u_1}|^2 - |x_{u_2}|^2 = 0 \text{ in } B, \tag{1.2}$$

$$x(\partial B) = \gamma, \text{ and } \quad x|_{\partial B} : \partial B \to \gamma \text{ is homeomorphic}, \tag{1.3}$$

where $x_{u_1} := (x^1_{u_1}, \cdots, x^n_{u_1})$ and $x_{u_2} := (x^1_{u_2}, \cdots, x^n_{u_2})$ are partial derivatives with respect to u_1, u_2 $((u_1, u_2) \in B)$, respectively, and $(\cdot, \cdot)$ and $|\cdot|$ are the usual inner product and Euclidian norm in $\mathbb{R}^n$.

If $n \geq 3$, mean curvature vanishes everywhere on solutions of the Plateau problem, and therefore all solutions of the Plateau problem are **minimal surfaces**. If $n = 2$, a solution

x of the Plateau problem is a **conformal mapping** from B to the domain defined by the Jordan curve γ (such domains are called *Jordan domains*) if x is orientation-preserving. In this sense, conformal mappings are minimal surfaces in $\mathbb{R}^2$.

For solutions of the Plateau problem, the following variational principle has been known (for example, see [3, pp.107–115], [4, Section 4.5]): Define the subset X_γ of $C(\overline{B};\mathbb{R}^n) \cap H^1(B;\mathbb{R}^n)$ by

$$X_\gamma := \left\{ \psi \in C(\overline{B};\mathbb{R}^n) \cap H^1(B;\mathbb{R}^n) \;\middle|\; \psi(\partial B) = \gamma \text{ and } \psi|_{\partial B} \text{ is monotone} \right\},$$

where $\psi|_{\partial B}$ being monotone means that $(\psi|_{\partial B})^{-1}(p)$ is connected for any $p \in \gamma$. We denote the Dirichlet integral (or the energy functional) on B for $\varphi = (\varphi^1, \cdots, \varphi^n) \in H^1(B;\mathbb{R}^n)$ by

$$D(\varphi) := \int_B |\nabla \varphi|^2 du = \int_B (|\nabla \varphi^1|^2 + \cdots + |\nabla \varphi^n|^2) du.$$

Then, we have that $\varphi \in X_\gamma$ *is a solution of the Plateau problem if and only if* $\varphi \in X_\gamma$ *is a stationary point of the functional* $D(\varphi)$ *in* X_γ.

The existence of solutions of the Plateau problem was proved by Douglas and Radó independently. Later on, the proof was significantly simplified by Courant (see [3] and [4]). Let z_1, z_2, $z_3 \in \partial B$ and ζ_1, ζ_2, $\zeta_3 \in \gamma$ be taken. We define $X_\gamma^{tp} \subset X_\gamma$ by

$$X_\gamma^{tp} := \left\{ \varphi \in X_\gamma \;\middle|\; \varphi(z_i) = \zeta_i, \quad i = 1,2,3 \right\}.$$

Since the Dirichlet integral is invariant under conformal transformation of B, we have

$$\inf_{y \in X_\gamma} D(y) = \inf_{y \in X_\gamma^{tp}} D(y).$$

Theorem 1.1 (Douglas-Radó-Courant) *If* $X_\gamma^{tp} \neq \emptyset$, *then there exists at least one* $x \in X_\gamma^{tp}$ *at which the minimum value of the Dirichlet integral in* X_γ^{tp} *is attained:*

$$D(x) = \inf_{y \in X_\gamma^{tp}} D(y).$$

Of course, such $x \in X_\gamma^{tp}$ *is a solution of the Plateau problem.*

The minimizers of the Dirichlet integral in X_γ^{tp} are called the **Douglas-Radó solutions**. In case of $n = 2$, the above existence theorem of solution of the Plateau problem is the **Riemann mapping theorem** for Jordan domains.

With the variational principle of the Plateau problem, we immediately think of the following strategy for approximating the Plateau problem: first, define the discretizations $\mathbb{S}_{\gamma,h}^{tp}$ of X_γ^{tp} and $D_h(\varphi_h)$ of $D(\varphi)$, respectively. Then define the discretized solutions of the Plateau problem as stationary points of D_h in $\mathbb{S}_{\gamma,h}^{tp}$. In [5, 6, 7, 8], the finite element method with piecewise linear triangle elements has been used to discretized the Plateau problem. In this paper we use the **boundary element method** with piecewise linear elements to define the discretized solutions of the Plateau problem. Since we have to discritized only ∂B in boundary element method, the work load for programming is much less than that of finite element method, and it seems a bit faster than FEM.

In Section 2 we define the boundary element minimal surfaces and conformal mappings. In Section 3 we prove convergence of BE solutions of the Plateau problems to the exact solutions. In Section 4 a numerical example is given.

2 Boundary Element Approximation

In this section, we consider a boundary element approximation of the Plateau problem. Let us consider the following Laplace problem with the Dirichlet boundary condition: for given $g \in H^{1/2}(\partial B)$, find $w \in H^1(B)$ such

$$\Delta w = 0, \text{ in } B, \qquad w = g, \text{ on } \partial B. \tag{2.1}$$

With the fundamental solution $K(u,v) := -\log|u-v|/(2\pi)$ for two-dimensional Laplacian Δ, we obtain from (2.1) the following integral equation on ∂B:

$$\frac{1}{2}g(u) + \int_{\partial B} \frac{\partial K(u,v)}{\partial n_v} g(v)ds_v = \int_{\partial B} K(u,v)\frac{\partial w}{\partial n}(v)ds_v, \tag{2.2}$$

for $u \in \partial B$. Solving (2.2) with the given data g on ∂B, we are able to obtain the Neumann data $\partial w/\partial n$ for the solution $w \in H^1(B)$. In other words, we can compute the Dirichlet-Neumann map $g \mapsto \partial w/\partial n$ associated with (2.1) by solving (2.2). By the Stokes theorem the Dirichlet integral $D(w)$ can be computed by

$$D(w) := \int_B |\nabla w|^2 du = \int_{\partial B} w\frac{\partial w}{\partial n} ds,$$

if the function $w \in H^1(B)$ is harmonic.

In this paper we always identify $x \in C \cap H^{1/2}(\partial B;\mathbb{R}^n)$ and the harmonic map $w \in C(\overline{B};\mathbb{R}^n) \cap H^1(B;\mathbb{R}^n)$ whose Dirichlet data is x (that is, $w|_{\partial B} = x$). From the above consideration, we use the equivalent form of the Dirichlet integral $D(x)$ for $x \in C \cap H^{1/2}(\partial B;\mathbb{R}^n)$ defined by

$$D(x) := \int_{\partial B} \left(x^1\frac{\partial w^1}{\partial n} + \cdots + x^n\frac{\partial w^n}{\partial n}\right) ds, \tag{2.3}$$

where the Neumann data $\partial w^i/\partial n$ are obtained by solving (2.2) with $g = x$. We have the following basic property of the Dirichlet integral:

Lemma 2.1 *Let $\psi, \psi_n \in H^{1/2}(\partial B;\mathbb{R}^n)$ be such that $\lim_{n\to\infty}\psi_n = \psi$ in $H^{1/2}(\partial B;\mathbb{R}^n)$. Then, we have $\lim_{n\to\infty} D(\psi_n) = D(\psi)$.*

We are now ready to describe the Plateau problem by the boundary integral equation (2.2). Let $\gamma \subset \mathbb{R}^n$ $(n \geq 2)$ be a given Jordan curve. Define the subset $X_\gamma \subset C \cap H^{1/2}(\partial B;\mathbb{R}^n)$ by

$$X_\gamma := \left\{ \psi \in C \cap H^{1/2}(\partial B;\mathbb{R}^n) \;\middle|\; \psi(\partial B) = \gamma,\ \psi \text{ is monotone} \right\}.$$

Take arbitrary $z_i \in \partial B$ and $\zeta_i \in \gamma$ $(i = 1,2,3)$. In the case $n = 2$, we take those points in the same orientation. Then define X_γ^{tp} by

$$X_\gamma^{tp} := \left\{ \psi \in X_\gamma \;\middle|\; \psi(z_i) = \zeta_i,\ i = 1,2,3 \right\}.$$

Let the Dirichlet integral $D(\psi)$ for $\psi \in C \cap H^{1/2}(\partial B;\mathbb{R}^2)$ be defined by (2.3). Then the Plateau problem is: *find $x \in X_\gamma^{tp}$ which is a stationary point of the Dirichlet integral $D(\psi)$ in the subset X_γ^{tp}.*

Now it is very clear how we can define the boundary element solutions of the Plateau problem.

First, we suppose that we have a family of triangulation $\{\triangle_h\}$ of the the 1-dimensional unit sphere ∂B, where h stands for the maximum size of triangles (that is, intervals) in the triangulation $\triangle_h$, and $h \to 0$. In this paper we always assume that $\partial B = \bigcup_{T\in\triangle_h} \overline{T}$ for simplicity. Let $S_h \subset C^0(\partial B)$ be the set of piecewise linear functions on each triangle. Here, the linearity is defined with respect to the arc-length parameter. We discretize X_γ as

$$\mathbb{S}_{\gamma,h} := \left\{\psi_h \in (S_h)^n \;\middle|\; \psi_h(\partial B \cap N_h) \subset \gamma \text{ and } \psi_h|_{\partial B} \text{ is } d\text{-monotone}\right\},$$

where N_h is the set of nodal points in $\triangle_h$, and $\psi_h|_{\partial B}$ being *d-monotone* means that the order of nodes on ∂B is preserved on γ by ψ_h. Suppose that the distinct points $z_i \in \partial B$ and $\zeta_i \in \gamma$ $(i = 1, 2, 3)$ are taken as above. We assume that $z_i \in \partial B$ are nodal points of $\triangle_h$ for each $h > 0$. Then we define

$$\mathbb{S}_{\gamma,h}^{tp} := \left\{\psi_h \in \mathbb{S}_{\gamma,h} \;\middle|\; \psi_h(z_i) = \zeta_i,\ i = 1,2,3\right\}.$$

For $\psi_h \in (S_h)^n$ we compute the discretized Dirichlet integral $D_h(\psi_h)$ by the following manner. First, we compute the solution of the Laplace equation

$$\Delta w = 0 \text{ in } B, \quad w = \psi_h \text{ on } \partial B,$$

by certain boundary element method on the space $(S_h)^n$, and obtain its approximated Neumann data $(\partial w/\partial n)_h = ((\partial w^1/\partial n)_h, \cdots, (\partial w^n/\partial n)_h) \in (S_h)^n$. Then, $D_h(\psi_h)$ is defined by

$$D_h(\psi_h) := \int_{\partial B} \left(\psi_h^1 \left(\frac{\partial w^1}{\partial n}\right)_h + \cdots + \psi_h^n \left(\frac{\partial w^n}{\partial n}\right)_h\right) ds. \tag{2.4}$$

A stationary point $x_h \in \mathbb{S}_{\gamma,h}^{tp}$ of $D_h(\psi_h)$ in the subset $\mathbb{S}_{\gamma,h}^{tp}$ is called a **boundary element minimal surface**. In the case $n = 2$ it is called a **boundary element conformal mapping**. In particular, the minimizer x_h of the discretized Dirichlet integral D_h in $\mathbb{S}_{\gamma,h}^{tp}$ is called the **boundary element Douglas-Radó solution**.

3 Convergence of BE Minimal Surfaces

In this section we consider convergence of the boundary element minimal surfaces. To do this we require the following reasonable assumption:

Assumption 3.1 *There exists a nonnegative function $g(h)$ for $h > 0$ such that $\lim_{h\to 0} g(h) = 0$ and, for sufficiently small $h > 0$,*

$$(1 - g(h))D_h(\psi_h) \le D(\psi_h) \le (1 + g(h))D_h(\psi_h), \tag{3.1}$$

for any $\psi_h \in (S_h)^n$, where the Dirichlet integrals D and D_h are defined by (2.3) *and* (2.4)*, respectively.*

In Assumption 3.1 we require that the boundary element method we use can attain sufficient accuracy so that the discretized Dirichlet integral D_h is a good approximation of the exact Dirichlet integral D. This is the only assumption we need for the boundary element method in this paper.

Lemma 3.2 *Suppose that Assumption 3.1 holds. Let $\{\psi_h \in (S_h)^n\}$ be a sequence such that $D_h(\psi_h) \leq M$ with some positive constant M. Suppose that $\{\psi_h\}$ converges uniformly to a continuous map $\psi \in C(\partial B; \mathbb{R}^n)$. Then we have $\psi \in H^{1/2}(\partial B; \mathbb{R}^n)$ and*

$$D(\psi) \leq \liminf_{h \to 0} D_h(\psi_h). \tag{3.2}$$

Proof. Let f, f_h be harmonic maps with $f = \psi$, $f_h = \psi_h$ on ∂B, respectively. Since ψ_h converges uniformly to ψ, and in view of well-known lower semicontinuity of the Dirichlet integral, we have $D(\psi) = D(f) \leq \liminf_{h\to 0} D(f_h) = \liminf_{h\to 0} D(\psi_h) \leq M$. Hence, $f \in H^1(B; \mathbb{R}^n)$ and $f|_{\partial B} = \psi \in H^{1/2}(\partial B; \mathbb{R}^n)$. By (3.1), we obtain (3.2). □

The following lemma is on the relative compactness of bounded subsets of $\mathbb{S}^{tp}_{\gamma,h}$, which is the most crucial in our convergence analysis.

Lemma 3.3 ([7], Lemma 6) *Suppose that Assumption 3.1 holds and the given Jordan curve is rectifiable. Take a sequence $\{\psi_h \in \mathbb{S}^{tp}_{\gamma,h}\}$. We assume that $D_h(\psi_h)$ are uniformly bounded. Then, there exists a subsequence $\{\psi_{h_i}\}$ such that ψ_{h_i} converges uniformly to a continuous map $\psi \in C \cap H^{1/2}(\partial B; \mathbb{R}^n)$ on ∂B. Moreover, $\psi \in X^{tp}_\gamma$.*

Theorem 3.4 *Suppose that Assumption 3.1 holds and the given Jordan curve γ is rectifiable. Let $\{x_h \in \mathbb{S}^{tp}_{\gamma,h}\}$ be a sequence of the boundary element Douglas-Radó solutions. Then there exists a subsequence $\{x_{h_i}\}$ which converges to one of the exact Douglas-Radó solutions $x \in X^{tp}_\gamma$ in the following sense:*

$$\lim_{h_i \to 0} \|x - x_{h_i}\|_{C(\partial B; \mathbb{R}^n)} = 0, \tag{3.3}$$

$$\lim_{h_i \to 0} \|x - x_{h_i}\|_{H^{1/2}(\partial B; \mathbb{R}^n)} = 0. \tag{3.4}$$

Proof. Since $\mathbb{S}^{tp}_{\gamma,h}$ are bounded closed subsets in a finite dimensional vector spaces, it is obvious that the boundary element Douglas-Radó solutions exist in each $\mathbb{S}^{tp}_{\gamma,h}$.

Let $y \in X^{tp}_\gamma$ be one of the Douglas-Radó solutions. Let $\Pi_h : C(\partial B; \mathbb{R}^n) \to (S_h)^n$ be the usual interpolant projection (see [2]), that is, $\Pi_h y \in (S_h)^n$ is defined so that $\Pi_h y(u_j) = y(u_j)$ for nodal points u_j of $\triangle_h$. It follows from Lemma 2.1 and (3.1) that $\lim_{h\to 0} D_h(\Pi_h y) = D(y)$.

Since $D_h(x_h) \leq D_h(\Pi_h y)$, $\{D_h(x_h)\}$ is uniformly bounded. Thus, by Lemma 3.3, there exists a subsequence $\{x_{h_i}\}$ which converges uniformly to a continuous map $x \in X^{tp}_\gamma$. By Lemma 3.2 we obtain

$$D(x) \leq \liminf_{h_i \to 0} D_{h_i}(x_{h_i}) \leq \lim_{h_i \to 0} D_{h_i}(\Pi_{h_i} y) = D(y). \tag{3.5}$$

Hence $x \in X^{tp}_\gamma$ is one of the Douglas-Radó solutions and (3.3) is proved.

Now, let w, $w_{h_i} \in H^1(B;\mathbb{R}^n)$ be harmonic maps with $w = x$, $w_{h_i} = x_{h_i}$ on ∂B, respectively. Since $\|w_{h_i}\|_{H^1(B;\mathbb{R}^n)}$ are uniformly bounded, $\{w_{h_i}\}$ has a weakly convergent subsequence. We know that, by the maximum principle of harmonic maps, $\{w_{h_i}\}$ converges to w uniformly on $\overline{B}$. Therefore, w_{h_i} converges to w weakly in $H^1(B;\mathbb{R}^n)$. Also we have

$$\lim_{h_i\to 0} \|w - w_{h_i}\|_{L^2(B;\mathbb{R}^n)} = 0. \tag{3.6}$$

On the other hand, we have $D(x) = D(w)$ and $D(x_{h_i}) = D(w_{h_i})$. With (3.1) and (3.5) we get

$$\lim_{h_i\to 0} |w_{h_i}|^2_{H^1(B;\mathbb{R}^n)} = \lim_{h_i\to 0} D(w_{h_i}) = D(w) = |w|^2_{H^1(B;\mathbb{R}^n)}. \tag{3.7}$$

Combining (3.6) and (3.7) we obtain

$$\lim_{h_i\to 0} \|w - w_{h_i}\|_{H^1(B;\mathbb{R}^n)} = \lim_{h_i\to 0} \|w|_{\partial B} - w_{h_i}|_{\partial B}\|_{H^{1/2}(\partial B;\mathbb{R}^n)} = 0.$$

Therefore (3.4) is proved.

If the Douglas-Radó solution is unique, the limit of convergent subsequence of $\{x_h\}$ is unique. Hence x_h converges to the unique Douglas-Radó solution in the sense of (3.3) and (3.4). □

Corollary 3.5 *Suppose that Assumption 3.1 holds and the given Jordan curve γ is rectifiable. Let $n = 2$ and $\{x_h \in \mathbb{S}^{tp}_{\gamma,h}\}$ the sequence of the boundary element conformal mappings. Then $\{x_h\}$ converges to the unique conformal mapping $x \in X^{tp}_\gamma$ in the sense of* (3.3) *and* (3.4).

A map $x \in X^{tp}_\gamma$ is said to be an *isolated stable minimal surface* if there exists a constant δ such that

$$0 < \|x - y\|_{C(\partial B;\mathbb{R}^n)} < \delta \quad \text{implies} \quad D(x) < D(y) \text{ for } y \in X^{tp}_\gamma.$$

Theorem 3.6 *Suppose that Assumption 3.1 holds and the given Jordan curve γ is rectifiable. Let $x \in X^{tp}_\gamma$ be an isolated stable minimal surface. Then there exists a sequence $\{x_h \in \mathbb{S}^{tp}_{\gamma,h}\}$ of stable boundary element minimal surfaces which converges to x in the sense of* (3.3) *and* (3.4).

Proof. As in the proof on Theorem 3.4, let $\Pi_h : C(\partial B;\mathbb{R}^n) \to (S_h)^n$ be the interpolant projection. We define δ-neighborhoods of $\Pi_h x$ by

$$U^\delta_h(\Pi_h x) := \left\{\psi_h \in \mathbb{S}^{tp}_{\gamma,h} \;\middle|\; \|\psi_h - \Pi_h x\|_{C(\partial B;\mathbb{R}^n)} \le \delta\right\}.$$

Since $U^\delta_h(\Pi_h x)$ is a bounded closed set in finite-dimensional Euclidean space, there exists $x_h \in \mathbb{S}^{tp}_{\gamma,h}$ such that $D_h(x_h)$ attains the minimum value of D_h in $U^\delta_h(\Pi_h x)$. By (3.5) $\{D_h(x_h)\}$ is uniformly bounded. Hence there exists a subsequence $\{x_{h_i}\}$ which converges uniformly to $\psi \in X^{tp}_\gamma$.

For arbitrary $\varepsilon > 0$ we take sufficiently small $h_i > 0$ so that $\|x - \Pi_{h_i} x\|_{C(\partial B;\mathbb{R}^n)} < \varepsilon/2$ and $\|\psi - x_{h_i}\|_{C(\partial B;\mathbb{R}^n)} < \varepsilon/2$. We then obtain

$$\|\psi - x\|_{C(\partial B;\mathbb{R}^n)} < \varepsilon + \|\Pi_{h_i} x - x_{h_i}\|_{C(\partial B;\mathbb{R}^n)} \leq \varepsilon + \delta.$$

Hence we show that $\|\psi - x\|_{C(\partial B;\mathbb{R}^n)} \leq \delta$ and, by the definition, $D(\psi) \geq D(x)$. On the other hand, from the lower-semicontinuity of the Dirichlet integral and (3.5) we have $D(\psi) \leq D(x)$. Thus we conclude that $D(\psi) = D(x)$, and, again by the assumption, $x = \psi$. By the exactly same way as in the proof of of Theorem 3.4 we can show that x_h converges to x in the sense of (3.3) and (3.4). Because of the convergence we have just proved, we now know that, for sufficiently small $h > 0$, x_h are inner points in $U_h^\delta(\Pi_h x)$. Hence they are boundary element minimal surfaces. □

4 A Numerical Example

In this section we give a numerical example. Let $n = 2$ and $\gamma = (\gamma_1, \gamma_2) \subset \mathbb{R}^2$ defined by

$$\gamma_1(\theta) := (1 + C\cos 3\theta)\cos\theta, \qquad \gamma_2(\theta) := (1 + C\sin 3\theta)\sin\theta,$$

for $\theta \in [0, 2\pi]$, where C is a constant. Let $z_i := \exp(\sqrt{-1}\theta_i)$, $\zeta_i := \gamma(\theta_i)$, and $\theta_i := 2(i-1)\pi/3$, $(i = 1, 2, 3)$. Let Ω be the Jordan domain bounded by γ. We compute conformal mapping $x \in X_\gamma^{tp}$ from B to Ω with $x(z_i) = \zeta_i$ $(i = 1, 2, 3)$. We do so by the finite element methods ([5, 6, 7, 8]) and the boundary element methods, and compare the results. The image of the finite element conformal mappings may be found in [5].

In Figure 4.1,4.2 we show the graphs of the function $y_h : [0, 2\pi] \to [0, 2\pi]$ with various C. The boundary element and finite element conformal mappings $x_h : \partial B \to \gamma$ are obtained as $x_h(\theta) := \Pi_h \gamma(y_h(\theta))$. For both methods the number of nodes on ∂B is 120. We notice that the boundary element and finite element conformal mappings are almost identical when $C \leq 0.4$. However, there are some gaps between them with $C > 0.4$. Probably, it is because boundary nodes tend to gather the narrow part of γ, and therefore the accuracy of the approximation becomes inferior on the rest of the boundary in one of (or both of) the methods.

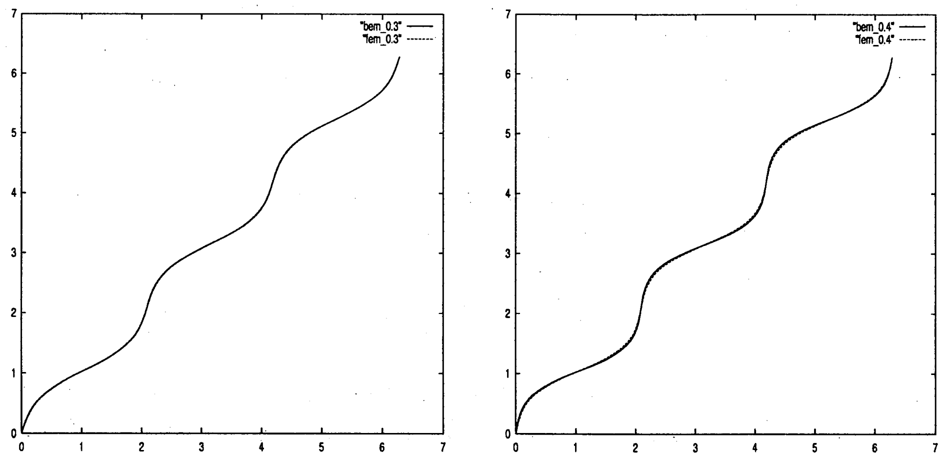

Figure 4.1: Comparison of FE and BE conformal mappings. $C = 0.3$ and $C = 0.4$.

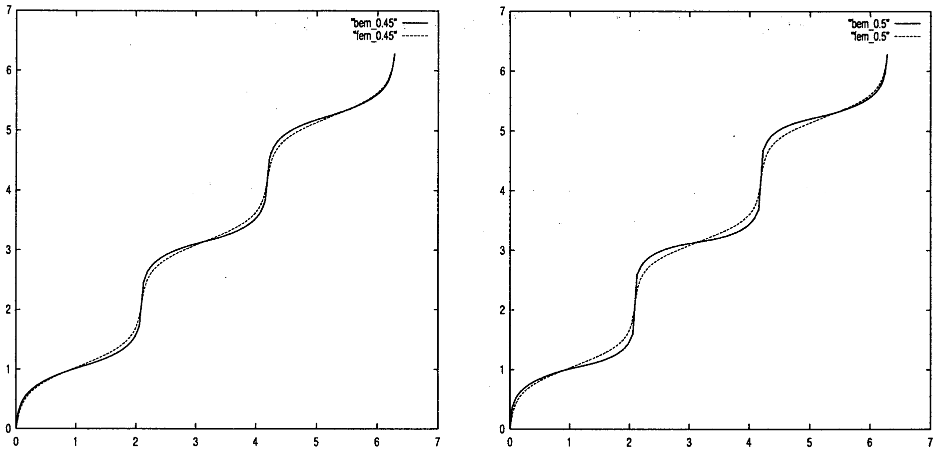

Figure 4.2: Comparison of FE and BE conformal mappings. $C = 0.45$ and $C = 0.5$.

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