

## Analysis for bifurcation phenomena of nonlinear vibrations

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### 0. Introduction

Combining some analytical methods and some numerical methods, we are able to analyse rigorously some (symmetry-breaking) bifurcation phenomena of periodic solutions of semilinear (hyperbolic and parabolic) partial differential equations (PDEs) describing some dissipative forced vibration systems. We adopt some methods in [O], where Oishi treats the Duffing equation. The outline of analysis is the following: In order to apply a bifurcation theorem ([WS], [K1]), we use some numerical method and find an approximate solution (= bifurcation point) with high accuracy of the extended system. Then, we show the unique existence of a real solution in a neighborhood of the approximate solution by using the convergence theorem of the Newton's method ([K1], [O]). The most difficult part in the analysis for concrete problems is to check the conditions of the convergence theorem of the Newton's method. More concretely, we have the following two difficult points:

- (I) to construct an highly accurate approximate solution of bifurcation point
- (II) to estimate the operator norm of the inverse of a linearized operator at bifurcation point

In this article we mainly intend to explain how to treat (I) and (II). For (I) we propose some methods by using the least square methods (see Section 2). For (II) we present some methods in which we approximate linearized operators by some operators with simpler structure (see Section 3). In view of the form, the author calls the latter operators '*diagonal operators*'. When we treat PDEs, the highly accurate approximate solution we need have in general very long terms. A lot of difficulties arise from such long length of approximate solutions. Even for one-dimensional vibration problem approximate solution we treat by the use of the spectral methods have more than one hundred terms. It seems that for higher-dimensional problems we need approximate solution with much more terms.

### 1. Our main example

Let  $f(\lambda, u) := u_{tt} - c^2 u_{xx} + \mu u_t + u^3 - \lambda \sin x \cos t$ . Here,  $c, \mu > 0$  are constants and  $\lambda > 0$  is a parameter. We consider the bifurcation phenomena of periodic solutions for the following semilinear wave equation:

$$(P) \quad \begin{cases} f(\lambda, u) = 0 & \text{in } (0, \pi) \times \mathbf{R}^+, \\ u(0, t) = u(\pi, t) = 0 & \text{for } t \geq 0. \end{cases}$$

This has some deep relations to the ordinary differential equation called the Duffing equation. By some numerical simulations we can observe rich bifurcation phenomena (such as the existence of turning points, symmetry-breaking bifurcations, chaos) for our problem (P) and the Duffing equation. It is difficult to analyse mathematically such phenomena only by the use of pure analytical methods, except for some problems with very restricted artificial forcing terms ([M], [Ko]). Oishi explains in [O] that we can analyse rigorously some bifurcation phenomena of the Duffing equations with natural forcing terms.

The equation in our problem (P) has some symmetry. Let  $S$  be the transformation defined by

$$S : u(x, t) \longmapsto -u(x, t + \pi).$$

Then we have  $f(\lambda, Su) = Sf(\lambda, u)$ . The symmetric periodic solution (resp. the asymmetric periodic solution) is a solution satisfying  $Su = u$  (resp.  $Su = -u$ ).

Let  $c = 1$ ,  $\mu = 0.1$ . Let us move the value of  $\lambda$  gradually larger from 0. Then we can observe that a branch of asymmetric  $2\pi$ -periodic solutions bifurcates from a branch of symmetric  $2\pi$ -periodic solutions at a certain value  $\lambda = \lambda_0 \in (2.7, 2.8)$  by some numerical simulations. We can prove this and the following holds:

**Proposition 1.1.** ([K2]) *There exists a symmetric branch  $\{(\lambda, u_\lambda)\}_\lambda$  of  $2\pi$ -periodic solutions of (P) which has a symmetry-breaking pitchfork bifurcation point  $(\lambda_0, u_0)$  such that*

$$|\lambda_0 - \Lambda_0|^2 + \|u_0 - U_0; H^1(D)\|^2 \leq (0.014)^2.$$

Here,  $D := (0, \pi) \times (0, 2\pi)$ ,  $\Lambda_0 := 2.767$  and

$U_0 := 1.675064 \cos t \sin x + \cdots + 6.878066 \times 10^{-6} \sin 21t \sin 21x$  has the form of a finite Fourier expansion consisting of 160 terms. We omit here the complete form of  $U_0$ .

In order to prove this result, we use a bifurcation theorem [K1, Theorem 3.1] which is a generalized version of [WS, Theorem 3.1]. We define a closed linear subspace  $X$  in  $H^1(D)$ :

$$X := \left\{ \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{N}-1}} a_{mn} e^{imt} \sin nx; \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{N}-1}} (m^2 + n^2) |a_{mn}|^2 < \infty \right\}$$

and define the symmetric subspace  $X_s$  and the anti-symmetric subspace  $X_a$ :

$$X_s := \left\{ \sum_{\substack{m \in 2\mathbf{Z}-1 \\ n \in 2\mathbf{N}-1}} a_{mn} e^{imt} \sin nx; \sum_{\substack{m \in 2\mathbf{Z}-1 \\ n \in 2\mathbf{N}-1}} (m^2 + n^2) |a_{mn}|^2 < \infty \right\},$$

$$X_a := \left\{ \sum_{\substack{m \in 2\mathbf{Z} \\ n \in 2\mathbf{N}-1}} a_{mn} e^{imt} \sin nx; \sum_{\substack{m \in 2\mathbf{Z} \\ n \in 2\mathbf{N}-1}} (m^2 + n^2) |a_{mn}|^2 < \infty \right\}.$$

Then, we have  $X = X_s \oplus X_a$ . Let  $Y := \overline{X}^{L^2(D)}$ . We define  $Y_s$  and  $Y_a$  similarly to the definition of  $X_s$  and  $X_a$ . Let  $L_0$  be a principal part of  $f$ , i.e.

$$L_0 := u_{tt} - c^2 u_{xx} + \mu u_t.$$

The operator  $L_0 : X \rightarrow Y$  is an *unbounded* operator with  $L_0^{-1} \in \mathcal{L}(Y, X)$ . The unboundedness of the operator is the special feature commonly in the analysis for PDEs with time variable. For this reason we formulate in [K1] generalize versions of some bifurcation theorems, which is applicable to non-Fréchet differentiable maps.

In view of bifurcation theorem [K1, Theorem 3.1], we should show that  $(\lambda_0, u_0)$  is a simple singular point of  $f(\lambda, u)$  and that an extended system:

$$F(\lambda, u, \phi) := \begin{pmatrix} l\phi - 1 \\ f(\lambda, u) \\ D_u f(\lambda, u)\phi \end{pmatrix} = 0$$

has an isolated solution  $(\lambda_0, u_0, \phi_0)$ , where  $l \in X^*$  is a functional to normalize  $\phi_0$  and  $F : \mathbf{R} \times X_s \times X_a \rightarrow \mathbf{R} \times Y_s \times Y_a$ . We should construct an approximate solution of  $(\lambda_0, u_0, \phi_0)$  and apply the convergence theorem of Newton's method [K1, Theorem 1.1].

## 2. Construction of approximate solution with high accuracy

In this section we will explain how to construct a highly accurate approximate solutions of partial differential equations. One of advantage of our method below is that we can construct a more highly accurate approximate solution by adding some

new high frequency terms provided we notice the accuracy of our approximate solution is not sufficient.

As an example we treat (P) in Section 1. We fix  $\lambda$  and set  $L(u) := f(\lambda, u)$  for simplicity. We will construct an approximate  $2\pi$ - periodic solution with high accuracy. We set the inner product:

$$(g, h) := \int_0^{2\pi} dt \int_0^\pi dx g(x, t) \overline{h(x, t)}.$$

We assume that we already obtain an approximate solution:

$$u^0 = \sum_{(k,l) \in A} a_{kl}^0 e^{ikt} \sin lx$$

by some methods (such as the finite difference methods or finite element methods). Here,  $i = \sqrt{-1}$  and

$$A := \{(k, l) \in \mathbf{N} \times \mathbf{N} ; |k|, l \leq M\}.$$

First let  $u_1$  be an 'optimized' function of  $u_0$  by the least-squares methods:

$$(2.0) \quad u^1 = \sum_{(k,l) \in A} a_{kl} e^{ikt} \sin lx.$$

Here, we determine unknown number  $a_{kl}$  such that  $(L(u^1), L(u^1))$  takes the minimum. Namely, we determine  $a_{kl}$  by applying the Newton method with the initial value:  $a_{kl} = a_{kl}^0$  to the simultaneous equations

$$(2.1) \quad \frac{\partial}{\partial a_{kl}} (L(u^1), L(u^1)) = 0, \quad (k, l) \in A.$$

In order to obtain an approximate solution with higher accuracy we set

$$(2.2) \quad u^2 = u^1 + \sum_{(k,l) \in B-A} \alpha_{kl} e^{ikt} \sin lx$$

$$\text{with } B := \{(k, l) \in \mathbf{N} \times \mathbf{N} ; |k|, l \leq N\}, \quad M < N$$

and we determine unknown number  $\alpha_{kl}$  such that  $(L(u^2), L(u^2))$  takes the minimum. Namely, We make a simultaneous equations for  $\alpha_{kl}$  similar to (2.1) and determine  $\alpha_{kl}$  by applying the Newton method with the initial value:  $\alpha_{kl} = 0$ . After the determination of  $\alpha_{kl}$  we optimize the part of  $u^1$  in the right-hand side of (2.2). If necessary, we repeat the similar procedure to obtain an approximate solution with higher accuracy.

### 3. To estimate the norm of inverse operator

First, we prepare two lemmas:

**Lemma 3.1.** *Let  $X$  and  $Y$  be Banach spaces. Let  $L : X \rightarrow Y$  be a (maybe unbounded) linear operator. Assume that there exists  $P \in \mathcal{L}(Y, X)$  such that*

$$\begin{cases} LP \in \mathcal{L}(Y, X), \\ \|LP - I\| < 1. \end{cases}$$

Then we have  $L^{-1} \in \mathcal{L}(Y, X)$  with estimates:

$$\begin{aligned} \|L^{-1} - P\| &\leq \frac{\|P\|\|LP - I\|}{1 - \|LP - I\|}, \\ \|L^{-1}\| &\leq \frac{\|P\|}{1 - \|LP - I\|}. \end{aligned}$$

**Lemma 3.2.** *Let  $X$  and  $Y$  be Banach spaces. Let  $L : X \rightarrow Y$  be a (maybe unbounded) linear operator. Assume that there exists a linear operator  $A : X \rightarrow Y$  such that*

$$\begin{cases} A^{-1} \in \mathcal{L}(Y, X), \\ LA^{-1} \in \mathcal{L}(Y, Y), \\ L - A \in \mathcal{L}(X, Y) \quad \text{with} \quad \|L - A\|\|A^{-1}\| < 1. \end{cases}$$

Then we have  $L^{-1} \in \mathcal{L}(Y, X)$  with estimates:

$$\begin{aligned} \|L^{-1} - A^{-1}\| &\leq \frac{\|L - A\|\|A^{-1}\|^2}{1 - \|L - A\|\|A^{-1}\|}, \\ \|L^{-1}\| &\leq \frac{\|A^{-1}\|}{1 - \|L - A\|\|A^{-1}\|}. \end{aligned}$$

Let  $X$  and  $Y$  be Banach spaces with  $X \hookrightarrow Y$  being dense. Let  $L : X \rightarrow Y$  be a (maybe unbounded) linear operator. In this section we will show how to estimate  $\|L^{-1}\|$ . We will approximate  $L$  by the following operator  $A$ :

$$(3.1) \quad A \simeq \begin{pmatrix} A_N & 0 \\ 0 & S \end{pmatrix},$$

where  $A_N$  is a finite  $N \times N$  matrix and  $S$  is an infinite diagonal matrix. The relation (3.1) means that there exists a complete orthogonormal base  $\{\phi_k\}_{k=1}^{\infty}$  of  $Y$  such that

$\{\phi_k\}_{k=1}^\infty \subset X$  and  $A_N X_N \subset X_N$  with  $X_N := [\phi_1, \dots, \phi_N]$  (which means the linear subspace generated by  $\phi_1, \dots, \phi_N$ ) and  $\phi_k$  ( $k \geq N+1$ ) is an eigenvector of  $S : X_r \rightarrow \overline{X_r}^Y$  with  $X_r := [\phi_{N+1}, \phi_{N+2}, \dots]$ . We set

$$(3.2) \quad P := A^{-1} = \begin{pmatrix} A_N^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix},$$

Let

$$x = \begin{pmatrix} x_N \\ x_r \end{pmatrix} \in X \quad \text{with} \quad x_N \in X_N \quad \text{and} \quad x_r \in X_r.$$

Then, we have

$$\begin{aligned} (LA^{-1} - I)x &= L \begin{pmatrix} A_N^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} x_N \\ x_r \end{pmatrix} - \begin{pmatrix} x_N \\ x_r \end{pmatrix} \\ &= (LA_N^{-1} - I)x_N + (LS^{-1} - I)x_r \end{aligned}$$

It follows that

$$\begin{aligned} \|(LA^{-1} - I)x\| &\leq \|LA_N^{-1} - I\| \|x_N\| + \|LS^{-1} - I\| \|x_r\| \\ &\leq (\|LA_N^{-1} - I\|^2 + \|LS^{-1} - I\|^2)^{1/2} \|x\| \end{aligned}$$

Therefore, we obtain that

$$\|LA^{-1} - I\| \leq (\|LA_N^{-1} - I\|^2 + \|LS^{-1} - I\|^2)^{1/2}$$

The following condition is sufficient to apply our Lemma 3.1:

$$(3.3) \quad \begin{cases} \|LA_N^{-1} - I\| \leq 1/2, \\ \|LS^{-1} - I\| \leq 1/2. \end{cases}$$

We remark that our method is actually applicable to problems with higher nonlinearity than our main problem (P). Therefore, we consider, as an example, the following simple problem:

$$L := -\frac{\partial^2}{\partial x^2} + p(x) \frac{\partial}{\partial x} + q(x) \quad \text{with} \quad p, q \in L^\infty(0, \pi).$$

We set

$$X := H_0^1(0, \pi) \quad \text{or} \quad H^2 \cap H_0^1(0, \pi), \quad \text{and} \quad Y := L^2(0, \pi).$$

We will show how to construct an operator  $A$  which approximates  $L$ . Let  $P(x)$  (resp.  $Q(x)$ ) be a finite Fourier Sine expansion (resp. a finite Fourier Cosine expansion) in  $(0, \pi)$  approximating  $p(x)$  (resp.  $q(x)$ ). We set

$$\tilde{L} := -\frac{\partial^2}{\partial x^2} + P(x) \frac{\partial}{\partial x} + Q(x).$$

*Assumption:*  $\tilde{L}$  sufficiently approximates  $L$ , i.e.  $\|L - \tilde{L}\|$  is sufficiently small.

We remark that this assumption holds in general for applications including the analysis for our problem (P). In view of Lemma 3.2 we can reduce the estimate of  $\|L^{-1}\|$  to that of  $\|\tilde{L}^{-1}\|$ . Therefore, we assume below without loss of generality that  $p(x)$  (resp.  $q(x)$ ) has the form of a finite Fourier Sine expansion (a finite Fourier Cosine expansion). We define  $S := -\partial^2/\partial x^2$  and  $\phi_n := \sin nx$  for  $n \in \mathbf{N}$ . Then, we have

$$(LS^{-1} - I) \sin nx = \frac{p(x)}{n} \cos nx + \frac{q(x)}{n^2} \sin nx.$$

We will later determine the value of  $N$ . We set

$$x_r := \sum_{n=N+1}^{\infty} a_n \sin nx \in X_r.$$

Then we have

$$\begin{aligned} \|(LS^{-1} - I)x_r\| &\leq \|p(x)\| \sum_{n=N+1}^{\infty} \frac{a_n \cos nx}{n} + \|q(x)\| \sum_{n=N+1}^{\infty} \frac{a_n \sin nx}{n^2} \\ &\leq \|p\|_{\infty} \left\| \sum_{n=N+1}^{\infty} \frac{a_n \cos nx}{n} \right\| + \|q\|_{\infty} \left\| \sum_{n=N+1}^{\infty} \frac{a_n \sin nx}{n^2} \right\| \\ &\leq \frac{\|p\|_{\infty}}{N+1} \left\| \sum_{n=N+1}^{\infty} a_n \cos nx \right\| + \frac{\|q\|_{\infty}}{(N+1)^2} \left\| \sum_{n=N+1}^{\infty} a_n \sin nx \right\| \\ &= \left( \frac{\|p\|_{\infty}}{N+1} + \frac{\|q\|_{\infty}}{(N+1)^2} \right) \left\| \sum_{n=N+1}^{\infty} a_n \sin nx \right\| \\ &= \left( \frac{\|p\|_{\infty}}{N+1} + \frac{\|q\|_{\infty}}{(N+1)^2} \right) \|x_r\| \end{aligned}$$

It follows that

$$(3.4) \quad \|LS^{-1} - I\| \leq \frac{\|p\|_{\infty}}{N+1} + \frac{\|q\|_{\infty}}{(N+1)^2}.$$

We choose the value of  $N$  such that the right-hand side of (3.4) is smaller than  $1/2$ . This condition satisfies the second inequality in (3.3). Let  $N = N_0$  be the smallest natural number satisfying such condition. We determine the value of  $N$  ( $\geq N_0$ ) such that

$$A_N X_N \subset X_N.$$

Here,  $A_N$  is the operator defined by

$$A_N \sin nx := \begin{cases} L \sin nx & \text{for } n \leq N_1, \\ S \sin nx = -n^2 \sin nx & \text{for } n > N_1. \end{cases}$$

Since the operator  $LA_N^{-1} - I : X_N \rightarrow Y$  is expressed by a finite matrix, we can compute the operator norm  $\|LA_N^{-1} - I\|$  by the use of computers. We determine  $N_1$  so large that the first inequality of (3.3) holds.

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