Title
Parametrization by fixed-points multipliers of the polynomials with degree $n$ (Theory and Application in Computer Algebra)

Author(s)
Nishizawa, Kiyoko

Citation
数理解析研究所講究録 1199: 127-131

Issue Date
2001-04

URL
http://hdl.handle.net/2433/64922

Type
Departmental Bulletin Paper

Textversion
publisher
Parametrization by fixed-points multipliers of the polynomials with degree $n$

城西大学理学部 西沢 清子(Kiyoko NISHIZAWA) *

Keywords and phrases: complex dynamical systems – topological conjugate – fixed points – multipliers coordinates – moduli space of the polynomials – algebraic curves – the group of automorphisms – holomorphic index formula.

1 Introduction

Let $\text{Poly}_n(\mathbb{C})$ be the the polynomials from the Riemann sphere, $\hat{\mathbb{C}}$, to itself, with degree $n$, and $\mathbb{M}_n$, called moduli space, the quotient space of $\text{Poly}_n(\mathbb{C})$ under the action of the affine transformation group $\mathfrak{A}(\mathbb{C})$.

We parametrize $\mathbb{M}_n$ by using multipliers of fixed points, and define a natural map $\Psi$ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$. A new coordinate system is called multiplier coordinates. Exhibiting the moduli space of a higher degree under this system deserves particular attention. For example, in study of geometry and topology of $\text{Poly}_n(\mathbb{C})$ from a viewpoint of complex dynamical systems, we make use of this system in order to express singular part, and dynamical loci as algebraic curves or surfaces([NF99], [NF00]).

The subject of this paper is surjectivity-problem of the map $\Psi$ from $\mathbb{M}_n$ to $\mathbb{C}^{n-1}$: a problem of characterization of exceptional part, $\mathcal{E}_n(=\mathbb{C}^{n-1}\setminus \mathbb{M}_n)$.

The initiator of the use of multiplier coordinates is J. Milnor ([Mil93]), to the case of the quadratic rational maps.

2 Polynomials of degree $n$

2.1 Polynomial maps and conjugacy

Let $\hat{\mathbb{C}}$ be the Riemann sphere, and $\text{Poly}_n(\mathbb{C})$ be the space of all polynomial maps of degree $n$ from $\hat{\mathbb{C}}$ to itself:

$$p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \quad (a_n \neq 0).$$

The group $\mathfrak{A}(\mathbb{C})$ of all affine transformations acts on $\text{Poly}_n(\mathbb{C})$ by conjugation:

$$g \circ p \circ g^{-1} \in \text{Poly}_n(\mathbb{C}) \quad \text{for} \quad g \in \mathfrak{A}(\mathbb{C}), \ p \in \text{Poly}_n(\mathbb{C}).$$

*kiyoko@math.josai.ac.jp
Two maps $p_1, p_2 \in \text{Poly}_n(\mathbb{C})$ are \textbf{holomorphically conjugate} if and only if there exists $g \in \mathcal{A}(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$.

Under this conjugacy of the action of $\mathcal{A}(\mathbb{C})$, any map in $\text{Poly}_n(\mathbb{C})$ is conjugate to a “monic” and “centered” map, i.e.,

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0.$$ 

We remark that this $p$ is determined up to the action of the group $G(n-1)$ of $(n-1)$-st roots of unity, where each $\eta \in G(n-1)$ acts on $p \in \text{Poly}_n(\mathbb{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$.

Every polynomial map from $\hat{\mathbb{C}}$ to itself is conjugate under an affine change of variable to a monic centered one, and this is uniquely determined up to conjugacy under the action of the group $G(n-1)$ of $(n-1)$-st roots of unity.

For example, in the case of $n = 3$, the following two monic and centered polynomials belong to the same conjugacy class:

$$z^3 + az + +c, \quad z^3 + az - c.$$ 

In the case of $n = 4$ the following three monic and centered polynomials belong to the same conjugacy class:

$$z^4 + az^2 + bz + c$$
$$z^4 + awz^2 + bz + c\omega^2$$
$$z^4 + aw^2z^2 + bz + c\omega$$

where $\omega$ is a third root of unity.

### 2.2 Moduli space of polynomial maps

The quotient space of $\text{Poly}_n(\mathbb{C})$ under the action $\mathcal{A}(\mathbb{C})$ will be denoted by $\mathbb{M}_n$, and called the \textbf{moduli space} of holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps $p$ of degree $n$.

Let $\mathcal{P}_1(n)$ be the affine space of all monic centered polynomials of degree $n$

$$p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0,$$

with coefficients-coordinate $(c_0, c_1, \cdots, c_{n-2})$.

Then we have an $(n-1)$-to-one canonical projection $\Phi$ from $\mathcal{P}_1(n)$ onto $\mathbb{M}_n$.

Hence the affine space $\mathcal{P}_1(n)$ is regarded as an $(n-1)$-sheeted covering space of $\mathbb{M}_n$. Thus we can use $\mathcal{P}_1(n)$ as a coordinate space for the moduli space $\mathbb{M}_n$, though it remains the ambiguity up to the group $G(n-1)$. This coordinate space has the advantages of being easy to be treated.

However, it would be also worthwhile to introduce another coordinate system having any merit different from $\mathcal{P}_1(n)$’s.

In fact, Milnor successfully introduced coordinates in the moduli space of the space of all quadratic rational maps using the elementary symmetric functions of the multipliers at the fixed points of a map ([Mil93]). To the case of $\text{Poly}_n(\mathbb{C})$, we try to explore an analogy.

### 2.3 Multiplier coordinates

Now we intend to explore another coordinate space for $\mathbb{M}_n$. For each $p(z) \in \text{Poly}_n(\mathbb{C})$, let $z_1, \ldots, z_n, z_{n+1}(= \infty)$ be the fixed points of $p$ and $\mu_i$ the multipliers of $z_i$; $\mu_i = p'(z_i) \ (1 \leq$
\(i \leq n\), and \(\mu_{n+1} = 0\). Consider the elementary symmetric functions of the \(n\) multipliers,

\[
\begin{align*}
\sigma_{n,1} &= \mu_1 + \cdots + \mu_n, \\
\sigma_{n,2} &= \mu_1 \mu_2 + \cdots + \mu_{n-1} \mu_n = \sum_{i=1}^{n-1} \mu_i \sum_{j>i}^{n} \mu_j, \\
&\quad \cdots \\
\sigma_{n,n} &= \mu_1 \mu_2 \cdots \mu_n, \\
\sigma_{n,n+1} &= 0.
\end{align*}
\]

Note that these are well defined on the moduli space \(\mathbb{M}_n\), since \(\mu_i\)'s are invariant by affine conjugacy.

2.3.1 The holomorphic index fixed point formula

For an isolated fixed point \(f(x_0) = x_0, \ x_0 \neq \infty\) we define the holomorphic index of \(f\) at \(x_0\) to be the residue

\[
\iota(f, x_0) = \frac{1}{2\pi i} \oint \frac{1}{z-f(z)} \, dz
\]

For the point at infinity, we define the residue of \(f\) at \(\infty\) to be equal to the residue of \(\phi \circ f \circ \phi\) at origin, where \(\phi(z) = \frac{1}{z}\). The Fatou index theorem (see [Mi190]) is as follows:

For any rational map \(f : \mathbb{C} \rightarrow \mathbb{C}\) with \(f(z)\) not identically equal to \(z\), we have the relation

\[
\sum_{f(z) = z} \iota(f, z) = 1
\]

This theorem can be applied to these \(\mu_i\)'s: \(\sum_{i=1}^{n} \frac{1}{1-\mu_i} + \frac{1}{1-0} = 1\), provided \(\mu_i \neq 1 (1 < i < n)\). Arranging this equation for the form of elementary symmetric functions, we have

\[
\gamma_0 + \gamma_1 \sigma_{n,1} + \gamma_2 \sigma_{n,2} + \cdots + \gamma_{n-1} \sigma_{n,n-1} = 0
\]

where

\[
\gamma_k = (-1)^k \binom{n-1}{k} \binom{n}{k} = (-1)^k (n-k).
\]

Note that \(\mu_i = 1 (1 \leq i \leq n)\) is allowable here. Then we have the following Linear Relation:

For the cubic case \((n = 3)\), we have \(3 - 2\sigma_{3,1} + \sigma_{3,2} = 0\)

For the quartic case \((n = 4)\), we have \(4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0\)

And in general the following linear relation holds:

**Theorem 1** Among \(\sigma_{n,i}\)'s, there is a linear relation

\[
\sum_{k=0}^{n-1} (-1)^k (n-k) \sigma_{n,k} = 0,
\]

where we put \(\sigma_{n,0} = 1\).

In view of Theorem 1, we have the natural map \(\Psi\) from \(\mathbb{M}_n\) to \(\mathbb{C}^{n-1}\) corresponding to

\[
\Psi(\langle p \rangle) = (\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n}).
\]

We remark that \(\Psi(\mathbb{M}_n) \subset \mathbb{C}^{n-1}\).
2.3.2 Characterization of exceptional set

To investigate whether this map $\Psi$ is surjective or not is our main subject: a problem of characterization of the part of $\mathbb{C}^{n-1} \setminus \Psi(M_n)$.

We call this set **exceptional set** and denote it by

$$\mathcal{E}_n = \mathbb{C}^{n-1} \setminus \Psi(M_n).$$

Our main subject is as follows:
For a given $(s_1, s_2, \cdots, s_{n-2}, s_n) \in \mathbb{C}^{n-1}$, we set $s_{n-1}$ a solution of

$$\sum_{k=0}^{n-1} (-1)^k (n-k) s_k = 0, \quad s_0 = 1.$$

Then for the point $(s_1, \cdots, s_n) \in \mathbb{C}^{n-1}$, we set a polynomial

$$m(z) = z^n + s_1 z^{n-1} + s_2 z^{n-2} + \cdots + s_{n-1} z + s_n$$

Then we denote the roots of this polynomial by

$$\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n.$$

**Can we obtain a polynomial $p(z) \in P_1(n)$ whose multiplier-coordinate $(\sigma_1, \cdots, \sigma_n)$ is corresponding to $(s_1, \cdots, s_n)$?**

Namely, can we find a polynomial satisfying that for fixed points $z_i$

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \text{ with } \mu_i = p'(z_i).$$

The case $n = 3$ is nicely solved: $\Psi$ is surjective. ([NF96], [FN97]. This fact is mentioned in [Mil93] without any details.)

We also solved this problem for the case $n = 4$ ([NF96], [FN97]):

**Theorem 2** $\Psi : M_4 \longrightarrow \mathbb{C}^3$ is not surjective:

$$\mathcal{E}_4 = \mathbb{C}^3 \setminus \Psi(M_4)$$

$$= (4, s, \frac{s^2}{4} - 2s + 4) \quad s \neq 4$$

As for the cases of general $n$, we expect analogous results.

Recently, we have a following result:

**Theorem 3** (*M. Fujimura*)
Let $\Omega = \{\mu_i\}_{i=1, \cdots, n}$ be the set of all roots of a polynomial $m(z)$. If $\Omega$ satisfies one of the following cases (A), (B) and (C), then there exists a polynomial $p(z) \in P_1(n)$ such that

$$p(z_i) = z_i, \quad (i = 1, \cdots, n) \text{ with } \mu_i = p'(z_i).$$

(A):
1. Any element of $\Omega$ is not equal 1 : $\mu_i \neq 1$,
2. $\sum_i \frac{1}{b_i} = 0, \quad b_i = 1 - \mu_i$. 
3. for any proper subset $\omega$ of roots, $\sum_{s \in \omega} \frac{1}{b_s} \neq 0$,

(B):
1. Let $\Omega' = \{\mu_i\}_{i=1, \ldots, m} \leq m \leq n - 2$ be a subset of $\Omega$ whose elements are not equal $1$ : $\mu_i \neq 1$,
2. for any subset $\omega$ of $\Omega'$, $\sum_{s \in \omega} \frac{1}{b_s} \neq 0$,

(C):
1. Any element of $\Omega$ is equal $1$ : $\mu_i = 1$.

2.3.3 Examples

We shall show some examples for our inverse problem. By these examples show that the Fujimura's theorem only gives a sufficient condition for surjectivity.

- For a set $\{\mu, 2 - \mu, \lambda, 2 - \lambda\}$, $\mu \neq \lambda$, $\mu \neq 1$ a corresponding polynomial exits in $P_1(4)$.
- For a set $\{\mu, 2 - \mu, \mu, 2 - \mu\}$ $\mu \neq 1$, no corresponding polynomial exits $P_1(4)$.
- For a set $\{\mu, \mu, \mu, \lambda, \lambda\}$, $\mu \neq 1$, $5 - 2\mu - 3\lambda = 0$ a corresponding polynomial exits $P_1(5)$.
- For a set $\{\mu, \mu, \mu, 2 - \mu, \frac{5-\mu}{2}\}$, $\mu \neq 1$, no corresponding polynomial exits $P_1(5)$.

References


