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Deciding Linear-Trigonometric Problems

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Abstract

In this paper, we present a decision procedure for certain linear-trigonometric problems for the reals and integers formalized in a suitable first-order language. The inputs are restricted to formulas, where all but one of the quantified variables occur linearly and at most one occurs both linearly and in a specific trigonometric function. Moreover we allow in addition the integer-part operation in formulas. Besides ordinary quantifiers, we allow also counting quantifiers. Furthermore we also determine the qualitative structure of the connected components of the satisfaction set of the mixed linear-trigonometric variable. We also consider the decision of these problems in subfields of the real algebraic numbers.

1 Introduction

Decision and quantifier elimination methods for the real numbers as ordered field have a large variety of applications in mathematics, computer science, and industrial engineering (see *e.g.* [3, 25]). In particular, many engineering and industrial applications can be reduced to the QE problems containing trigonometric functions (see [9, 6]).

Unfortunately, the general first-order theory of the real numbers with sine or cosine and the constant π is undecidable: this is because the zeros of sine function are exactly the integer multiples of π. As a consequence one can code Hilbert's tenth problem in existential formulas involving $+, -, \cdot$, the sine function and the constants $0, 1, \pi$ (compare [13]). In fact as remarked already in [19] even the first-order theory of the real numbers with sine (without the constant π) is undecidable, since one can code the first-order theory of rational numbers in this theory by representing rational numbers as quotients of arbitrary zeros of the sine function. In the special case, where the quantified variables of the given formula split into purely polynomial ones and purely trigonometric ones the decision problem and the quantifier elimination problem can be reduced to the theory of real numbers as ordered field. It suffices to replace for a purely trigonometric variable x the terms $\sin(x)$ and $\cos(x)$ by new variables u, v, respectively, with the additional relation $u^2+v^2 = 1$ (compare [7, 4, 5]). The introduction of two new variables in place of one trigonometric variables creates, however, serious problems in practical computational complexity. A more efficient alternative was proposed by Pau *et al.*; it adapts Collins' CAD

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method [2] to the trigonometric case [12]. And trigonometric problems of special types have been studied by D. Richardson by a variety of approaches [14, 15, 16, 17].

Here we consider the decision problem for a fragment of the full first-order theory of the reals as ordered field with one specific trigonometric function. In contrast to the special case, where quantified variables are separated into polynomial and trigonometric variables, we allow one mixed quantified variable x occurring both in the trigonometric function and in linear polynomials. All other quantified variables x_i may occur only linearly. Parameters may occur in arbitrary degree. We assume moreover that the mixed variable x is quantified last (*i.e.* outermost) in the given formula. Closed formulas of this restricted type are called *normal*. We present an decision procedure for normal formulas. The procedure has certain parallels in its approach to the decision procedure for linear-exponential problems announced in [26]. In fact by a simple coding trick we could also allow a whole block of quantifiers with respect to mixed variables in place of a single mixed variable.

Both extend also to the mixed real-integer case, *i.e*, to formulas, where terms may involve the integer-part operation and hence implicitly quantifiers over the integers. Our results may be viewed as a first step towards decidable classes of problems with mixed algebraic-trigonometric real and integer variables that are motivated by the study of control theory of hybrid systems [8].

The proofs involve real and mixed real-integer linear quantifier elimination [21, 23, 10, 27], elementary analysis, the idea of checking the real satisfiability of formulas by finitely many symbolic test points, and Lindemann's theorem from transcendental number theory in order to decide relations between constant terms. In contrast to the purely linear case considered in [10, 27] the test points considered here have no straightforward mathematical representation. We invent a symbolic notation for them, and show, how to sort these points in their natural order, and how to evaluate the sign of other terms at these symbolically represented points in an indirect manner.

We also solve the decision problem for normal formulas, where quantifiers range over an arbitrary subfield F of the field of real algebraic numbers. In contrast to purely linear problems, mixed linear-trigonometric problems may have different truth values over the reals and over F. Besides ordinary quantifiers, we may also allow counting quantifiers.

In contrast to the general decision procedure for the first order theory of the reals as exponential field that relies on Schanuel's conjecture [11], we require no open conjecture from transcendence theory. Instead we use only Lindemann's theorem and the transcendency of π in order to decide relations between constants.

The classical theorem of Lindemann (see [18], page 70) asserts that the values of the complex exponential function for non-zero complex algebraic arguments are transcendental. As a consequence a corresponding assertion holds for real trigonometric functions as well. This is seen as follows: Call a real or complex function $f(z)$ is *strongly transcendental* (with exceptional point p) if for all real or complex numbers ($z \neq p$) it is not the case that both z and $f(z)$ are algebraic. Then by Lindemann's theorem e^z is strongly transcendental with exceptional point 0. As an immediate consequence we see that the real logarithm $\ln(x)$ is strongly transcendental with exceptional point 1.

Next we claim that the real sinus and cosinus functions are strongly transcendental with exceptional point 0. Indeed $e^{ix} = \cos(x) + i\sin(x)$ and $\cos^2(x) + \sin^2(x) = 1$. So if $\cos(x)$ is algebraic, then by the second equation $\sin(x)$ is algebraic; since i is algebraic, this implies that $i\sin(x)$, and hence e^{ix} is algebraic. By Lindemann's theorem this implies that ix, and hence x is transcendental, or that $x = 0$. As a consequence the inverse functions $\arccos(x)$ and

$\arcsin(x)$ are also strongly transcendental with exceptional point 1 and 0, respectively. In order to see that the real tangent function is strongly transcendental with exceptional point 0, we use the well-known formula for the arcustangent: $\arctan(x) = -\frac{i}{2}\ln(\frac{1+ix}{1-ix})$. It shows that for algebraic $x \neq 0$ $\arctan(x)$ is transcendental. As a consequence $\arctan(x)$ and $tan(x)$ are strongly transcendental with exceptional point 0. In a similar way one sees that the real arcuscotangent function and hence the real cotangent function are strongly transcendental without exceptional point.

2 The formal framework

Let F be an arbitrary subfield F of the field of real algebraic numbers. We consider the following fragment L of a first-order language for the real numbers. Let x be a distinguished variable. Linear terms are expressions of the form $s := cx + \sum_{i=1}^{n} c_i x_i + c_0$ with $c; c_i \in F$. The trigonometric term is an expressions of the form $t := trig(ax + b)$, where $trig$ denotes a real trigonometric function sine, cosine or tangent, and $a(\neq 0)$, b are fixed elements of F. Atomic formulas are equations or inequalities of the form $s = 0, s = t, s < t, t < s$, where s is a linear term and t is an trigonometric term. Arbitrary formulas are obtained from atomic formulas as usual in first order logic, *i.e.* by iterated application of the propositional operations $\wedge, \vee, \neg$ and quantifiers $\exists x, \exists x_i, \forall x, \forall x_i$ with respect to the variable x or to some x_i. Every formula can be rewritten equivalently in prenex normal form. If all occurrences of x in a formula φ are quantified, then φ is a *closed formula.* In the natural interpretation of formulas over the real numbers, every closed formula is either true or false. We call a closed formula φ *normal* if it is prenex and either does not contain the variable x or the quantifier $\exists x$ or $\forall x$ binding x is outermost in the prefix of φ. Our goal is a *decision procedure* for normal formulas in L *i.e.* an algorithm that takes normal formulas in L as inputs and Output the truth value of the input formula over the reals. This goal is analogous to that in [22] for almost linear integer problems.

In fact we could allow in normal formulas a whole block of quantifiers referring to mixed variables, say $y_1, \ldots, y_m$ in place of a single mixed variable x. Then trigonometric terms would be of the form $trig(a_0 + a_1 y_1 + \ldots a_m y_m)$ for fixed real constants $a_0, \ldots, a_m$. Such a formula $\exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n(\psi)$ with arbitrary quantifiers Q_i can be equivalently replaced by the following normal formula

$$\exists x \exists y_1 \ldots \exists y_m Q_1 x_1 \ldots Q_n x_n (x = a_0 + a_1 y_1 + \ldots a_m y_m \ \wedge \ \psi'),$$

where ψ' results from ψ by replacing each occurence of $trig(a_0 + a_1 y_1 + \ldots a_m y_m)$ by $trig(x)$.

In fact we will consider also formulas in an extended sense, where we admit besides the "ordinary" quantifiers $\exists, \forall$ also the "counting quantifiers" $\exists^{\leq n}, \exists^{n}, \exists^{\geq n}, \exists^{\leq \infty}, \exists^{\infty}$ interpreted as "there exist at most n, exactly n, at least n, only finitely many, infinitely many", respectively for $n \in \mathbf{N}$. Formulas, closed formulas, normal formulas in this extended sense will be referred to as formulas, closed formulas, normal formulas with counting quantifiers. Notice that the first three types of counting quantifiers are definable using ordinary quantifiers, while the last two are not definable in this way.

Let $\varphi := Qx(\psi(x))$ be a normal formula (Q a quantifier $\exists$ or $\forall$). Let ψ_1 result from ψ by replacing the trigonometric term $trig(ax + b)$ by a new variable z. Then ψ_1 is a linear formula in the sense of [21, 10]. So by linear real quantifier elimination, ψ_1 is equivalent to a quantifier-free linear formula ψ_2 with x and z as the only linear variables. Let ψ_3 result from

ψ_2 by backsubstitution of the trigonometric term $t = trig(ax+b)$ for z. Then $\varphi' := Qx(\psi_3)$ is a normal formula with only one quantifier that is equivalent to φ. Notice that rational linear combinations of trigonometric terms can be rewritten as a rational multiple of a trigonometric term. So by dividing by rational constants the atomic formulas of φ' can assumed to be normal.

For normal formulas with counting quantifiers one can argue similarly, using linear elimination by test points. So for example a counting quantifier $\exists^{\infty}x_i(\sigma(x_i))$ can be eliminated by restricting the disjunction over test terms used for the elimination of the quantifier $\exists x_i(\sigma(x_i))$ to test terms of the form $s \pm \epsilon$, where s is a linear term.

So the problem of deciding the truth value of normal formulas is reduced to normal input formulas of the form $Qx(\psi)$, where ψ is quantifier-free. Moreover by passing to the negation - if necessary - we may assume that Q is an existential quantifier. So it suffices to decide such univariate normal formulas with a single existential quantifier $\exists x$ over the reals. In fact we will do more: In case the truth value of $\exists x(\psi)$ is 'true', we will provide a sample point $r \in \mathbb{R}$ for x. In fact if M is the set of reals satisfying $\psi(x)$, then we will determine the number of connected components of M and their types as a non-empty intervals $[b,c]$, (b,c), $[b,c)$ or $(b,c]$.

As a consequence we can also in an obvious way decide a univariate normal formula $Qx(\psi)$ with quantifier-free ψ, where Q is a counting quantifier.

3 Geometric properties of trigonometric functions and straight lines

In order to study the solutions of equations and inequalities between straight lines and trigonometric functions, we now consider functions of the type $f(x) = trig(x) - (cx+d)$, and focus on the case of $trig(x) = \cos(x)$ or $trig(x) = \sin(x)$. In the following, we only deal with the case of $\cos(x)$ because the case of $\sin(x)$ is very analogous to that of $\cos(x)$. Let the basic intervals for be $\mathcal{I}_n := (-\frac{\pi}{2} + n\pi, -\frac{\pi}{2} + (n+1)\pi)$ for $n \in \mathbf{N}$. For any straight line $y = cx + d$ the equation $\cos(x) = cx + d$ has at most two solutions in the interval $\mathcal{I}_n$. We denote these solutions (if they exist) by $\xi_{n,1}$ and $\xi_{n,2}$.

Definition 1 *Consider the equation* $\cos(x) = cx + d$ *with real numbers* c, d *and assume that* $c > 0$. *Then let* $x_{+1} = \frac{1-d}{c}$, $x_0 = -\frac{d}{c}$ *and* $x_{-1} = \frac{-1-d}{c}$. *Then we call the region* $[x_{-1}, x_{+1}]$ *from* x_{-1} *to* x_{+1} *the* domain *of the line* $cx+d$, *in particular, we call the region* $[x_0, x_{+1}]$ *from* x_0 *to* x_{+1} *the* positive *domain and the region* $[x_{-1}, x_0]$ *from* x_{-1} *to* x_0 *the* negative *domain of the line. For* $c < 0$ *the definitions are analogous.*

Lemma 2 *Consider the equation* $\cos(x) = cx + d$ *with real numbers* c, d. *Then the following hold:*

1. *If* $c = 0$ *and* $|d| < 1$, *then the equation has infinitely many solutions.*

   If $1 > d \geq 0$ *then the solutions are* $\xi_{n,1}, \xi_{n,2}$, *where* $-\frac{\pi}{2} + n\pi \leq \xi_{n,1} < \xi_{n,2} \leq -\frac{\pi}{2} + (n+1)\pi$ *for all even integer* n. *Moreover* $\cos(x) \geq cx + d$ *for* $\xi_{n,1} \leq x \leq \xi_{n,2}$, *otherwise* $\cos(x) < cx + d$ *in* $\mathcal{I}_n$.

   If $0 > d > -1$ *then the solutions are* $\xi_{n,1}, \xi_{n,2}$ *where* $-\frac{\pi}{2} + n\pi \leq \xi_{n,1} < \xi_{n,2} \leq -\frac{\pi}{2} + (n+1)\pi$ *for all odd integer* n. *Moreover* $\cos(x) \leq cx + d$ *for* $\xi_{n,1} \leq x \leq \xi_{n,2}$, *otherwise* $\cos(x) > cx + d$ *in* $\mathcal{I}_n$.

2. *If* $c = 0$ *and* $|d| = 1$, *then the equation has infinitely many solutions* $n\pi$ *where* n *is even for* $d = 1$ *and odd for* $d = -1$. *Moreover* $\cos(x) \geq cx + d$ *for* $d = -1$, $\cos(x) \leq cx + d$ *for* $d = 1$.

3. *If* $c = 0$ *and* $|d| > 1$, *then the equation has no solution. Moreover* $\cos(x) > cx + d$ *for* $d < -1$, $\cos(x) < cx + d$ *for* $d < 1$.

4. *If* $|c| \geq 1$ *then the equation has exactly one solution* ξ. *When* $c \geq 1$ *then* $\cos(x) \geq cx + d$ *for* $x \leq \xi$, $\cos(x) < cx + d$ *for* $x > \xi$. *And when* $c \leq -1$ *then* $\cos(x) \geq cx + d$ *for* $x \geq \xi$, $\cos(x) < cx + d$ *for* $x < \xi$.

5. *If* $0 < c < 1$ *then the following cases occur in the positive domain where* m, n *are integers s.t.* $m < n$*:*

   (i) If $-\frac{\pi}{2} + 2m\pi < x_0 < -\frac{\pi}{2} + (2m+1)\pi$ *and* $-\frac{\pi}{2} + 2n\pi < x_{+1} < 2n\pi$ *then the equation has the solutions* $\xi_{2m,1} < \xi_{2m+2,1} < \xi_{2m+,2} < \cdots < \xi_{2n-2,1} < \xi_{2n-2,2}$.
   Moreover $\cos(x) \geq cx + d$ *for* $x_0 \leq x \leq \xi_{2m,1}$, $\cos(x) < cx + d$ *for* $\xi_{2m,1} < x < \xi_{2m+2,1}$, $\cos(x) \geq cx + d$ *for* $\xi_{2m+2,1} \leq x \leq \xi_{2m+2,2}$, $\cdots$, $\cos(x) \geq cx + d$ *for* $\xi_{2n-2,1} \leq x \leq \xi_{2n-2,2}$, $\cos(x) < cx + d$ *for* $\xi_{2n-2,2} \leq x_{+1}$.

   (ii) If $-\frac{\pi}{2} + 2m\pi < x_0 < -\frac{\pi}{2} + (2m+1)\pi$ *and* $2n\pi < x_{+1} < -\frac{\pi}{2} + (2n+1)\pi$ *then the equation has the solutions* $\xi_{2m,1} < \xi_{2m+2,1} < \xi_{2m+2,2} < \cdots < \xi_{2n-2,1} < \xi_{2n-2,2} < \xi_{2n,1} < \xi_{2n,2}$.
   Moreover $\cos(x) \geq cx + d$ *for* $x_0 \leq x \leq \xi_{2m,1}$, $\cos(x) < cx + d$ *for* $\xi_{2m,1} < x < \xi_{2m+2,1}$, $\cos(x) \geq cx + d$ *for* $\xi_{2m+2,1} \leq x \leq \xi_{2m+2,2}$, $\cdots$, $\cos(x) \geq cx + d$ *for* $\xi_{2n-2,1} \leq x \leq \xi_{2n-2,2}$, $\cos(x) < cx + d$ *for* $\xi_{2n-2,2} < x < \xi_{2n,1}$, $\cos(x) \geq cx + d$ *for* $\xi_{2n,1} \leq x \leq \xi_{2n,2}$, $\cos(x) < cx + d$ *for* $\xi_{2n,2} < x \leq x_{+1}$.

   (iii) If $-\frac{\pi}{2} + (2m+1)\pi < x_0 < -\frac{\pi}{2} + (2m+2)\pi$ *and* $-\frac{\pi}{2} + 2n\pi < x_{+1} < 2n\pi$ *then the equation has the solutions* $\xi_{2m+2,1} < \xi_{2m+2,2} < \cdots < \xi_{2n-2,1} < \xi_{2n-2,2}$.
   Moreover $\cos(x) < cx + d$ *for* $x_0 < x < \xi_{2m+2,1}$, $\cos(x) \geq cx + d$ *for* $\xi_{2m+2,1} \leq x \leq \xi_{2m+2,2}$, $\cdots$, $\cos(x) \geq cx + d$ *for* $\xi_{2n-2,1} \leq x \leq \xi_{2n-2,2}$, $\cos(x) < cx + d$ *for* $\xi_{2n-2,2} \leq x_{+1}$.

   (vi) If $-\frac{\pi}{2} + (2m+2)\pi < x_0 < -\frac{\pi}{2} + (2m+2)\pi$ *and* $2n\pi < x_{+1} < -\frac{\pi}{2} + (2n+1)\pi$ *then the equation has the solutions* $\xi_{2m+2,1} < \xi_{2m+2,2} < \cdots < \xi_{2n-2,1} < \xi_{2n-2,2} < \xi_{2n,1} < \xi_{2n,2}$,
   Moreover $\cos(x) < cx + d$ *for* $x_0 < x < \xi_{2m+2,1}$, $\cos(x) \geq cx + d$ *for* $\xi_{2m+2,1} \leq x \leq \xi_{2m+2,2}$, $\cdots$, $\cos(x) \geq cx + d$ *for* $\xi_{2n-2,1} \leq x \leq \xi_{2n-2,2}$, $\cos(x) < cx + d$ *for* $\xi_{2n-2,2} < x < \xi_{2n,1}$, $\cos(x) \geq cx + d$ *for* $\xi_{2n,1} \leq x \leq \xi_{2n,2}$, $\cos(x) < cx + d$ *for* $\xi_{2n,2} < x \leq x_{+1}$.

   For the negative domain we have an analogous case distinction.

6. *If* $-1 < c < 0$ *then we have analogous subcases as in case 5.*

Proof Obvious from the graphs of $\cos(x)$ and of $cx + d$. ∎

4 Decision Procedures for univariate formulas

In this section we decide the validity of univariate closed normal formulas in the language L of the form $\exists x(\psi)$, where ψ is a quantifier-free. In view of the results in the section 2 this will solve the general decision problems for normal formulas.

Let the input formula φ be of the form $\exists x(\psi(x))$, where ψ is quantifier-free. The only trigonometric term occurring in ψ is $trig(ax+b)$ where a, b are elements of F and $a \neq 0$. By the linear substitution $x \mapsto a^{-1}x - a^{-1}b$, φ is equivalent to $\exists x(\psi')$, where $\psi' := \psi(a^{-1}x - a^{-1}b))$. Notice that the only trigonometric term occuring in ψ' is $trig(x)$. So ψ' is a propositional combination of atomic formulas

$$\alpha_i := c_i x + d_i \;\rho_i\; trig(x) \;or\; c_i x + d_i \;\rho_i\; 0 \tag{1}$$

with $i \in I$, $\rho_i \in \{=, <, >\}$. Let s be 2π plus the maximal absolute value of the domain bounds of all lines $c_i x + d_i$ with $c_i \neq 0$ occurring in ψ' and of the first components of all intersection points between all lines $y = c_i x + d_i$ and $y = c_j' x + d_j'$ with $c_i \neq 0$ or with $c_j \neq 0$. Consider only the region $[-s, s]$. For every atomic formula α_i in ψ', the set M_{α_i} of reals defined by it in $[-s, s]$ consists of one interval or finitely many disjoint intervals described in lemma 2. So the set $M_{\psi'}$ of reals in $[-s, s]$ defined by ψ' is a finite union of intervals, whose endpoints are among solutions of the equations

$$c_i x + d_i = trig(x) \;or\; c_i x + d_i = 0 \tag{2}$$

obtained from the atomic formulas α_i in (1).

Lemma 3 *If $\exists x(\psi')$ holds in $\mathbb{R}$, then it holds in $[-s, s]$.*

Proof This is an immediate consequence from the definition of s and lemma 2. ∎

Let $p_1 < p_2 < \ldots < p_n$ be the solutions of the equations (2) ordered in increasing order. Then the truth value of each atomic formula α_j of ψ' and hence of ψ' remains invariant in each of the intervals $(-\infty, p_1)$, (p_1, p_2), $\ldots$, (p_{n-1}, p_n), $(p_n . \infty)$. So following the ideas in [10] in the present context, it suffices to determine the truth value of each α_i at the test points $-s, p_i$, and $p_i + \epsilon$, where ϵ is a positive infinitesimal. From the result of these evaluations one can then determine the truth-value of ψ' at all these points, and thus determine the structure of the set of real values x for which $\psi'(x)$ holds. In particular one can then decide, whether or not φ holds. More precisely, one obtains the exact number of connected components of the set of real values x for which $\psi'(x)$ holds.

Two problems arise in this procedure:

1. First, one needs to determine the order relations between the solutions of the equations (2).

2. Second, one must be able to evaluate all atomic formulas α_j of ψ' at these solutions, closely to the right of these solutions, and at $-s$.

In contrast to the purely linear situation in [10], the solutions of the equations $c_i x + d_i = trig(x)$ $(i \in I)$ are no longer given in a standard mathematical notation. So the evaluation of α_j at or closely to the right of these solutions has to be done in a roundabout way. It turns out that both problems can be overcome by case distinctions following lemma 2 and evaluation of $trig(x)$ at the solution of the equations (2) in $[-s, s]$.

The next two lemmas specify the details depending on the cases specified in lemma 2. Moreover, here we also restrict to the case of $trig(x) = \cos(x)$.

Lemma 4 *Let ℓ and ℓ' be a distinct straight lines defined by $y = cx + d$ and $y = c'x + d'$, respectively, where $c, c'd, d' \in F$. Let D be the interval $[-t, t]$ where*

$$t := max\{-\frac{\pi}{2} + (n+1)\pi \mid n \in \mathbf{N} \text{ and } -\frac{\pi}{2} + n\pi < s\}.$$

Let $\mathcal{I}_{2n} = (-\frac{\pi}{2} + 2n\pi, -\frac{\pi}{2} + (2n+1)\pi)$ for integers n. We denote as before by $\xi_{2n,1} < \xi_{2n,2}$ the potential solutions of the equation $\cos(x) = cx + d$ in $\mathcal{I}_{2n} \cap D$. Similarly we denote by $\xi'_{2n,1} < \xi'_{2n,2}$ the potential solutions of the equation $\cos(x) = c'x + d'$ in $\mathcal{I}_{2n} \cap D$. We let $(\alpha, \beta) \in F^2$ denote the potential intersection point of the lines ℓ and ℓ'. Notice that $\xi_{2n,1}, \xi_{2n,2}, \xi'_{2n,1}, \xi'_{2n,2}$ are all different from α, since $\cos(x)$ is strongly transcendental. If the line $cx + d > 0$ in an interval, then it is called positive *in the interval.*

If one of the two lines ℓ and ℓ' has no intersection point with $\cos(x)$, the ordering of the solutions $\xi_{i,j}$ are obvious from lemma 2. Hence we only consider the cases where both two lines have at least one solution with $\cos(x)$ in $\mathcal{I}_{2n}$. Then the order relation among the solutions $\xi_{i,j}$ are decided as follows:

1. *If the two lines ℓ and ℓ' are both positive and have intersection points with $\cos(x)$ in $\mathcal{I}_{2n}$, then let $\xi_{2n,1}, \xi_{2n,2}$ be the intersection points with $\cos(x)$ for the line ℓ and $\xi'_{2n,1}, \xi'_{2n,2}$ be those for the line ℓ'. (Note that the case where $\cos(\alpha) = \beta$ is impossible, since $\cos(x)$ is strongly transcendental.)*

   (a) *If there is no intersection point of ℓ and ℓ' with α in $\mathcal{I}_{2n}$ or the intersection point (α, β) has α in $\mathcal{I}_{2n}$ and $\cos(\alpha) < \beta$, then we have $\xi'_{2n,1} < \xi_{2n,1} < \xi_{2n,2} < \xi'_{2n,2}$ for $c(2n\pi) + d > c'(2n\pi) + d'$ and $\xi_{2n,1} < \xi'_{2n,1} < \xi'_{2n,2} < \xi_{2n,2}$ for $c(2n\pi) + d < c'(2n\pi) + d'$.*

   (b) *If the intersection point of ℓ and ℓ' has α in $\mathcal{I}_{2n}$ and $\cos(\alpha) > \beta$, then we have $\xi'_{2n,1} < \xi_{2n,1} < \xi'_{2n,2} < \xi_{2n,2}$ for $c < c'$, and $\xi_{2n,1} < \xi'_{2n,1} < \xi_{2n,2} < \xi'_{2n,2}$ for $c > c'$.*

2. *If the line ℓ is positive and has intersection points with $\cos(x)$ in $\mathcal{I}_{2n}$ and ℓ' has an intersection point $(\gamma, 0)$ with the x-axis in $\mathcal{I}_{2n}$, let $\xi_{2n,1}, \xi_{2n,2}$ be the intersection points with $\cos(x)$ for the line ℓ and $\xi'_{2n,1}$ be that for the line ℓ'. (The case where the line ℓ' is positive and has intersection points with $\cos(x)$ in $\mathcal{I}_{2n}$ and ℓ has an intersection point $(\gamma, 0)$ with the x-axis in $\mathcal{I}_{2n}$ is similar to this case.)*

   (a) *Consider the case $c' > 0$: If $\cos(\alpha) < \beta$, then we have $\xi_{2n,1} < \xi_{2n,2} < \xi'_{2n,1}$. If $\cos(\alpha) > \beta$, then we have $\xi_{2n,1} < \xi'_{2n,1} < \xi_{2n,2}$.*

   (b) *Consider the case $c' < 0$: If $\cos(\alpha) < \beta$, then we have $\xi'_{2n,1} < \xi_{2n,1} < \xi_{2n,2}$. If $\cos(\alpha) > \beta$, then we have $\xi_{2n,1} < \xi'_{2n,1} < \xi_{2n,2}$.*

3. *If the two lines ℓ and ℓ' have the intersection points $(\gamma, 0), (\gamma', 0)$ with the x-axis in $\mathcal{I}_{2n}$, respectively, let $\xi_{2n,1}$ be the intersection point with $\cos(x)$ for the line ℓ and $\xi'_{2n,1}$ be that for the line ℓ'.*

   (a) *If there is no intersection point of ℓ and ℓ' with α in $\mathcal{I}_{2n}$ or the intersection point (α, β) has α in $\mathcal{I}_n$ and $\cos(\alpha) < \beta$, then the following cases occur:*

i. *If* $c, c' > 0$ *then* $\xi_{2n,1} < \xi'_{2n,1}$ *for* $c(2n\pi) + d > c'(2n\pi) + d'$ *and* $\xi'_{2n,1} < \xi_{2n,1}$ *for* $c(2n\pi) + d < c'(2n\pi) + d'$.

ii. *If* $c, c' < 0$ *then* $\xi'_{2n,1} < \xi_{2n,1}$ *for* $c(2n\pi) + d > c'(2n\pi) + d'$ *and* $\xi_{2n,1} < \xi'_{2n,1}$ *for* $c(2n\pi) + d < c'(2n\pi) + d'$.

(b) *If the intersection point* ℓ *and* ℓ' *has* α *in* $\mathcal{I}_{2n}$ *and* $\cos(\alpha) > \beta$, *then the following cases occur:*

i. *If* $c, c' > 0$ *then* $\xi'_{2n,1} < \xi_{2n,1}$ *for* $c(-\frac{\pi}{2} + 2n\pi) + d > c'(-\frac{\pi}{2} + 2n\pi) + d'$ *and* $\xi_{2n,1} < \xi'_{2n,1}$ *for* $c(-\frac{\pi}{2} + 2n\pi) + d < c'(-\frac{\pi}{2} + 2n\pi) + d'$.

ii. *If* $c, c' < 0$ *then* $\xi_{2n,1} < \xi'_{2n,1}$ *for* $c(-\frac{\pi}{2} + 2(n+1)\pi) + d > c'(-\frac{\pi}{2} + 2(n+1)\pi) + d'$ *and* $\xi'_{2n,1} < \xi_{2n,1}$ *for* $c(-\frac{\pi}{2} + 2(n+1)\pi) + d < c'(-\frac{\pi}{2} + 2(n+1)\pi) + d'$.

iii. *If* $cc' < 0$ *then* $\xi_{2n,1} < \xi'_{2n,1}$ *for* $c < 0, c' > 0$ *and* $\xi'_{2n,1} < \xi_{2n,1}$ *for* $c > 0, c' < 0$.

The interval $\mathcal{I}_{2n+1}$ *for an integer* n *is treated in analogous way to the interval* $\mathcal{I}_{2n}$.

Lemma 5 *Let* $\xi_{2n,1} < \xi_{2n,2}$ *be the potential solutions of the equation* $cx + d = \cos(x)$ *in* $\mathcal{I}_{2n} \cap D$. *Let the potential intersection point of two lines* ℓ *and* ℓ' *be* (α, β). *We only consider the cases where both two lines have at least one solution with* $\cos(x)$ *in* $\mathcal{I}_{2n} \cap D$. *Then the order relation between* $c'\xi_{2n,i} + d'$ *and* $\cos(\xi_{2n,i})$ $\mathcal{I}_{2n}$ *is decided as follows:*

1. *If the two lines* ℓ *and* ℓ' *are both positive on* $\mathcal{I}_{2n}$ *and have intersection points with* $\cos(x)$ *in* $\mathcal{I}_{2n}$, *then let* $\xi_{2n,1}, \xi_{2n,2}$ *be the intersection points with* $\cos(x)$ *for the line* ℓ.

(a) *If there is no intersection point of* ℓ *and* ℓ' *with* α *in* $\mathcal{I}_{2n}$ *or the intersection point* (α, β) *has* α *in* $\mathcal{I}_{2n}$ *and* $\cos(\alpha) < \beta$, *then we have* $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$, $c'\xi_{2n,2} + d' < \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' < \cos(\xi_{2n,2})$, *for* $c(2n\pi) + d > c'(2n\pi) + d'$ *and* $c'\xi_{2n,1} + d' > \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' > \cos(\xi_{2n,1})$, $c'\xi_{2n,2} + d' > \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' > \cos(\xi_{2n,2})$ *for* $c(2n\pi) + d < c'(2n\pi) + d'$.

(b) *If the intersection point of* ℓ *and* ℓ' *has* α *in* $\mathcal{I}_{2n}$ *and* $\cos(\alpha) > \beta$, *then we have* $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$, $c'\xi_{2n,2} + d' > \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' > \cos(\xi_{2n,2})$, *for* $c < c'$, $c'\xi_{2n,1} + d' > \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' > \cos(\xi_{2n,1})$, $c'\xi_{2n,2} + d' < \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' < \cos(\xi_{2n,2})$ *for* $c > c'$.

2. *If the line* ℓ *is positive and has intersection points with* $\cos(x)$ *in* $\mathcal{I}_{2n}$ *and* ℓ' *has an intersection point* $(\gamma, 0)$ *with the* x*-axis in* $\mathcal{I}_{2n}$, *let* $\xi_{2n,1}, \xi_{2n,2}$ *be the intersection points with* $\cos(x)$ *for the line* ℓ. *(The case where the line* ℓ' *is positive on* $\mathcal{I}_{2n}$ *and* ℓ *has an intersection point* $(\gamma, 0)$ *with the* x*-axis in* $\mathcal{I}_{2n}$ *is similar to this case.)*

(a) *Consider the case* $c' > 0$*: If* $\cos(\alpha) < \beta$, *then we have* $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ $c'\xi_{2n,2} + d' < \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' < \cos(\xi_{2n,2})$. *If* $\cos(\alpha) > \beta$, *then we have* $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ $c'\xi_{2n,2} + d' > \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' > \cos(\xi_{2n,2})$.

(b) *Consider the case* $c' < 0$*: If* $\cos(\alpha) < \beta$, *then we have* $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ $c'\xi_{2n,2} + d' < \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' < \cos(\xi_{2n,2})$. *If* $\cos(\alpha) > \beta$, *then we have* $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ $c'\xi_{2n,2} + d' > \cos(\xi_{2n,2})$, $c'(\xi_{2n,2} + \epsilon) + d' > \cos(\xi_{2n,2})$.

3. If the two lines ℓ and ℓ' have the intersection points $(\gamma, 0), (\gamma', 0)$ with x-axis in $\mathcal{I}_{2n}$ respectively, let $\xi_{2n,1}$ be the intersection point with $\cos(x)$ for the line ℓ.

(a) If there is no intersection point of ℓ and ℓ' with α in $\mathcal{I}_{2n}$ or the intersection point (α, β) has α in $\mathcal{I}_{2n}$ and $\cos(\alpha) < \beta$, then the following cases occur:

i. If $c, c' > 0$ then $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ for $c(2n\pi) + d > c'(2n\pi) + d'$ and $c'\xi_{2n,1} + d' > \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' > \cos(\xi_{2n,1})$ for $c(2n\pi) + d < c'(2n\pi) + d'$.

ii. If $c, c' < 0$ then $c'\xi_{2n,1} + d' > \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' > \cos(\xi_{2n,1})$ for $c(2n\pi) + d > c'(2n\pi) + d'$ and $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ for $c(2n\pi) + d < c'(2n\pi) + d'$.

(b) If the intersection point of ℓ and ℓ' has α in $\mathcal{I}_{2n}$ and $\cos(\alpha) > \beta$, then the following cases occur:

i. If $c, c' > 0$ then $c'\xi_{2n,1} + d' > \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' > \cos(\xi_{2n,1})$ for $c(-\frac{\pi}{2} + 2n\pi) + d > c'(\frac{\pi}{2} + 2n\pi) + d'$ and $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ for $c(-\frac{\pi}{2} + 2n\pi) + d < c'(\frac{\pi}{2} + 2n\pi) + d'$.

ii. If $c, c' < 0$ then $c'\xi_{2n,1} + d' > \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' > \cos(\xi_{2n,1})$ for $c(-\frac{\pi}{2} + 2(n+1)\pi) + d > c'(\frac{\pi}{2} + 2(n+1)\pi) + d'$ and $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$ for $c(-\frac{\pi}{2} + 2(n+1)\pi) + d < c'(\frac{\pi}{2} + 2(n+1)\pi) + d'$.

iii. If $cc' < 0$ then $c'\xi_{2n,1} + d' < \cos(\xi_{2n,1})$, $c'(\xi_{2n,1} + \epsilon) + d' < \cos(\xi_{2n,1})$.

The interval $\mathcal{I}_{2n+1}$ for integers n is treated in analogous way to the interval $\mathcal{I}_{2n}$.

Proof of lemma 4 and 5 These facts are an immediate consequence of lemma 2. Draw a diagram for each case. ∎

Armed with the two lemmas we can now determine the exact number of non-empty connected components of the set M of all real values x, where $\psi'(x)$ holds. Since all coefficients occurring in ψ' are real algebraic numbers in F and $\cos(x)$ is strongly transcendental we can decide equations and order inequalities between the elements of F and between the values of $\cos(x)$ for $x \in F$. Then all the case distinctions made in the two proceeding lemmas can be evaluated algorithmically. So we can determine algorithmically the exact number of real solutions of every equation arising from an atomic subformula of ψ'. Next we can order all these solutions by the first lemma. By the second lemma we can evaluate the truth value of every atomic subformula of ψ' and hence of ψ' at all these points in $[-s, s]$, and closely to the right of all these points. From these evaluations we obtain immediately the number of non-empty connected components of the set $M_{\psi'}$, and moreover the type of intervals that these components constitute in increasing order.

5 The mixed real-integer case

In this section we consider a considerable extension of linear-exponential problems given by an extension of the concept of term used so far. We admit as new operation symbol in our extension language the integer-part operation $[\_]$ (compare [27]). Similar as in the language L'' considered in [27], extended terms are now expressions obtained from terms by means of addition, scalar multiplication by rational constants and the integer-part operation. Notice that these operations may be nested in an arbitrary way. Atomic formulas in the extended sense are then of the form $t = 0,\ t > 0,\ t \geq 0$, for extended terms t. Formulas, closed formulas and normal formulas in the extended sense are obtained from atomic formulas as before with the additional restriction that the quantifier over the mixed variable x should range now only over a bounded interval $[-S, S]$. By means of the integer-part operation we are now able to code also quantifiers ranging over the integers. Indeed a quantifier $(\exists x \in \mathbb{Z})(\varphi)$ can be equivalently expressed by the extended formula $\exists x(x = [x] \ \wedge\ \varphi)$. So the problems that can be expressed by normal formulas in the extended sense are now mixed real-integer linear-trigonometric problems.

We are now going to describe a *decision procedure for normal formulas in the extended sense*: By an argument similar to the one in section 2 we can use the mixed real-integer linear quantifier elimination of [27] to reduce the decision problem for normal formulas to the case of normal formulas with just one existential quantifier

$$\exists x(-S \leq x \leq S \ \wedge\ \psi(x)) \tag{3}$$

where $\psi(x)$ is quantifier-free. From the given bound S we can compute an integer bound T for the absolute values of all extended terms and subterms $t(x)$ occurring in $\psi(x)$ for $-S \leq x \leq S$. Next, we successively eliminate all occurrences of the integer-part operation from (3) : Let $[t]$ be an innermost occurrence of integer-part operation in $\psi(x)$. We write $\psi(x)$ as $\tau([t])$. Then (3) is equivalent to $\bigvee_{i=-T}^{T} \exists x(-S \leq x \leq S \ \wedge\ i1 \leq t(x) < (i+1)1 \ \wedge\ \tau(i1))$ This decreases the number of occurrences of the integer part operation by one. By iteration of the procedure all occurrences of the integer part operation can be eliminated from the given formula. At this point the resulting formula can be decided as in section 4.

6 The Tangent case

So far the trigonometric function in normal formulas was essentially the sine or the cosine function. What happens if we replace this function by the tangent function?

The obvious changes are the following: $tan(x)$ is defined only for $x \neq \frac{\pi}{2} + n\pi$ for integer n, $tan(x)$ has period π, and $tan(x)$ is unbounded. We extend $tan(x)$ to a total function on $\mathbb{R}$ by setting $tan(\frac{\pi}{2} + n\pi) = 0$. So the new tasks are the following:

1. Compute the real zeros of the functions $f_{c,d}(x) := tan(x) + cx + d$ where $c, d \in F$ and determine the sign of $f_{c,d}(x)$ infinitesimally to the right of these zeros.

2. Eliminate the occurrence of the integer part operation from an univariate extended normal formulas.

For the first task we note to begin with that in each basic interval $\mathcal{I}_n$ the function $f_{c,d}(x)$ has exactly one zero, with the possible exception of the interval $\mathcal{I}_m$ containing the zero $-d/c$ of the line $cx + d$ in case $c \neq 0$. In the exceptional interval $\mathcal{I}_m$ the function has exactly one zero if $c > -1$, or $c = -1$ and $d = 0$, or $c < -1$ and the values of $tan(\pm acos(\frac{1}{\sqrt{-c}} + m\pi))$ have the same sign, and exactly three zeros otherwise. This follows from the following facts: $f'_{c,d}(x) = (\cos(x))^{-2} + c > 0$ for $c > -1$. For $c < -1$, $f'_{c,d}(x)$ has exactly two real zeros $\pm acos(\frac{1}{\sqrt{-c}} + n\pi)$ in each of the intervals $\mathcal{I}_n$. By the strong transcendence of $tan(x)$, $f_{c,d}(x)$

has a multiple zero iff $c = -1$ and $d = 0$. Using these facts we can solve the first task similarly as in section 4. As a consequence we can decide normal formulas with $trig(x) = tan(x)$.

We do not know how to solve the second task, since $tan(x)$ is unbounded. Instead we modify the tangent function by truncating its values at some fixed positive integer k as follows: Put $tan_k(x) = tan(x)$; if $|tan(x)| \leq k$, $tan_k(x) = -k$ for $tan(x) < -k$, $tan_k(x) = k$ for $tan(x) > k$. Then the second task can be solved similarly as in section 6. So for each fixed k we obtain a decision procedure for extended normal formulas with $trig(x) = tan_k(x)$.

Again these decision procedures can be modified as indicated before for quantifiers ranging over a subfield of the field of real algebraic numbers.

7 Deciding linear-trigonometric problems over subfields

So far we have considered the decision problem for (extended) normal formulas, where all quantifiers range over the field of real numbers. It is well-known that the truth value of linear formulas does not change when quantifiers are restricted to an arbitrary subfield of the reals [10]. By [27] the same applies to mixed real-integer linear formulas. For linear-trigonometric problems there is, however, a big difference between validity in $\mathbb{R}$ and in some subfield F of the real algebraic numbers: Consider for example the normal formula $\exists x(\cos(x) = x)$. In the reals this formula is true, since the equation $\cos(x) = x$ has a unique real solution ξ in the interval $(0, \pi/2)$. By the strong trancendency of $\cos(x)$ this solution is transcendental, and hence the formula is false in F.

So over F the elimination of linear variables from (extended) normal formulas remains valid, while the decision of univariate normal formulas involving a trigonometric function in section 4 may become incorrect.

In the following we sketch a modified decision algorithm that will be valid for arbitrary subfields F of the field of real algebraic numbers. As remarked above it suffices to modify the decision of univariate normal formulas $\exists x(\psi(x)$. Since the decision works by substituting finitely many test points, it suffices to delete all test point that are not in F. These are exactly the test points of the form $\xi_{n,i}$ arising as zeros of functions $f(x) = trig(x) + cx + d$ different from zero. Notice, however, that we keep all the test point of the form $\xi_{n,i} + \epsilon$, since they could be replaced by an element of F.

Thus in our example $\exists x(\cos(x) = x)$ the decision in the real line uses as test points $-1 - 2\pi$, the solution ξ of the equation $\cos(x) = x$, and the point $\xi + \epsilon$ for a positive infinitesimal ϵ, whereas in F the test point ξ is dropped. Hence the result of the decision is "true" in the reals and "false" in F.

Remark 6 (Complexity of the decision problem) *The complexity of the decision problem for (extended) normal formulas has not been determined yet exactly. We expect our decision method to run in doubly exponential time for non-extended normal formulas and in exponential time for existential non-extended normal formulas.*

8 Conclusions

We have shown that the decision problem for several classes of formulas involving real and integer variables, linear polynomials and a trigonometric function is decidable. This is in contrast to the undecidability of the full elementary theory of the reals with a specific trigonometric function. The formulas may involve one mixed linear-trigonometric variable and several linear variables. The decision method uses linear quantifier elimination and a symbolic test point method. It is quite explicit and implementable. Variants of it work over subfields of the real algebraic numbers. In its approach it is similar to the decision method for linear-exponential problems announced in [26]. We expect that the approach can be generalized to formulas involving larger classes of elementary functions.

Our decision method can be construed as a modified quantifier elimination method, when the coefficients of terms are taken as parameters. Notice, however, that in this case the output formula will no longer be normal, but will include more complicated terms.

References

[1] BAREISS, E. Sylvester's identity and multistep integer preserving Gaussian elimination. *Mathematics of Computation 22* (1968), 565–578.

[2] COLLINS, G. E. *Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition, LNCS 32.* Springer Verlag, 1975.

[3] DOLZMANN, A., STURM, T., AND WEISPFENNING, V. Real quantifier elimination in practice. Technical Report MIP-9720, FMI, Universität Passau, D-94030 Passau, Germany, Dec. 1997.

[4] GUTIERREZ, J., AND RECIO, T. Advances on the simplification of sine-cosine equations. *J. Symbolic Computation 26*, 1 (1998), 31–71.

[5] HONG, H., AND SCHICHO, J. Algorithms for trigonometric curves (simplification, implicitization, parameterization). *J. Symbolic Computation 26*, 3 (1998), 279–300.

[6] JIRSTRAND, M. Nonlinear control system design by quantifier elimination. *J. Symbolic Computation 24*, 2 (Aug. 1997), 137–152.

[7] KOVÁCS, P. Computer algebra in robot-kinematics. In *Proceeding of the workshop:"Computer Algebra in Science and Engineering", Bielefeld, Aug'94* (Singapore, 1994), World Scientific, pp. 303–316.

[8] LAFFERRIERE, G., PAPPAS, G. J., AND YOVINE, S. Decidable hybrid systems. Tech. rep., VERIMAG, Univ. of Grenoble, 1998.

[9] LISKA, R., AND STEINBERG, S. Applying quantifier elimination to stability analysis of difference schemes. *The Computer Journal 36*, 5 (Oct. 1993), 497–503.

[10] LOOS, R., AND WEISPFENNING, V. Applying linear quantifier elimination. *The Computer Journal 36*, 5 (1993), 450–462. Special issue on computational quantifier elimination.

[11] MACINTYRE, A., AND WILKIE, A. On the decidability of the real exponential field. In *Kreiseliana: About and around Georg Kreisel.* A.K. Peters, 1996, pp. 441–467.

[12] PAU, P., AND SCHICHO, J. Quantifier elimination for trigonometric polynomials by cylindrical trigonometric decomposition. *RISC-Linz Technical Report (submitted to Journal of Symbolic Computation* (1999).

[13] RICHARDSON, D. Some undecidable problems involving elementary functions of a real variable. *J. Symbolic Logic 33* (1968), 514–520.

[14] RICHARDSON, D. Towards computing non algebraic cylindrical decompositions. In *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation* (Bonn, Germany, July 1991), S. M. Watt, Ed., pp. 247–255.

[15] RICHARDSON, D. Wu's method and the Khovanskii finiteness theorem. *Journal of Symbolic Computation 12* (1991), 127–141.

[16] RICHARDSON, D. The elementary constant problem. In *Proceedings of the 1992 International Symposium on Symbolic and Algebraic Computation* (Berkeley, California, July 1992), P. S. Wang, Ed., pp. 108–116.

[17] RICHARDSON, D. An ISSAC'95 tutorial: Algorithmic methods for finding real solution of systems involving exponential and other elementary functions. Preprint, July 1995.

[18] SHIDLOVSKII, A. B. *Transcendental Numbers.* Walter de Gruyter, Berlin, New York, 1989.

[19] TARSKI, A. A decision method for elementary algebra and geometry. Tech. rep., University of California, 1948. Second edn., rev. 1951.

[20] WALDSCHMIDT, M. Simultaneous approximation of numbers connected with the exponential function. *J. Australian Math. Society (Series A) 25* (1978), 466–478.

[21] WEISPFENNING, V. The complexity of linear problems in fields. *Journal of Symbolic Computation 5*, 1–2 (Feb.–Apr. 1988), 3–27.

[22] WEISPFENNING, V. The complexity of almost linear Diophantine problems. *Journal of Symbolic Computation 10*, 5 (Nov. 1990), 395–403.

[23] WEISPFENNING, V. Existential equivalence of ordered abelian groups with parameters. *Archive for Mathematical Logic 29* (1990), 237–248.

[24] WEISPFENNING, V. Parametric linear and quadratic optimization by elimination. Technical Report MIP-9404, FMI, Universität Passau, D-94030 Passau, Germany, Apr. 1994.

[25] WEISPFENNING, V. Simulation and optimization by quantifier elimination. *Journal of Symbolic Computation 24*, 2 (Aug. 1997), 189–208. Special issue on applications of quantifier elimination.

[26] WEISPFENNING, V. Deciding linear-exponential problems. Poster presentation at ISSAC'99, Vancouver, July 1999.

[27] WEISPFENNING, V. Mixed real-integer linear quantifier elimination. In *ISSAC'99* (1999), S. Dooley, Ed., ACM-Press, pp. 129–136.