数理解析研究所講究録 1351 巻 2004 年 189-200

Numeration systems, fractals and stochastic processes

大阪市立大学・理学研究科 釜江哲朗 (Teturo Kamae)
Faculty of Science
Osaka City University

1 Numeration systems

By a *numeration system*, we mean a compact metrizable space Θ with at least 2 elements as follows:

1. There exists a nontrivial closed multiplicative subgroup G of $\mathbb{R}_+$ such that $(\mathbb{R}, G)$ *acts numerically* to Θ in the sense that there exist continuous mappings $\chi_1 : \Theta \times \mathbb{R} \to \Theta$ and $\chi_2 : \Theta \times G \to \Theta$, where we denote $\omega + t := \chi_1(\omega, t)$, $\lambda\omega := \chi_2(\omega, \lambda)$, satisfying that

$$\omega + 0 = \omega, \ (\omega + t) + s = \omega + (t + s)$$
$$1\omega = \omega, \ \eta(\lambda\omega) = (\eta\lambda)\omega$$
$$\lambda(\omega + t) = \lambda\omega + \lambda t$$

for any $\omega \in \Theta$, $t, s \in \mathbb{R}$ and $\lambda, \eta \in G$.

2. The additive action of $\mathbb{R}$ to Θ is minimal and uniquely ergodic having 0-topological entropy.

3. The multiplicative action of $\lambda(\in G)$ to Θ has $|\log \lambda|$-topological entropy. Moreover, the unique invariant probability measure under the additive action is invariant under the G-action and is the unique probability measure attaining the topological entropy of the multiplication by $\lambda \neq 1$.

Note that if Θ is a numeration system, then Θ is a connected space with the continuum cardinality. Also, note that the multiplicative group G as above is either $\mathbb{R}_+$ or $\{\lambda^n;\ n \in \mathbb{Z}\}$ for some $\lambda > 1$. Moreover, the additive action is faithful, that is $\omega + t = \omega$ implies $t = 0$ for any $\omega \in \Theta$ and $t \in \mathbb{R}$. This is because if there exist $\omega_1 \in \Theta$ and $t_1 \neq 0$ such that $\omega_1 + t_1 = \omega_1$. Let $\lambda_n \in G$ tends to 0 as $n \to \infty$. Take a limit point ω_∞ of $\lambda_n \omega$. Then, ω_∞ becomes a fix point with respect to the additive action by the distributive law and the continuity of the additive action, which contrdicts with the minimality of the additive action together with $\sharp\Theta \geq 2$.

We construct Θ as above as a colored tiling space corresponding to a weighted substitution. Then, we study α-homogeneous cocycles on it with respect to the addition. They are interesting from the point of views of fractal functions or sets as well as self-similar processes. We obtain the zeta-functions of Θ with respect to the multiplication.

Let Σ be a nonempty finite set. An element in Σ is called a *color*. A rectangle $(a, b] \times [c, d)$ in $\mathbb{R}^2$ is called an *admissible tile* if $d - c = e^{-b}$ is satisfied. A *colored tiling* ω is a mapping from $dom(\omega)$ to Σ, where $dom(\omega)$ consists of admissible tiles which are disjoint each other and the union of which is $\mathbb{R}^2$. For $S \in dom(\omega)$, $\omega(S)$ is considered as the color painted on the admissible tile S. In another word, a colored tiling is a partition of $\mathbb{R}^2$ by admissible tiles with colors in Σ.

A topology is introduced on $\Omega(\Sigma)$ so that a net $\{\omega_n\}_{n \in I} \subset \Omega(\Sigma)$ converges to $\omega \in \Omega(\Sigma)$ if for every $S \in dom(\omega)$, there exist $S_n \in dom(\omega_n)$ $(n \in I)$ such that

$$\omega(S) = \omega_n(S_n) \text{ for any } n \in I \text{ and } \lim_{n \to \infty} \rho(S, S_n) = 0,$$

where ρ is the Hausdorff metric.

For an admissible tile $S := (a, b] \times [c, d)$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we denote

$$\begin{aligned} S + t &:= (a, b] \times [c - t, d - t) \\ \lambda S &:= (a - \log \lambda, b - \log \lambda] \times [\lambda c, \lambda d). \end{aligned}$$

Note that they are also admissible tiles.

For $\omega \in \Omega(\Sigma)$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we define $\omega + t \in \Omega(\Sigma)$ and

$\lambda\omega \in \Omega(\Sigma)$ as follows:

$$\begin{aligned} dom(\omega + t) &:= \{S + t;\ S \in dom(\omega)\} \\ (\omega + t)(S + t) &:= \omega(S) \text{ for any } S \in dom(\omega) \\ dom(\lambda\omega) &:= \{\lambda S;\ S \in dom(\omega)\} \\ (\lambda\omega)(\lambda S) &:= \omega(S) \text{ for any } S \in dom(\omega). \end{aligned}$$

Thus, $(\mathbb{R}, \mathbb{R}_+)$ acts numerically to $\Omega(\Sigma)$. We construct compact metrizable subspaces of $\Omega(\Sigma)$ corresponding to weighted substitutions which are numeration systems.

2 Weighted substitutions

A *weighted substitution* (φ, η) on Σ is a mapping $\Sigma \to \Sigma^+ \times (0,1)^+$, where $\Sigma^+ = \bigcup_{\ell=1}^{\infty} \Sigma^\ell$, such that $|\varphi(\sigma)| = |\eta(\sigma)|$ and $\sum_{i<|\eta(\sigma)|} \eta(\sigma)_i = 1$ for any $\sigma \in \Sigma$, where $|\quad|$ implies the length of the word. Note that φ is a substitution on Σ in the usual sense. We define $\eta^n : \Sigma \to (0,1)^+$ $(n = 2, 3, ...)$ inductively by

$$\eta^n(\sigma)_k = \eta(\sigma)_i \eta^{n-1}(\varphi(\sigma)_i)_j$$

for any $\sigma \in \Sigma$ and i, j, k with

$$0 \le i < |\varphi(\sigma)|,\ 0 \le j < |\varphi^{n-1}(\varphi(\sigma)_i)|,\ k = \sum_{h<i} |\varphi^{n-1}(\varphi(\sigma)_h)| + j.$$

Then, (φ^n, η^n) is also a weighted substitution for $n = 2, 3, \cdots$.

A substitution φ on Σ is called *mixing* if there exists a positive integer n such that for any $\sigma, \sigma' \in \Sigma$, $\varphi^n(\sigma)_i = \sigma'$ holds for some i with $0 \le i < |\varphi^n(\sigma)|$, which we always assume.

We define the *base set* $B(\varphi, \eta)$ as the closed, multiplicative subgroup of $\mathbb{R}_+$ generated by the set

$$\left\{ \begin{array}{ll} \eta^n(\sigma)_i\ ; & \sigma \in \Sigma,\ n = 0, 1, \cdots \text{ and } 0 \le i < |\varphi^n(\sigma)| \\ & \qquad\qquad \text{such that } \varphi^n(\sigma)_i = \sigma \end{array} \right\}.$$

Let $G := B(\varphi, \eta)$. Then, there exists a function $g : \Sigma \to \mathbb{R}_+$ such that $g(\varphi(\sigma)_i)G = g(\sigma)\eta(\sigma)_i G$ for any $\sigma \in \Sigma$ and $0 \le i < |\varphi(\sigma)|$.

Note that if $G = \mathbb{R}_+$, then we can take $g \equiv 1$. In another case, we can define g by $g(\sigma_0) = 1$ and $g(\sigma) := \eta^n(\sigma_0)_i$ for some n and i such that $\varphi^n(\sigma_0)_i = \sigma$, where σ_0 is any fixed element in Σ.

Let (φ, η) be a weighted substitution. Let $G = B(\varphi, \eta)$. Let g satisfy the above equality. Let $\Omega(\varphi, \eta, g)'$ be the set of all elements ω in $\Omega(\Sigma)$ such that

(I) if $(a, b] \times [c, d) \in dom(\omega)$, then $e^{-b} \in g(\omega((a, b] \times [c, d)))G$, and

(II) if $(a, b] \times [c, d) \in dom(\omega)$ and $\omega((a, b] \times [c, d)) = \sigma$, then for $i = 0, 1, \cdots, |\varphi(\sigma)| - 1$, $S^i \in dom(\omega)$ and $\omega(S^i) = \varphi(\sigma)_i$, where

$$S^i := (b, b - \log \eta(\sigma)_i] \times [c + (d-c)\sum_{j=0}^{i-1} \eta(\sigma)_j, c + (d-c)\sum_{j=0}^{i} \eta(\sigma)_j).$$

A horizontal line $\gamma := (-\infty, \infty) \times \{y\}$ is called a *separating line* of $\omega \in \Omega(\varphi, \eta, g)'$ if for any $S \in dom(\omega)$, $S^\circ \cap \gamma = \emptyset$, where S° denotes the set of inner points of S. Let $\Omega(\varphi, \eta, g)''$ be the set of all $\omega \in \Omega(\varphi, \eta, g)'$ which do not have a separating line and $\Omega(\varphi, \eta, g)$ be the closure of $\Omega(\varphi, \eta, g)''$. Then, $(\mathbb{R}, G)$ acts to $\Omega(\varphi, \eta, g)$ numerically. We usually denote $\Omega(\varphi, \eta, 1)$ simply by $\Omega(\varphi, \eta)$.

Theorem 1. *The space $\Omega(\varphi, \eta, g)$ is a numeration system with $G = B(\varphi, \eta)$.*

Example 1. Let $\Sigma = \{+, -\}$ and (φ, η) be a weighted substitution such that

$$\begin{aligned} + &\to (+, 4/9)(-, 1/9)(+, 4/9) \\ - &\to (-, 4/9)(+, 1/9)(-, 4/9), \end{aligned}$$

where we express a weighted substitution (φ, η) by

$$\sigma \to (\varphi(\sigma)_0, \eta(\sigma)_0)(\varphi(\sigma)_1, \eta(\sigma)_1) \cdots \ (\sigma \in \Sigma).$$

Then, $4/9 \in B(\varphi, \eta)$ since $\varphi(+)_0 = +$ and $\eta(+)_0 = 4/9$. Moreover, $1/81 \in B(\varphi, \eta)$ since $\varphi^2(+)_4 = +$ and $\eta^2(+)_4 = 1/81$. Since $4/9$ and $1/81$ do not have a common multiplicative base, we have

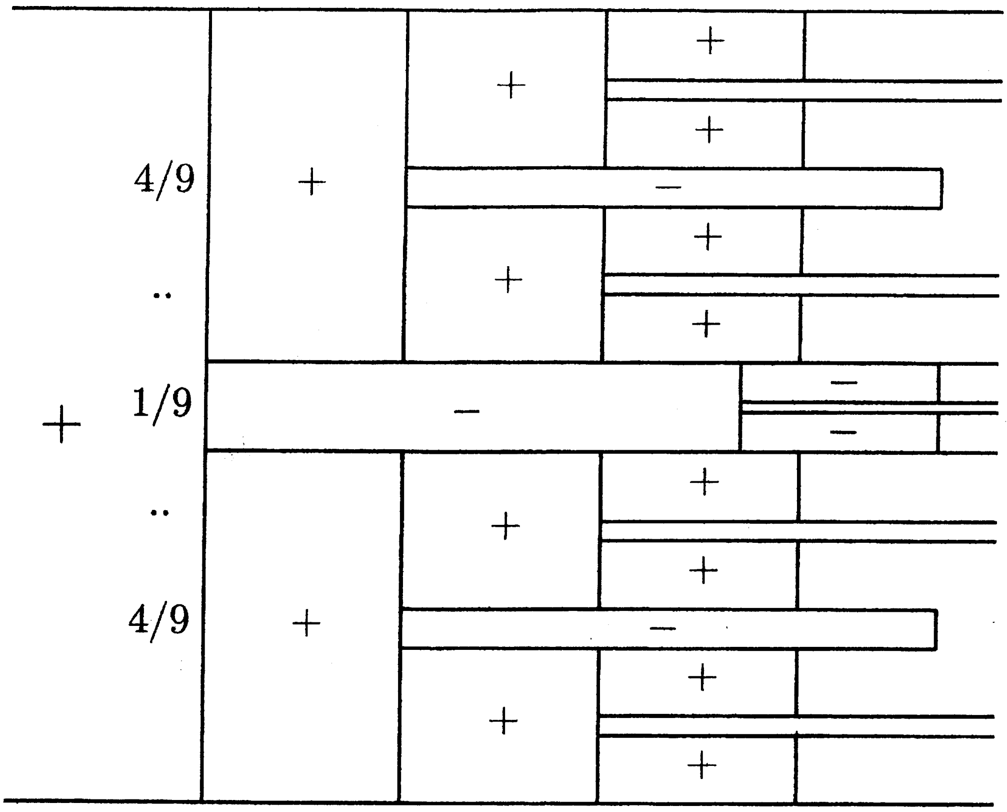

Figure 1: a colored tiling in in Example 1

$B(\varphi, \eta) = \mathbb{R}_+$. Therefore with $g \equiv 1$, we can define a numeration system $\Omega(\varphi, \eta)$. A colored tiling belonging to this space is shown in Figure 1. The vertical size of tiles are proportional to the weights and the horizontal sizes are the minus of the logarithm of the weights. This example is discussed later.

3 ζ-function

Let $\Omega := \Omega(\varphi, \eta, g)$. For $\alpha \in \mathbb{C}$, we define the associated matrices on the suffix set $\Sigma \times \Sigma$ as follows:

$$\begin{aligned} M_\alpha = M_\alpha(\varphi, \eta) &:= \left(\sum_{i;\varphi(\sigma)_i = \sigma'} \eta(\sigma)_i^\alpha \right)_{\sigma, \sigma' \in \Sigma} \\ M_{\alpha,+} = M_{\alpha,+}(\varphi, \eta, g) &:= \left(1_{\varphi(\sigma)_0 = \sigma'} \, \eta(\sigma)_0^\alpha \right)_{\sigma, \sigma' \in \Sigma} \\ M_{\alpha,-} = M_{\alpha,-}(\varphi, \eta, g) &:= \left(1_{\varphi(\sigma)_{|\varphi(\sigma)|-1} = \sigma'} \, \eta(\sigma)_{|\varphi(\sigma)|-1}^\alpha \right)_{\sigma, \sigma' \in \Sigma} . \end{aligned} \tag{1}$$

Let $CO(\Omega)$ be the set of *closed orbits* of Ω with respect to the action of G. That is, $CO(\Omega)$ is the family of subsets ξ of Ω such that $\xi = G\omega$ for some $\omega \in \Omega$ with $\lambda\omega = \omega$ for some $\lambda \in G$ with $\lambda > 1$. We call λ as above a *multiplicative cycle* of ξ. The minimum multiplicative cycle of ξ is denoted by $cy(\xi)$.

Define the ζ-function of G-action to Ω by

$$\zeta_\Omega(\alpha) := \prod_{\xi \in CO(\Omega)} (1 - cy(\xi)^{-\alpha})^{-1}, \tag{2}$$

where the infinite product converges for any $\alpha \in \mathbb{C}$ with $\mathcal{R}(\alpha) > 1$. It is extended to the whole complex plane by the analytic extension.

Theorem 2. *We have*

$$\zeta_\Omega(\alpha) = \frac{\det(I - M_{\alpha,+}) \det(I - M_{\alpha,-})}{\det(I - M_\alpha)} \zeta_{SL_0(\Omega)}(\alpha),$$

where

$$\zeta_{SL_0(\Omega)}(\alpha) := \prod_{\xi \in CO_0(\Omega)} (1 - cy(\xi)^{-\alpha})^{-1}$$

is a finite product with respect to $\xi \in CO(\Omega)$ which has a separating line.

4 β-expansion system

Let β be an algebraic integer with $\beta > 1$ such that 1 has the following periodic β-expansion

$$\begin{aligned} &1 = (b_1 0^{i_1-1} b_2 0^{i_2-1} \cdots b_k 0^{i_k-1})^\infty \\ &b_1, b_2, \cdots, b_k \in \{1, 2, \cdots, \lfloor \beta \rfloor\} \\ &i_1, i_2, \cdots, i_k \in \{1, 2, \cdots\}, \end{aligned}$$

where $(\quad)^\infty$ implies the infinite time repetition of $(\quad)$. Let $n := i_1 + i_2 + \cdots + i_k \geq 1$ and assume that n is the minimum period of the above sequence. Since the above sequence is the expansion of 1, we have the solution of the following equation in $a_1, a_2, \cdots, a_{k+1}$ with $a_1 = a_{k+1} = 1$ and $0 < a_j < 1$ $(j = 2, \cdots, k)$:

$$a_j = b_j \beta^{-1} + a_{j+1} \beta^{-i_j} \ (j = 1, 2, \cdots, k).$$

Let $\Sigma := \{1, 2, \cdots, k\}$ and define a weighted substitution (φ, η) by

$$\begin{aligned} j &\to (1,\ (1/a_j)\beta^{-1})^{b_j}\ (j+1,\ (a_{j+1}/a_j)\beta^{-i_j}) \\ &\qquad (j = 1, 2, \cdots, k-1) \\ k &\to (1,\ (1/a_k)\beta^{-1})^{b_k}\ (1,\ (a_{k+1}/a_k)\beta^{-i_k}) \quad . \end{aligned}$$

Then, φ is mixing and $B(\varphi, \eta) = \{\beta^n;\ n \in \mathbb{Z}\}$. Define $g : \Sigma \to \mathbb{R}_+$ by $g(j) := a_j$. Then, $\Omega(\varphi, \eta, g)$ is a numeration system by Theorem 1. We denote $\Theta(\beta) := \Omega(\varphi, \eta, g)$ and $\Theta(\beta)$ is called the *β-expansion system.*

Theorem 3. *We have*

$$\zeta_{\Theta(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \sum_{j=1}^{k} b_j \beta^{-(i_1 + \cdots + i_{j-1} + 1)\alpha} - \beta^{-n}}.$$

Example 2. Let us consider the β-expansion system with $\beta > 1$ such that $\beta^3 - \beta^2 - \beta - 1 = 0$. Then the expansion of 1 is $(110)^\infty$ and the corresponding weighted substitution is

$$\begin{aligned} 1 &\to (1,\ \beta^{-1})(2,\ \beta^{-2} + \beta^{-3}) \\ 2 &\to (1,\ \frac{\beta^{-2}}{\beta^{-2} + \beta^{-3}})(1,\ \frac{\beta^{-3}}{\beta^{-2} + \beta^{-3}}) \end{aligned}$$

By Theorem 3, we have

$$\zeta_{\Theta(\beta)}(\alpha) = \frac{1-\beta^{-\alpha}}{1-\beta^{-\alpha}-\beta^{-2\alpha}-\beta^{-3\alpha}}.$$

We will discuss this example in the next section.

5 homogeneous cocycles and fractals

Let $\Omega := \Omega(\varphi, \eta, g)$. A continuous function $F : \Omega \times \mathbb{R} \to \mathbb{C}$ is called a *cocycle* on Ω if

$$F(\omega, t+s) = F(\omega, t) + F(\omega + t, s) \tag{3}$$

holds for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. A cocycle F on Ω is called α-*homogeneous* if

$$F(\lambda\omega, \lambda t) = \lambda^{\alpha} F(\omega, t)$$

for any $\omega \in \Omega$, $\lambda \in G$ and $t \in \mathbb{R}$, where α is a given complex number. A cocycle $F(\omega, t)$ on Ω is called *adapted* if there exists a function $\Xi : \Sigma \times \mathbb{R}_+ \to \mathbb{C}$ such that

$$F(\omega, d) - F(\omega, c) = \Xi(\omega(S), d-c) \tag{4}$$

for any tile $S := (a, b] \times [c, d) \in dom(\omega)$.

In [1], nonzero adapted α-homogeneous cocycles on Ω with $0 < \alpha < 1$ is characterized. In fact, we have

Theorem 4. *A nonzero adapted α-homogeneous cocycle on Ω is characterized by (4) with α and Ξ satisfying that $\mathcal{R}(\alpha) > 0$ and there exists a nonzero vector $\xi = (\xi_\sigma)_{\sigma\in\Sigma}$ such that $M_\alpha \xi = \xi$ and $\Xi(\omega(S), d-c) = (d-c)^{\alpha}\xi_{\omega(S)}$ for any tile $S := (a, b] \times [c, d) \in dom(\omega)$. Hence, a nonzero adapted α-homogeneous cocycle exists if and only if $\mathcal{R}(\alpha) > 0$ and α is a pole of $\zeta_\Omega(\alpha)$.*

Let Ω_{int} be the set of $\omega \in \Omega$ such that there exists $(a, b] \times [c, d) \in dom(\omega)$ satisfying that $c = 0$ and $a < 0 \le b$. An element $\omega \in \Omega_{int}$ is called an *integer* in Ω. Let

$$\tilde{\Omega}_{int} := \{(\omega, t) \in \Omega_{int} \times \mathbb{R};\ \omega + t \in \Omega_{int}\}.$$

A continuous function $F : \tilde{\Omega}_{int} \to \mathbb{C}$ is called a cocycle on Ω_{int} if (3) is satisfied for any $\omega \in \Omega_{int}$ and $t, s \in \mathbb{R}$ such that $(\omega, t) \in \tilde{\Omega}_{int}$ and $(\omega, t+s) \in \tilde{\Omega}_{int}$.

A cocycle F on Ω_{int} is called adapted if there exists a function $\Xi : \Sigma \times \mathbb{R}_+ \to \mathbb{C}$ such that (4) is satisfied for any $\omega \in \Omega_{int}$ and $c, d \in \mathbb{C}$ such that $(\omega, c) \in \tilde{\Omega}_{int}$, $(\omega, d) \in \tilde{\Omega}_{int}$ and $(a, b] \times [c, d) \in dom(\omega)$ for some $a < b$. This forces to imply that $a < 0$.

Let $\alpha \in \mathbb{C}$. A cocycle F on Ω_{int} is called α-homogeneous if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $(\omega, t) \in \tilde{\Omega}_{int}$ and $\lambda \in G$ with $(\lambda\omega, \lambda t) \in \tilde{\Omega}_{int}$. Note that if $(\omega, t) \in \tilde{\Omega}_{int}$, then for any $\lambda \in G$ with $\lambda > 1$, $(\lambda\omega, \lambda t) \in \tilde{\Omega}_{int}$ holds.

A cocycle F on Ω_{int} is called a *coboundary* on Ω_{int} if there exists a continuous function $G : \Omega_{int} \to \mathbb{R}^k$ such that

$$F(\omega, t) = G(\omega + t) - G(\omega)$$

for any $(\omega, t) \in \tilde{\Omega}_{int}$.

The following theorem is proved in [3].

Theorem 5. *A nonzero adapted α-homogeneous cocycle on Ω_{int} with $\mathcal{R}(\alpha) < 0$ is characterized by (4) with Ξ satisfying that there exists a nonzero vector $\xi = (\xi_\sigma)_{\sigma \in \Sigma}$ such that $M_\alpha \xi = \xi$ and $\Xi(\omega(S), d - c) = (d - c)^\alpha \xi_{\omega(S)}$ for any tile $S := (a, b] \times [c, d) \in dom(\omega)$ with $a < 0$. Hence, a nonzero adapted α-homogeneous cocycle on Ω_{int} with $\mathcal{R}(\alpha) < 0$ exists if and only if α is a pole of $\zeta_\Omega(\alpha)$. Moreover, any cocycle as this is a coboundary.*

Example 3. Let us consider the β-expansion system in Example 2. Denote $\Omega := \Theta(\beta)$. The associated matrix is

$$M_\alpha = \begin{pmatrix} \beta^{-\alpha} & (\beta^{-2} + \beta^{-3})^\alpha \\ \frac{\beta^{-2\alpha} + \beta^{-3\alpha}}{(\beta^{-2} + \beta^{-3})^\alpha} & 0 \end{pmatrix}$$

Let γ be one of the complex solutions of the equation $z^3 - z^2 - z - 1 = 0$. Then, $|\gamma| < 1$. Let $\alpha \in \mathbb{C}$ be such that $\gamma = \beta^\alpha$. Then, $\mathcal{R}(\alpha) < 0$.

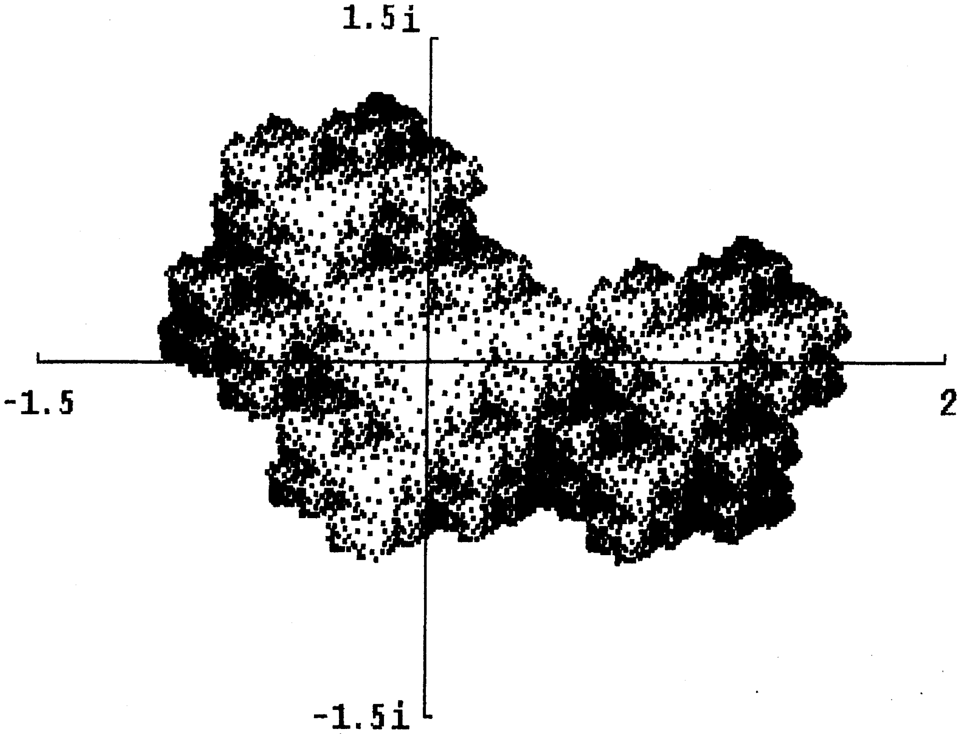

Figure 2: $G(\Omega_{int})$

Since M_1 and M_α are algebraically conjugate and

$$M_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we have

$$M_\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, there exists an α-homogeneous adapted cocycle F on Ω_{int} satisfying that

$$F(\omega, d) - F(\omega, c) = (d - c)^\alpha$$

if there exists $(a, b] \times [c, d) \in dom(\omega)$ with $a < 0$.

For $\omega \in \Omega_{int}$, let $S_0(\omega)$ be the tile $(a, b] \times [c, d) \in \omega$ such that $c = 0$ and $a < 0 \leq b$. We will define a continuous function $G : \Omega_{int} \to \mathbb{C}$ such that

$$F(\omega, t) = G(\omega + t) - G(\omega) \tag{5}$$

for any $(\omega, t) \in \tilde{\Omega}_{int}$. For $i = 0, 1, 2, \cdots$, let S_i be the i-th ancestor of $S_0(\omega)$. Let $Corner(S_i) =: (b_i, c_i)$. Let

$$G(\omega) := \sum_{i=0}^{\infty} (c_i - c_{i+1})^{\alpha}.$$

Then, we can prove (5). The set $G(\Omega_{int})$ is known as Rauzy fractal which is shown in Figure 2.

6 N-process

We consider the $\Omega = \Omega(\varphi, \eta)$ defined in Example 1. Since

$$M_{\alpha} = \begin{pmatrix} 2(4/9)^{\alpha} & (1/9)^{\alpha} \\ (1/9)^{\alpha} & 2(4/9)^{\alpha} \end{pmatrix}$$

and that

$$M_{1/2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

we have a $(1/2)$-homogeneous cocycle F by Theorem 4 with the above ξ. That is, F is defined by

$$F(\omega, d) - F(\omega, c) = \pm (d - c)^{1/2} \tag{6}$$

if there is a tile $(a, b] \times [c, d) \in dom(\omega)$, where $\pm$ corresponds to the color of the tile.

Consider the stochastic process $(\mathbf{N}_t)_{t \in \mathbb{R}}$ defined by $\mathbf{N}_t(\omega) = F(\omega, t)$, where ω comes from the probability space (Ω, μ), μ being the unique invariant probability measure invariant under the additive action. This process was called the N-process and studied in [2]. A prediction theory based on the N-process was developed. A process $Y_t = H(\mathbf{N}_t, t)$, where the function $H(x, s)$ is an unknown function which is twice continuously differentiable in x and once continuously differentiable in s and $H_x(x, s) > 0$ is considered. The aim is to predict the value Y_c from the observation $Y_J := \{Y_t;\ t \in J\}$, where $J = [a, b]$ and $a < b < c$.

Theorem 6. ([2]) *There exists an estimator $\hat{Y}_c$ which is a measurable function of the observation Y_J such that*

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = O((c-b)^2)$$

as $c \downarrow b$.

References

[1] Teturo Kamae, Linear expansions, strictly ergodic homogeneous cocycles and fractals, Israel J. Math. 106 (1998) pp.313-337.

[2] Teturo Kamae, Stochastic analysis based on deterministic Brownian motion, Israel J. Math. 125 (2001), pp.317-346.

[3] Nertila Gjini and Teturo Kamae, Coboundary on colored tiling space as Rauzy fractal, Indagationes Mathematicae 10-3 (1999) pp.407-421.

[4] Teturo Kamae, Numeration systems, fractals and stochastic processes, submitted to Israel J. Math.