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Term Structure Modelling and Monetary Policy

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1 Introduction

Since March 2001, the Bank of Japan's main operating target of monetary policy is the outstanding balance of current account deposits at the Bank. Under this quantitative easing scheme, the present monetary policy in Japan goes beyond the so-called zero interest rate policy (hereafter ZIRP). Under the ZIRP, particularly the short-term and mid-term interest rates are so low that we have difficulty applying traditional yield curve models such as Vasicek model to them. We examine Japan's financial markets, especially the bond markets, and discuss the market structure.

Marumo, Nakayama, Nishioka, and Yoshida [2003] constructs a term structure model with the following properties to take "policy duration effect" into consideration.

1. Instantaneous spot rate follows the traditional Vasicek augmented by incorporating the probability of policy duration as one of the risk factors.
2. Long-term interest rate zone is mainly determined by risk prices perceived in markets.

Meanwhile, in many countries, short rates are used as major instruments by monetary authorities for realization of their monetary policies. In Japan, the Bank of Japan uses the overnight call rate. Consequently, the possibility to take interaction between the interest rate controlled by the monetary policy

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and market rates into models explicitly seems to be one of advantages in modelling term structure by short rate models.

Along this line, Otaka and Yoshida [2000] and Yoshida [2003] propose the term structure model in the framework of Forward Backward Stochastic Differential Equations (hereafter FBSDE's). Additional advantage of this short rate modelling is the possibility that the theoretical discussion of determining the level of the target short rate can be embedded in the mechanism of generating whole term structure explicitly by using the theory of monetary policy reaction functions (hereafter MPRF's).

In this paper we try to explain term structure models, which explicitly capture the relation between a short rate and the monetary authority based on the above three papers.

2 Overview of Recent Monetary Policy in Japan

2.1 Zero Interest Rate Policy ; February 12, 1999 - August 11, 2000

In February 12, 1999, the BOJ decided that the BOJ would encourage the uncollateralized overnight call rate to move as low as possible by providing ample liquidity. In April 13, 1999, the Governor of the Bank of Japan stated that that the policy would be maintained until such time as when deflationary fears are removed, which is the concept of "policy duration effect."

In August 11, 2000, the BOJ decided to encourage the uncollateralized overnight call rate to move on average around 0.25% because that the BOJ felt confident that Japan's economy has reached the stage where deflationary concern has been dispelled, the condition for lifting the zero interest rate policy.

2.2 New Monetary Policy - Quantitative Easing - ; March 19, 2001 -

Although the policy was once abandoned in August 2000, in March 19, 2001, the BOJ decided to reactivate the policy in the following enhanced manner:

1. the BOJ changed the main operating target of monetary policy operation from the uncollateralized overnight call rate to the outstanding balance of current account deposits and increase the average from ¥ 4 trillion to ¥ 5 trillion.

2. the BOJ committed that it will continue to pursue the new monetary policy framework "until the Consumer Price Index registers stably a zero percent or an increase year on year."

3. the BOJ increases the BOJ's outright purchases of long-term government bonds.

3 A Term Structure Model under ZIRP

- Marumo, Nakayama, Nishioka, and Yoshida [2003] -

3.1 Dynamics of Instantaneous Spot Rate

We fix the probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\})$[1]. Q is the equivalent martingale measure with respect to the original measure P. We assume that an instantaneous spot rate is zero in the period of the ZIRP and follows the OU process once the policy is abandoned. Let τ be the stopping time until the BOJ ends the ZIRP under Q. Therefore, under Q, an instantaneous spot rate r_t^τ at time t given τ satisfies

$$dr_t^\tau = 1_{\{\tau \le t\}}\{\kappa(\mu - r_t^\tau)dt + \sigma dW_t^*\}, \tag{1}$$

where W^* is the Wiener process and τ and W^* are assumed to be independent under Q. From the theory of change of measures,

$$\mu = m - \frac{\sigma}{\kappa}\lambda,$$

where m is the mean-reverting level under the original measure P and λ is the market price of risk. Furthermore, we assume that τ obeys the standard Gamma distribution under Q^2, i.e.,

$$\Pr[\tau \le t] = \Psi(t) = \frac{\Gamma_t(\alpha)}{\Gamma(\alpha)}, \quad \alpha > 0,$$

where $\Gamma_t(\alpha) = \int_0^t u^{\alpha-1}e^{-u}du$ and $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}du$.

If we regard $N_t = \{1_{\{\tau \le t\}}\}$ as the F-adapted non-negative process, the hazard rate l_t of N_t is given by

$$l_t = \frac{\psi(t)}{1 - \Psi(t)}.$$

From the theory of *Doob* $-$ *Meyer* decomposition, $M_t = N_t - \int_0^t l_s(1 - N_s)ds$ becomes a martingale under the EMM Q.

3.2 Discount Bond Price Formula

Next, we derive a yield curve model in the period of the ZIRP. Let $P(t,T)$ be a discount bond price with maturity T, i.e.,

$$\begin{aligned} P(t,T) &= E^Q\left[e^{-\int_t^T r_s^\tau ds}|\mathcal{F}_t\right] \qquad (2) \\ &= E^Q\left[1_{\{\tau<t\}}e^{-\int_t^T r_s^\tau ds}|\mathcal{F}_t\right] + E^Q\left[1_{\{t\le\tau\le T\}}e^{-\int_t^T r_s^\tau ds}|\mathcal{F}_t\right] + E^Q\left[1_{\{T<\tau\}}|\mathcal{F}_t\right], \end{aligned}$$

[1]See Schönbucher [1999] for details.
[2]See Marumo et al. [2003] for details

where $E^Q[\cdot]$ denotes an expectation operator under the EMM Q.

Under the condition $\tau > t$,

$$
\begin{aligned}
&E^Q\left[1_{\{\tau<t\}}e^{-\int_t^T r_s^\tau ds}|\mathcal{F}_t\right] = 0,\\
&E^Q\left[1_{\{T<\tau\}}|\mathcal{F}_t\right] = 1_{\{\tau\le t\}}E^Q\left[1_{\{T<\tau\}}|\mathcal{F}_t\right] + 1_{\{t<\tau\}}E^Q\left[1_{\{T<\tau\}}|\mathcal{F}_t\right]\\
&\qquad = 1_{\{t<\tau\}}\Pr[T<\tau|t<\tau]\\
&\qquad = 1_{\{t<\tau\}}\frac{\Pr[T<\tau]}{\Pr[t<\tau]}\\
&\qquad = 1_{\{t<\tau\}}\frac{1-\Psi(T)}{1-\Psi(t)}.
\end{aligned}
$$

On the other hand, the second part of the righthand side (2) can be written as

$$
\begin{aligned}
E^Q\left[1_{\{t\le\tau\le T\}}e^{-\int_t^T r_s^\tau ds}|\mathcal{F}_t\right] &= E^Q\left[1_{\{t\le\tau\le T\}}E^Q\left[e^{-\int_\tau^T r_s^\tau ds}|\mathcal{F}_\tau\right]|\mathcal{F}_t\right]\\
&= E^Q\left[1_{\{t\le\tau\le T\}}Z_\tau|\mathcal{F}_t\right],
\end{aligned}
$$

where $Z_t = P^t(t,T)$ is a discount bond price given stopping time t.

Using $1_{\{t\le\tau\le T\}}Z_\tau = \int_t^T Z_s dN_s$, we can obtain that

$$
\begin{aligned}
E^Q\left[1_{\{t\le\tau\le T\}}Z_\tau|\mathcal{F}_t\right] &= E^Q\left[\int_t^T Z_s dN_s|\mathcal{F}_t\right]\\
&= E^Q\left[\int_t^T Z_s l_s(1-N_s)ds|\mathcal{F}_t\right]\\
&= \int_t^T Z_s l_s E^Q[1-N_s|\mathcal{F}_t]ds\\
&= \int_t^T Z_s l_s E^Q[1_{\{s<\tau\}}|\mathcal{F}_t]ds\\
&= 1_{\{t<\tau\}}\int_t^T Z_s l_s \frac{1-\Psi(s)}{1-\Psi(t)}ds\\
&= 1_{\{t<\tau\}}\int_t^T P^s(s,T)\frac{\psi(s)}{1-\Psi(s)}\frac{1-\Psi(s)}{1-\Psi(t)}ds,
\end{aligned}
$$

$$
\begin{aligned}
&P^t(t,T) = H_1(T-t),\\
&H_1(t) = \exp\left\{\frac{(H_2(t)-t)(\kappa^2\mu-\sigma^2/2)}{\kappa^2} - \frac{\sigma^2 H_2^2(t)}{4\kappa}\right\},\\
&H_2(t) = \frac{1-e^{-\kappa t}}{\kappa}.
\end{aligned}
$$

Hence, under the condition $\tau > t$,

$$
P(t,T) = \int_t^T \frac{\psi(s)}{1-\Psi(t)}H_1(T-s)ds + \frac{1-\Psi(T)}{1-\Psi(t)}. \tag{3}
$$

3.3 Empirical Analysis

The daily instantaneous spot rates data are estimated from the Japanese Government Bond market using the Vasicek and Fong method, which is globally fitted to yield curves.

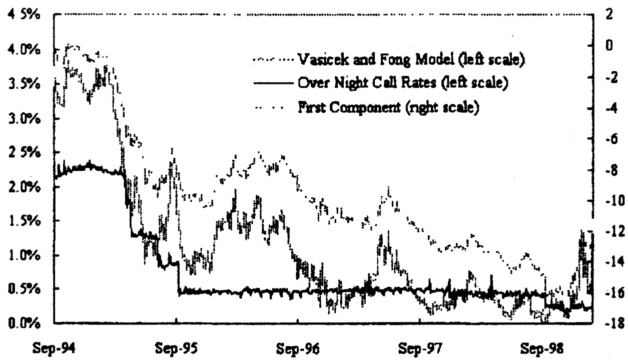

Figure 1: Estimated Instantaneous Spot Rates
(Cited from Marumo et al. [2003])

Parameters of the SDE (1) are estimated using the ordinary least-squares (OLS) method.

Table 1: Estimated Parameters of the Vasicek Model

	κ	m	σ
1992/1/6 ~1999/1/29	0.7131	0.006476	0.01017
	(1.974) **	(0.8477)	(89.40) ***

The figures in the parentheses denote t-value. *** denotes 99%, ** 95% and * 90% confidence level, respectively.
(Cited from Marumo et al. [2003])

Our model has still two parameters to be estimated: the shape parameter in the standard gamma distribution α and the market price of risk λ. We estimate these parameters daily by fitting the model to an actually yield curve using the OLS method.

Fig.2 compares between the yield curves estimated by our model and the Vasicek model. Fig.3 shows the time series of residual between the predicted value of our model and the market interest rates. The fitting of our model is satisfactory across any remaining maturity.

Fig.4 shows examples of the estimated distribution. In the period before August 2000, when the ZIRP was abandoned by the BOJ, the peak of the probability density function shifts leftward toward the end of the ZIRP. In the

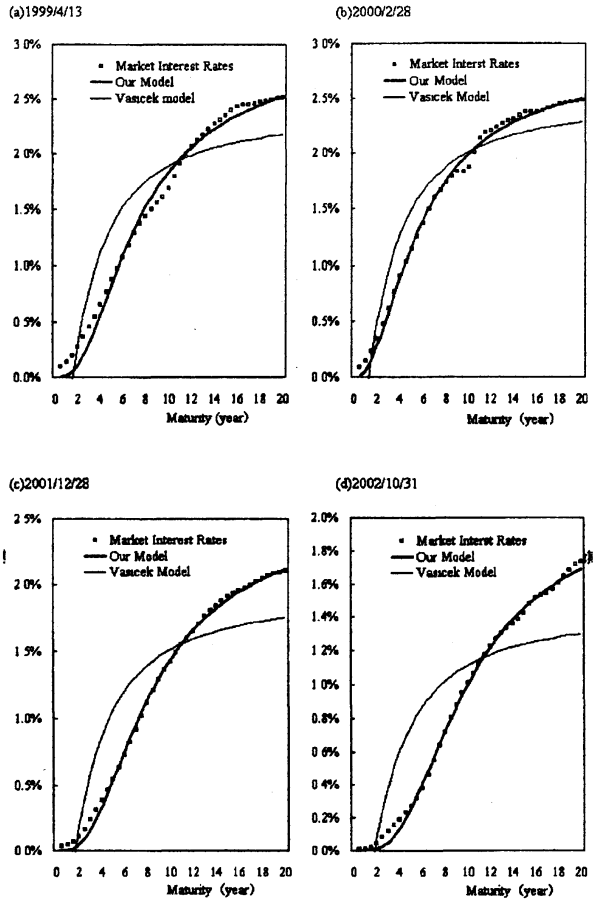

Figure 2: Estimated Yield Curves
(Cited from Marumo et al. [2003])

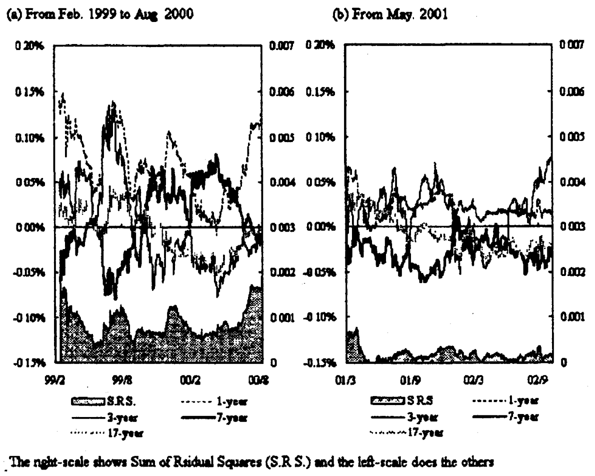

Figure 3: Residual of the model
(Cited from Marumo et al. [2003])

period after March 2001, when the quantitative monetary easing policy was adopted, the peak shifts rightward as time goes by.

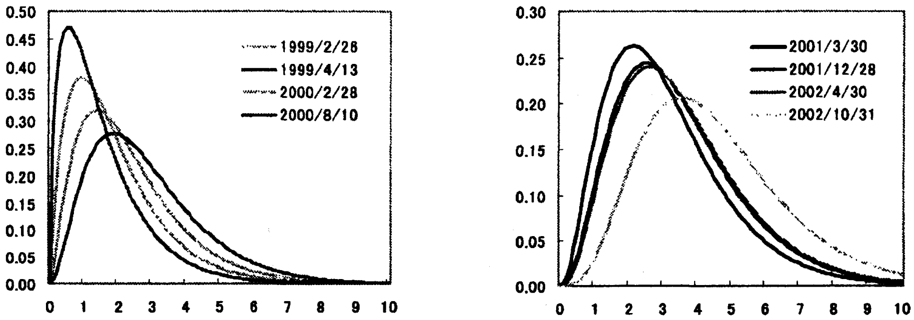

Figure 4: Probability Distribution of the Policy Duration Effect
(Cited from Marumo et al. [2003])

4 A Term Structure Model with MPRF

- Otaka and Yoshida [2000], Yoshida [2003] -

4.1 Forward Backward Stochastic Differential Equations

The Forward and Backward SDE's are described as the following system.

$$
\begin{aligned}
X_t &= x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s)^* \cdot dW_s^P, \qquad X_0 = x, \quad (4) \\
Y_t &= g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s^* \cdot dW_s^P. \quad (5)
\end{aligned}
$$

Theorem 4.1 *Solvability of FBSDE's* Under the following assumptions 1-3, FBSDE's (4)-(5) have a unique solution (X, Y, Z).

1. $b(\cdot)$, $h(\cdot)$, $\sigma(\cdot)$, and $g(\cdot)$ are smooth functions whose first derivatives are bounded by some constant L. Furthermore, $g \in C^{2+\alpha}(\mathbf{R}^n)$ for some $\alpha \in (0,1)$.

2. There exist some positive function $\mu(\cdot)$ and positive constant ν such that

$$\mu(|y|)I \leq \sigma(t,x,y)\sigma(t,x,y)^* \leq \nu I.$$

3. For all $(t,x,z) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^{m\times d}$, there exist some positive constant ν such that

$$|b(t,x,0,0)| + |h(t,x,0,z)| \leq \nu.$$

Proof See Ma and Yong [1999] Chapter 4 Theorem 2.2.

The Backward SDE's are thought as a special case of the FBSDE's, in which the forward processes are independent of the backward processes.

$$\xi = Y_t + \int_t^T h(s,Y_s,Z_s)ds - \int_t^T Z_s^* \cdot dW_s^P, \qquad Y_T = \xi. \tag{6}$$

Theorem 4.2 *Solvability of BSDE* BSDE(6) has an unique solution (Y,Z), given the standard parameters (ξ, h) which satisfy the following conditions.

1. $\xi \in \mathbf{L}_T^2(\mathbf{R}^m)$: Let $\mathbf{L}_T^2(\mathbf{R}^m)$ be the space of all $\mathcal{F}_T$-measurable random variables satisfying $E(|X|^2) < \infty$.
2. $h(\cdot,0,0) \in \mathbf{H}_T^2(\mathbf{R}^m)$: Let $\mathbf{H}_T^2(\mathbf{R}^m)$ be the space of the all predictable processes $\psi : \Omega \times [0,T] \to \mathbf{R}^m$ such that $E\int_0^T |\psi_t|^2 dt < \infty$ and h is uniformly Lipschitz.

Proof See Pardoux and Peng (1990) Theorem 3.1.

4.2 Term Structure Models in the FBSDE's Framework

In this subsection a short rate model with the instrument (e.g., the overnight call rate in Japan) is developed. We fix the probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$[3] satisfying the usual conditions. Let $\{r^{inst}(t), t \geq 0\}$ denote the instrument process and $\{r(t), t \geq 0\}$ denote the prospective instantaneous spot rate process in the JGB market, respectively. A price process of the T-maturity discount bond $\{P(t,T), 0 \leq t \leq T\}$ is given by the expected discounted formula $P(t,T) = E\left[\exp(-\int_t^T r_u du)|\mathcal{F}_t\right]$.

We consider term structure models in the FBSDE's framework as follows[4]. Let $\{W^1(t), t \geq 0\}$ and $\{W^2(t), t \geq 0\}$ be two independent Wiener process.

[3]See Duffie, Ma and Yong (1995) for the discussion about the probability measure.
[4]See Yoshida [2003] for details.

Suppose that state variables $X_t = (X_t^1, X_t^2)$ satisfies

$$
\begin{aligned}
dX_t &= b(t, X_t, P(t,T))dt + \sigma(t, X_t)\cdot dW_t, && (7)\\
dP(t,T) &= -h(t, X_t, P(t,T))dt - Z_t^* \cdot dW_t, && (8)\\
X_0^1 = x_0^1, \quad X_0^2 &= x_0^2, \quad P(T,T) = g(X(T)) = 1, && (9)\\
h(t, X(t), P(t,T)) &= -l(X_t^1)P(t,T) = -r_t P(t,T), && (10)\\
\sigma(t, X_t) &= \begin{pmatrix} \sigma_{11}(t,X_t) & \sigma_{12}(t,X_t) \\ 0 & \sigma_{22}(t,X_t) \end{pmatrix}, && (11)\\
& \sigma_{11}(\cdot), \sigma_{12}(\cdot), \sigma_{22}(\cdot) : \text{positive functions}, \\
Z_t &= (Z_t^1, Z_t^2)^*, \quad W_t = (W_t^1, W_t^2)^*. && (12)
\end{aligned}
$$

Remarks In this model setting, we assume that

$$
b(t, X_t, P(t, X(t)) = \begin{pmatrix} \kappa_1(t,X_t)(X_t^2 - X_t^1) \\ \kappa_2(t,X_t)(\psi(t, X_t, P(t,T)) - X_t^2) \end{pmatrix}. \quad (13)
$$

Then,

1. *Vasicek Model* :
   $X_t^1 = r_t$, $\kappa_2(\cdot) = 0$, $l(X_t^1) = X_t^1$, σ_{11} is a positive constant, and 0 otherwise.

2. *Cox, Ingersoll, and Ross model* :
   Same as the Vasicek model, except $\sigma_{11}(t, X_t) = \sigma\sqrt{X_t^1}$.

3. *Hull - White model* :
   $\kappa_1(t, X_t) = \kappa_1(t)$, $\kappa_2(t, X_t)\psi(t, X_t, P(t,T)) = \psi(t)$, $\sigma_{22}(\cdot) = 0$ in the Vasicek Model setting.

4. *Black - Karasinski model* :
   $l(X_t^1) = \exp(X_t^1)$ in the Hull - White model setting.

5. *Duffie - Kan model* :
   Set $h(t, X_t, P(t,T))$ as a function of $P(t,T)'s$ with various maturities.

4.3 Term Structure Model with MPRF

The state variables $X(t) = (X_1(t), X_2(t)) = (\ln r(t), \ln r^{inst}(t))$ are assumed to satisfy the stochastic differential equation of the form

$$\begin{aligned} dX(t) &= b(t, X(t), P(t,T))dt + \sigma \cdot dW(t), \\ dP(t,T) &= -h(X(t), P(t,T))dt + Z(t)^* \cdot dW(t), \\ h(t, X(t), P(t,T)) &= -\exp(X_t^1)P(t,T) = -r_t P(t,T), \\ X_1(0) &= \ln r(0) = \ln r_0, \quad X_2(0) = \ln r^{inst}(0) = \ln r_0^{inst}, \\ P(T,T) &= g(X(T)) = 1, \end{aligned}$$

where

$$b(t, X(t), P(t, X(t))) = \begin{pmatrix} \kappa_1(X_2(t) - X_1(t)) \\ \kappa_2(\psi(t, X(t), P(t,T)) - X_2(t)) \end{pmatrix},$$

κ_1, κ_2 : constants,

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ 0 & \sigma_{22} \end{pmatrix}, \quad \sigma_{11}, \sigma_{12}, \sigma_{22} : \text{positive constants},$$

$$Z(t) = (Z_1(t), Z_2(t))^*, W(t) = (W^1(t), W^2(t))^*.$$

In the above formulation, the existence of some deterministic "monetary policy reaction" function $\psi(\cdot)$ is assumed, by which the monetary authority decides the level of the target rate.

As for the function $\psi(\cdot)$, we use the instrument rule as the monetary policy reaction function. We show some examples of instrument rules.

- Taylor rule

$$r_{target}^{inst}(t) = \bar{i} + \pi_t + 0.5(\pi_t - \hat{\pi}) + 0.5y_t,$$

  where $\bar{i}$ is the average of the instrument.

- Henderson-McKibbin rule

$$r_{target}^{inst}(t) = \bar{i} + 2(\pi_t + y_t - (\widehat{\pi + y})).$$

- QPM model (Bank of Canada, Reserve Bank of New Zealand)

$$r_{target}^{inst}(t) = r_t^L + \gamma(E_Q[\pi_{t+T}|\mathcal{F}_t] - \hat{\pi}),$$

  where $r^L(t)$ is a long nominal rate.

In the context of monetary policy rules, the equation (13) means that the instrument is given by the weighted average of the process and the target level in practice.

We assume that

$$r^{inst}_{target} = -\ln P(t,T)/(T-t) - \beta^{T-t} = Y(t,T) - \beta^{T-t} \tag{14}$$

where $Y(t,T)$ is a $(T-t)$ yield function and β^{T-t} is a term premium function.
Based on the observed data, we assume

$$\beta^{T-t} = Y(t,T)(1 - e^{-\beta_1(T-t)})$$

where β_1 is a positive constant. Hence,

$$\psi(t, X(t), P(t,T)) = \ln r^{inst}_{target} = \ln Y(t,T) - \beta_1(T-t).$$

Remark As for the long rate, the sliding bond yield $-\ln P(t,t+T)/T$ may be used instead of the bond yield in (14). But in the JGB market, the yield of some fixed series with enough liquidity is used to represent the market level while it exists in some period of time to maturity. Furthermore, in the HJM framework, the SDE of the sliding bond price $D(t,T) = P(t,t+T)$ contains the forward rate explicitly (Rutkowski (1997)), i.e.,

$$dD(t,T) = D(t,T)((r(t) - f(t,t+T))dt + \sigma(t,t+T)\cdot dW(t))$$

where $f(t,T)$ is the instantaneous forward rate starting at T. Hence, it is hard to derive or solve the corresponding PDE in our approach.

Using the Jensen inequality, we obtain

$$\begin{aligned} Y(t,T) &= -\frac{\ln P(t,T)}{T-t} = -\frac{1}{T-t}\ln E_Q\left[e^{-\int_t^T r(u)du}|\mathcal{F}_t\right] \\ &\leq E_Q\left[r^*(\omega)|\mathcal{F}_t\right] = C^*(r_t,t,T), \end{aligned}$$

where $r^*(\omega) = \sup_{u\in[t,T]} r(u,\omega)$, $\omega \in \Omega$ and $\lim_{T-t\to 0} C^*(r_t,t,T) = r_t$. On the other hand,

$$\begin{aligned} C_*(r_t,t,T) &= \hat{r}_*(t,T) = -\frac{1}{T-t}\ln\left[e^{-\hat{r}_*(t,T)(T-t)}\right] \\ &= -\frac{1}{T-t}\ln E_Q\left[e^{-r_*(\omega)(T-t)}|\mathcal{F}_t\right] \leq Y(t,T), \end{aligned}$$

where $r_*(\omega) = \inf_{u\in[t,T]} r(u,\omega)$, $\omega \in \Omega$ and $\lim_{T-t\to 0} C_*(r_t,t,T) = r_t$.

Assumption Based on the above estimates, we assume that the possibility that $X_1(t)$ and $X_2(t)$ are out of some interval $[L_1^T, U_1^T]$ and $[L_2^T, U_2^T]$ is negligible, respectively, for some numbers L_1^T, U_1^T, L_2^T and U_2^T which depend on parameters of processes and T.

The base model in the stationary state can be described as

Problem B_s
$dX(t) = b(t, X(t), P(t,T))dt + \sigma \cdot dW(t),$
$dP(t,T) = -h(X(t), P(t,T))dt + Z(t)^* \cdot dW(t),$
$h(t, X(t), P(t,T)) = -\exp(X_t^1)P(t,T) = -r_t P(t,T),$
$X_1(0) = \ln r(0) = \ln r_0, \quad X_2(0) = \ln r^{inst}(0) = \ln r_0^{inst},$
$P(T,T) = g(X(T)) = 1,$
$X(t) \in \mathcal{D} = \{X(t) | L_1^T \le X_1(t) \le U_1^T, \ L_2^T \le X_2(t) \le U_2^T\},$
where

$$b(t, X(t), P(t, X(t))) = \begin{pmatrix} \kappa_1(X_2(t) - X_1(t)) \\ \kappa_2 (\ln Y(t,T) - \beta_1(T-t) - X_2(t)) \end{pmatrix},$$

$\kappa_1 > 0,$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ 0 & \sigma_{22} \end{pmatrix}, \quad \sigma_{11}, \sigma_{12}, \sigma_{22} > 0,$$

$Z(t) = (Z_1(t), Z_2(t))^T, W(t) = (W^1(t), W^2(t))^T.$

To solve this problem B_s, we can use the Four Step Scheme. In general, the Four Step Scheme consists of the four major steps, in this case, they are reduced to the three steps because the volatility functions are constant[5].

1. Solve the PDE

$$\begin{aligned} \theta_t &+ \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)\theta_{x_1x_1} + \sigma_{12}\sigma_{22}\theta_{x_1x_2} + \frac{1}{2}\sigma_{22}^2\theta_{x_2x_2} + \kappa_1(x_2(t) - x_1(t))\theta_{x_1} \\ &+ \kappa_2(\ln(-\ln\theta/(T-t)) - \beta_1(T-t) - x_2(t))\theta_{x_2} - e^{x_1(t)}\theta = 0, \\ &\theta(x,T) = 1. \end{aligned}$$

2. Using θ obtained in the previous step to solve the forward SDE

$$\begin{aligned} dX_1(t) &= \kappa_1(X_2(t) - X_1(t))dt + \sigma_{11}dW^1(t) + \sigma_{12}dW^2(t), \\ dX_2(t) &= \kappa_2(\ln(-\ln\theta(X(t),t)/(T-t)) - \beta_1(T-t) - X_2(t))\,dt \\ &\quad +\sigma_{22}dW^2(t), \\ &X_1(0) = \ln r(0) = \ln r_0, \quad X_2(0) = \ln r^{inst}(0) = \ln r_0^{inst}. \end{aligned}$$

3. Set

$$\begin{aligned} P(t,T) &= \theta(X(t),t), \\ Z_t &= \theta_x(X(t),t)^T \cdot \sigma. \end{aligned}$$

[5]See Ma and Yong (1999) for details.

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