Molecular Gas Dynamics

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Supplementary Notes and Errata

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Bibliography

Update of bibliography

- [19] Arkeryd, L. and A. Nouri [2006]:
  (to be published) → 401–443
- [29] Bardos, C., F. Golse, and Y. Sone [2006]:
  (to be published) → 275–300
- [59] Chen, C.-C., I.-K. Chen, T.-P. Liu, and Y. Sone [2006]:
  [2006] → [2007]
  59, (to be published) → **60**, 147–163
- [136] Liu, T.-P. and S.-H. Yu [2006a]:
  [2006a] → [2006]
- [137] Liu, T.-P. and S.-H. Yu [2006b]:
  [2006b] → [2007]
  59, (to be published) → **60**, 295–356

The corresponding corrections in the text

- p. 166, the first line in the third paragraph:
  [2004b, 2006a] → [2004b, 2006]
- p. 166, the last line in the third paragraph:
  [2006b] → [2007]
- p. 183, the second line:
  [2006] → [2007]

Errata

- p. 9, the 7th line:
  specular condition → specular reflection
- p. 27, the 3rd line of Footnote 26:
  Eq. (1.99) → a linear combination of Eqs. (1.99) and (1.101)
- p. 27, the 6th line of Footnote 26:
  except for a common constant factor → except for a common constant factor and additive functions (say, f_a in $\hat{H}$ and f_{bi} in $\hat{H}_i$ in their second order) satisfying $\mathrm{Sh}\,\partial f_a/\partial \hat{t} + \partial f_{bi}/\partial x_i = 0$
- p. 81, the 4th line in Footnote 7:
  u_{iGm} → $u_{iGm} - u_{jGm} n_j n_i$
  or ϕ_{eGm} → of ϕ_{eGm}
- p. 83, the first line in Footnote 14:
  u_{iGm} → $u_{iGm} - u_{jGm} n_j n_i$
- p. 504, the first line in Footnote 24:
  damin → domain
- p. 505, Eq. (A.60):

$$\left|\frac{1}{\sin\theta_c}\frac{\partial \sin^2\theta_e}{\partial\theta_c}\right| \to \left|\frac{1}{\sin\theta_c}\frac{\mathrm{d}\sin^2\theta_e}{\mathrm{d}\theta_c}\right|$$

$$\left|\frac{1}{\sin\theta_c}\frac{\partial b^2}{\partial\theta_c}\right| \to \left|\frac{1}{\sin\theta_c}\frac{\mathrm{d}b^2}{\mathrm{d}\theta_c}\right|$$

- p. 506, the 13th line [The line next to Eq. (M-A.63)]:
  with respect to θ_c → with respect to θ_α
- p. 617, the right-hand side of Eq. (C.2b):
  In order to avoid misunderstanding, $\frac{2(n+1)!}{\beta^{n+2}}\pi$ is better expressed as

$$\frac{2\pi(n+1)!}{\beta^{n+2}}.$$

- p. 628, Reference [110]:
  Reference [110] should be placed after Reference [112].
- p. 639, the 3rd line in Reference [262]:
  gs → gas

Supplementary Notes

In the present supplementary notes, the letter M is attached to the labels of sections, equations, etc. in the book "Molecular Gas Dynamics" to avoid confusion.

1 Chapter M-1

1.1 Supplement to Footnote M-9 in Section M-1.3

We will explicitly show the process of derivation of the conservation equations (M-1.12)–(M-1.14) by taking into account the discontinuity of the velocity distribution function $f(\boldsymbol{X}, \boldsymbol{\xi}, t)$ for a typical case.

Let $S(\boldsymbol{X})$ be a continuous and sectionally smooth function of $\boldsymbol{X}$, and let the surface in the $\boldsymbol{X}$ space consisting of the points $\boldsymbol{X}_0$ that satisfy $S(\boldsymbol{X}_0) = 0$ be indicated by S_0.[1] The surface S_0 may be an infinite surface or a bounded surface separating the space $\boldsymbol{X}$ into two regions. The velocity distribution function f at time t_0 is assumed to be discontinuous across the surface S_0 and to be smooth except on S_0. The discontinuity propagates along the characteristics of the Boltzmann equation (M-1.5), i.e., $X_i - \xi_i(t - t_0) = X_{0i}$, for each $\boldsymbol{\xi}$.[2] Take a point $(\boldsymbol{X}, t)$ in the space and time, where $t > t_0$. At this point or at $(\boldsymbol{X}, t)$, the discontinuity of f lies on the surface $S^{(\boldsymbol{\xi})}(\boldsymbol{X}, t)$ in the $\boldsymbol{\xi}$ space that consists of the points $\boldsymbol{\xi}_D$ satisfying

$$S(X_i - \xi_{Di}(t - t_0)) = 0. \tag{1}$$

The point $\boldsymbol{\xi}_D$ is determined by $\boldsymbol{X}$, t, and $\boldsymbol{X}_0$, i.e., $\boldsymbol{\xi}_D(\boldsymbol{X}, t; \boldsymbol{X}_0)$. Let the side of the domain in the $\boldsymbol{\xi}$ space that satisfies $S(X_i - \xi_i(t - t_0)) > 0$ be indicated by V_+, and the other side of the domain by V_-; let the outward unit normal to the surface $S^{(\boldsymbol{\xi})}(\boldsymbol{X}, t)$ with respect to V_+ be indicated by $n_{Di}(\boldsymbol{\xi}_D; \boldsymbol{X}, t)$. Then,

$$n_{Di}(\boldsymbol{\xi}_D; \boldsymbol{X}, t) = -\left.\frac{\partial S(\boldsymbol{X} - \boldsymbol{\xi}(t - t_0))/\partial \xi_i}{|\partial S(\boldsymbol{X} - \boldsymbol{\xi}(t - t_0))/\partial \xi_j|}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}_D} = \left.\frac{\partial S(\boldsymbol{Y})/\partial Y_i}{|\partial S(\boldsymbol{Y})/\partial Y_j|}\right|_D, \tag{2}$$

where $|a_i| = (a_i^2)^{1/2}$ and the subscript D to $\partial S(\boldsymbol{Y})/\partial Y_j$ indicates $\boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{\xi}_D(t - t_0)$. The variations of $\boldsymbol{\xi}_D$ with respect to $\boldsymbol{X}$ or t for a given $\boldsymbol{X}_0$, i.e., $\partial \boldsymbol{\xi}_D/\partial X_i$ and $\partial \boldsymbol{\xi}_D/\partial t$, are determined from Eq. (1) as

$$\left.\frac{\partial S(\boldsymbol{Y})}{\partial Y_j}\right|_D \left(\delta_{ij} - \frac{\partial \xi_{Dj}}{\partial X_i}(t - t_0)\right) = 0, \qquad \left.\frac{\partial S(\boldsymbol{Y})}{\partial Y_j}\right|_D \left(\frac{\partial \xi_{Dj}}{\partial t}(t - t_0) + \xi_{Dj}\right) = 0.$$

Thus, with the aid of Eq. (2),

$$n_{Dj}\frac{\partial \xi_{Dj}}{\partial X_i} = \frac{n_{Di}}{t - t_0}, \qquad n_{Dj}\frac{\partial \xi_{Dj}}{\partial t} = -\frac{n_{Dj}\xi_{Dj}}{t - t_0}. \tag{3}$$

[1] It is assumed that $(\partial S/\partial X_i)^2 \neq 0$ on S_0. The normal to the surface S_0 is defined except at special points.

[2] For simplicity of explanation, we consider the case where $F_i = 0$ here.

The integral of such a discontinuous function with respect to $\boldsymbol{\xi}$ over its whole space is split into two parts as

$$\int \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int_{V_+} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} + \int_{V_-} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi},$$

where $\psi(\boldsymbol{\xi})$ is a smooth function of $\boldsymbol{\xi}$. Then, the integrand is smooth in each of V_+ and V_-. According to Lemma in page M-492, the following derivatives of integrals over the domain V_+ are transformed as[3]

$$\frac{\partial}{\partial t}\int_{V_+} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int_{V_+} \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial t} \mathbf{d}\boldsymbol{\xi} + \int_{S^{(\boldsymbol{\xi})}} \psi(\boldsymbol{\xi}) f \frac{\partial \xi_{Dj}}{\partial t} n_{Dj} \mathrm{d}^2\boldsymbol{\xi},$$

$$\frac{\partial}{\partial X_i}\int_{V_+} \xi_i \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int_{V_+} \xi_i \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial X_i} \mathbf{d}\boldsymbol{\xi} + \int_{S^{(\boldsymbol{\xi})}} \xi_i \psi(\boldsymbol{\xi}) f \frac{\partial \xi_{Dj}}{\partial X_i} n_{Dj} \mathrm{d}^2\boldsymbol{\xi},$$

where the integral over the surface $S^{(\boldsymbol{\xi})}$ of the second term on the right-hand side of each equation is due to the variation of the domain V_+ with t or X_i. Summing the above two derivatives and noting Eq. (3), we have

$$\frac{\partial}{\partial t}\int_{V_+} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} + \frac{\partial}{\partial X_i}\int_{V_+} \xi_i \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int_{V_+} \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial t} \mathbf{d}\boldsymbol{\xi} + \int_{V_+} \xi_i \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial X_i} \mathbf{d}\boldsymbol{\xi},$$

where the surface integrals over $S^{(\boldsymbol{\xi})}$ are cancelled. Similarly,

$$\frac{\partial}{\partial t}\int_{V_-} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} + \frac{\partial}{\partial X_i}\int_{V_-} \xi_i \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int_{V_-} \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial t} \mathbf{d}\boldsymbol{\xi} + \int_{V_-} \xi_i \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial X_i} \mathbf{d}\boldsymbol{\xi}.$$

Thus, we have

$$\frac{\partial}{\partial t}\int \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} + \frac{\partial}{\partial X_i}\int \xi_i \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial t} \mathbf{d}\boldsymbol{\xi} + \int \xi_i \psi(\boldsymbol{\xi}) \frac{\partial f}{\partial X_i} \mathbf{d}\boldsymbol{\xi}. \quad (4)$$

It may be noted that the interchange of differentiation and integration is possible only for the above combination of the integrals. With this formula, the conservation equations are derived by choosing 1, ξ_i, and ξ_i^2 as $\psi(\boldsymbol{\xi})$.

When the surface S_0, i.e., $S(\boldsymbol{X}) = 0$, is a finite surface or semi-infinite surface which does not divide the $\boldsymbol{\xi}$ space into V_+ and V_-, we can take it as a special case where some part of S_0 joins to its other part and V_- degenerates empty. When there is a body in a gas, the discontinuity as shown in Section M-3.1.6 generally exists. The analysis can be carried out in a similar way; that is, determine the position of the discontinuity in the $\boldsymbol{\xi}$ space first, carry out the differentiations in each region where the velocity distribution function is smooth with the aid of the lemma in page M-492, and sum up the results.

(Section 1.1: Version 6-00)

[3]The correspondence of the variables here and those in the lemma is as follows: $\boldsymbol{\xi} \leftrightarrow \boldsymbol{X}$, t or $X_i \leftrightarrow \vartheta$, $n_{Di} \leftrightarrow \boldsymbol{n}_w$, $\mathbf{d}\boldsymbol{\xi} \leftrightarrow \mathbf{d}\boldsymbol{X}$, $\mathrm{d}^2\boldsymbol{\xi} \leftrightarrow \mathrm{d}^2\boldsymbol{X}$, $V_+ \leftrightarrow \mathcal{D}(\vartheta)$, $S^{(\boldsymbol{\xi})} \leftrightarrow \partial\mathcal{D}$.

1.2 Note on the equality condition of Eq. (M-1.38)

The statement of the equality condition of Eq. (M-1.38), i.e., "The equality in Eq. (1.38) holds when and only when f is the Maxwellian that satisfies the boundary condition (1.26)... ", needs supplementary explanation. Some condition is required of the scattering kernel K_B in the boundary condition (M-1.26) for f to be limited to the Maxwellian. For some K_B, the equality holds in Eq. (M-1.38) for f other than the Maxwellian. See Section 6.4.1 for more detailed discussion.

(Section 1.2: Version 5-00)

1.3 Supplement to Footnote M-26 in Chapter M-1

Footnote M-26 is supplemented with more explicit mathematical expressions for the process given there. Take the non-dimensional form of the equation for the H function, i.e., Eq. (M-1.72):

$$\mathrm{Sh}\frac{\partial \hat{H}}{\partial \hat{t}} + \frac{\partial \hat{H}_i}{\partial x_i} = \frac{1}{k}\hat{G}, \tag{5}$$

where

$$\left.\begin{aligned} &\hat{H}(x_i,\hat{t}) = \int \hat{f}\ln(\hat{f}/\hat{c}_0)\mathbf{d}\boldsymbol{\zeta}, \quad \hat{H}_i(x_i,\hat{t}) = \int \zeta_i \hat{f}\ln(\hat{f}/\hat{c}_0)\mathbf{d}\boldsymbol{\zeta}, \\ &\hat{G} = -\frac{1}{4}\int (\hat{f}'\hat{f}'_* - \hat{f}\hat{f}_*)\ln\left(\frac{\hat{f}'\hat{f}'_*}{\hat{f}\hat{f}_*}\right)\widehat{B}\mathrm{d}\Omega\mathbf{d}\boldsymbol{\zeta}_*\mathbf{d}\boldsymbol{\zeta} \le 0, \end{aligned}\right\} \tag{6}$$

with $\hat{c}_0 = c_0(2RT_0)^{3/2}/\rho_0$. The perturbed form of the velocity distribution function $\hat{f}$ is defined by

$$\hat{f} = E(1+\phi), \tag{7}$$

where

$$E = \frac{1}{\pi^{3/2}}\exp(-\zeta^2).$$

Let ε be a small quantity. Here, we take the case in which ϕ is of the order of ε, and examine the terms of the order of ε^2 of Eq. (5). The perturbed function ϕ is expressed as

$$\phi = \phi_1\varepsilon + \phi_2\varepsilon^2 + \cdots . \tag{8}$$

Corresponding to the expansion, the macroscopic variables, i.e., ω, u_i, P, etc., $\hat{H}$, $\hat{H}_i$, and $\hat{G}$ are also expressed as

$$h = h_1\varepsilon + h_2\varepsilon^2 + \cdots , \tag{9a}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1\varepsilon + \hat{H}_2\varepsilon^2 + \cdots , \tag{9b}$$

$$\hat{H}_i = \hat{H}_{i0} + \hat{H}_{i1}\varepsilon + \hat{H}_{i2}\varepsilon^2 + \cdots , \tag{9c}$$

$$\hat{G} = \hat{G}_0 + \hat{G}_1\varepsilon + \hat{G}_2\varepsilon^2 + \cdots , \tag{9d}$$

where h represents the perturbed macroscopic variables, ω, u_i, P, etc., and the quantities ϕ_n, h_n, $\hat{H}_n$, $\hat{H}_{in}$, and $\hat{G}_n$ are of the order of unity. Then, with the aid of the expanded forms of Eqs. (M-1.78a)–(M-1.78f), $\hat{H}_n$, $\hat{H}_{in}$, and $\hat{G}_n$ are expressed as

$$\hat{H}_0 = -\frac{3}{2} - \ln \pi^{3/2}\hat{c}_0, \tag{10a}$$

$$\begin{aligned}\hat{H}_1 &= (1 - \ln \pi^{3/2}\hat{c}_0) \int E\phi_1 \mathbf{d}\boldsymbol{\zeta} - \int \zeta^2 E\phi_1 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)\omega_1 - \frac{3}{2}P_1, \end{aligned}\tag{10b}$$

$$\begin{aligned}\hat{H}_2 &= (1 - \ln \pi^{3/2}\hat{c}_0) \int E\phi_2 \mathbf{d}\boldsymbol{\zeta} - \int \zeta^2 E\phi_2 \mathbf{d}\boldsymbol{\zeta} + \frac{1}{2}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)\omega_2 - \left(\frac{3}{2}P_2 + u_{i1}^2\right) + \frac{1}{2}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}, \end{aligned}\tag{10c}$$

$$\hat{H}_{i0} = 0, \tag{11a}$$

$$\begin{aligned}\hat{H}_{i1} &= (1 - \ln \pi^{3/2}\hat{c}_0) \int \zeta_i E\phi_1 \mathbf{d}\boldsymbol{\zeta} - \int \zeta_i \zeta^2 E\phi_1 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)u_{i1} - \left(Q_{i1} + \frac{5}{2}u_{i1}\right), \end{aligned}\tag{11b}$$

$$\begin{aligned}\hat{H}_{i2} &= (1 - \ln \pi^{3/2}\hat{c}_0) \int \zeta_i E\phi_2 \mathbf{d}\boldsymbol{\zeta} - \int \zeta_i \zeta^2 E\phi_2 \mathbf{d}\boldsymbol{\zeta} + \frac{1}{2}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} \\ &= (1 - \ln \pi^{3/2}\hat{c}_0)(u_{i2} + \omega_1 u_{i1}) - \left(Q_{i2} + \frac{5}{2}u_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) \\ &\quad + \frac{1}{2}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}, \end{aligned}\tag{11c}$$

$$\hat{G}_0 = 0, \tag{12a}$$

$$\hat{G}_1 = 0, \tag{12b}$$

$$\hat{G}_2 = -\frac{1}{4}\int EE_*(\phi_1' + \phi_{1*}' - \phi_1 - \phi_{*1})^2 \widehat{B}\mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \le 0. \tag{12c}$$

With the aid of these expressions, the ε and ε^2-order expressions of Eq (5) are given as

$$\mathrm{Sh}\frac{\partial \hat{H}_1}{\partial \hat{t}} + \frac{\partial \hat{H}_{i1}}{\partial x_i} = (1 - \ln \pi^{3/2}\hat{c}_0)\left(\mathrm{Sh}\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial u_{i1}}{\partial x_i}\right) - \left[\frac{3}{2}\mathrm{Sh}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i1} + Q_{i1}\right)\right], \tag{13a}$$

$$\mathrm{Sh}\frac{\partial \hat{H}_2}{\partial \hat{t}} + \frac{\partial \hat{H}_{i2}}{\partial x_i} = (1 - \ln \pi^{3/2}\hat{c}_0)\left(\mathrm{Sh}\frac{\partial \omega_2}{\partial \hat{t}} + \frac{\partial (u_{i2} + \omega_1 u_{i1})}{\partial x_i}\right) - \mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left(\frac{3}{2}P_2 + u_{i1}^2\right) - \frac{\partial}{\partial x_i}\left(Q_{i2} + \frac{5}{2}u_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) + \frac{1}{2}\left(\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}\right). \tag{13b}$$

Substituting the series expansion (9a) into the conservation equation (M-1.87), we have

$$\mathrm{Sh}\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial u_{i1}}{\partial x_i} = 0, \tag{14a}$$

$$\mathrm{Sh}\frac{\partial \omega_2}{\partial \hat{t}} + \frac{\partial (u_{i2} + \omega_1 u_{i1})}{\partial x_i} = 0. \tag{14b}$$

Similarly, from the conservation equation (M-1.89), we have

$$\frac{3}{2}\mathrm{Sh}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i1} + Q_{i1}\right) = 0, \tag{15a}$$

$$\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left(\frac{3}{2}P_2 + u_{i1}^2\right) + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i2} + Q_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) = 0. \tag{15b}$$

With the aid of the expanded forms (14a)–(15b) of the conservation equations (M-1.87) and (M-1.89), Eqs. (13a) and (13b) are reduced to, for the solution of the Boltzmann equation (M-1.47) or (M-1.75a),

$$\mathrm{Sh}\frac{\partial \hat{H}_1}{\partial \hat{t}} + \frac{\partial \hat{H}_{i1}}{\partial x_i} = 0, \tag{16a}$$

$$\mathrm{Sh}\frac{\partial \hat{H}_2}{\partial \hat{t}} + \frac{\partial \hat{H}_{i2}}{\partial x_i} = \frac{1}{2}\left(\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}\right). \tag{16b}$$

Thus, the $o(\varepsilon^2)$ terms being neglected in Eq. (5), it is reduced to

$$\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} = -\frac{1}{2k}\int EE_*(\phi_1' + \phi_{1*}' - \phi_1 - \phi_{1*})^2 \widehat{B}\,\mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \le 0. \tag{17}$$

This expression does not contain ϕ_2.

(Section 1.3: Version 4-00)

2 Chapter M-2

2.1 Section M-2.5

2.1.1 Section M-2.5.1

The following form:

$$\sigma = -\frac{2}{\pi} \int_{0<\xi<\infty,\, l_i n_i<0} \xi^3 l_j n_j f(\boldsymbol{X}, \xi \boldsymbol{l}) \mathrm{d}\xi \mathrm{d}\Omega(\boldsymbol{l}),$$

is more appropriate as Eq. (M-2.39b) than the one in the book. Then, the explanation of $\mathrm{d}\Omega(\boldsymbol{l})$, i.e.,

$\mathrm{d}\Omega(\boldsymbol{l})$ is the solid angle element in the direction of $\boldsymbol{l}$,

has to be inserted between ' where ' and ' T_w ' just after Eq. (M-2.39c).

(Section 2.1.1: Version 6-00)

3 Chapter M-3

3.1 Processes of solution of the systems in Section M-3.7.2 (July 2007)

The processes of solutions of the fluid-dynamic-type equations derived in Section M-3.7.1 are straightforward and may not need explanation. For the equations in Section M-3.7.2, some explanation may be better to be given. The discussion will be made on the basis of the boundary conditions in Section M-3.7.3 for a simple boundary where the shape of the boundary is invariant and its velocity component normal to it is zero.

3.1.1 "Incompressible Navier–Stokes set"

Consider the initial and boundary-value problem of Eqs. (M-3.265)–(M-3.268), i.e.,

$$\frac{\partial P_{S1}}{\partial x_i} = 0, \tag{18}$$

$$\frac{\partial u_{iS1}}{\partial x_i} = 0, \tag{19a}$$

$$\frac{\partial u_{iS1}}{\partial \tilde{t}} + u_{jS1}\frac{\partial u_{iS1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_{S2}}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{iS1}}{\partial x_j^2}, \tag{19b}$$

$$\frac{5}{2}\frac{\partial \tau_{S1}}{\partial \tilde{t}} - \frac{\partial P_{S1}}{\partial \tilde{t}} + \frac{5}{2}u_{jS1}\frac{\partial \tau_{S1}}{\partial x_j} = \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S1}}{\partial x_j^2}, \tag{19c}$$

$$\frac{\partial u_{iS2}}{\partial x_i} = -\frac{\partial \omega_{S1}}{\partial \tilde{t}} - \frac{\partial \omega_{S1} u_{iS1}}{\partial x_i}, \tag{20a}$$

$$\frac{\partial u_{iS2}}{\partial \tilde{t}} + u_{jS1}\frac{\partial u_{iS2}}{\partial x_j} + u_{jS2}\frac{\partial u_{iS1}}{\partial x_j}$$
$$= -\frac{1}{2}\left(\frac{\partial P_{S3}}{\partial x_i} - \omega_{S1}\frac{\partial P_{S2}}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{iS2}}{\partial x_j} + \frac{\partial u_{jS2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{kS2}}{\partial x_k}\delta_{ij}\right)$$
$$- \frac{\gamma_1 \omega_{S1}}{2}\frac{\partial^2 u_{iS1}}{\partial x_j^2} + \frac{\gamma_4}{2}\frac{\partial}{\partial x_j}\left[\tau_{S1}\left(\frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i}\right)\right] - \frac{\gamma_3}{3}\frac{\partial}{\partial x_i}\frac{\partial^2 \tau_{S1}}{\partial x_j^2}, \tag{20b}$$

$$\frac{3}{2}\frac{\partial P_{S2}}{\partial \tilde{t}} + \frac{3}{2}u_{jS1}\frac{\partial P_{S2}}{\partial x_j} + \frac{5}{2}\left(\frac{\partial P_{S1} u_{jS2}}{\partial x_j} - \frac{\partial \omega_{S2}}{\partial \tilde{t}} - \frac{\partial(\omega_{S2} u_{jS1} + \omega_{S1} u_{jS2})}{\partial x_j}\right)$$
$$= \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S2}}{\partial x_j^2} + \frac{5\gamma_5}{4}\frac{\partial}{\partial x_j}\left(\tau_{S1}\frac{\partial \tau_{S1}}{\partial x_j}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i}\right)^2, \tag{20c}$$

where

$$P_{S1} = \omega_{S1} + \tau_{S1}, \quad P_{S2} = \omega_{S2} + \omega_{S1}\tau_{S1} + \tau_{S2}. \tag{21}$$

From Eq. (18), P_{S1} is a function of $\tilde{t}$, i.e.,

$$P_{S1} = f_1(\tilde{t}). \tag{22}$$

In an unbounded-domain problem where the pressure at infinity is specified (or the pressure is specified at some point), $P_{S1} = f_1(\tilde{t})$ is known, but in a bounded-domain problem of a simple boundary, $f_1(\tilde{t})$ is unknown at this moment and is determined later. Let u_{iS1} and τ_{S1} as well as $f_1(\tilde{t})$ be given at time $\tilde{t}$ in such a way that u_{iS1} satisfies Eq. (19a). Taking the divergence of Eq. (19b) and using Eq. (19a), we have

$$\frac{\partial^2 P_{S2}}{\partial x_i^2} = -2\frac{\partial u_{jS1}}{\partial x_i}\frac{\partial u_{iS1}}{\partial x_j}. \tag{23}$$

On a simple boundary, the derivative of P_{S2} normal to it is found to be expressed with u_{iS1} and its space derivatives by multiplying Eq. (19b) by the normal vector to the boundary.[4] In the unbounded-domain problem, where $f_1(\tilde{t})$ is known, P_{S2} is determined by Eq. (23). In the bounded-domain problem, P_{S2} is determined by Eq. (23) except for an additive function of $\tilde{t}$ [say, $f_2(\tilde{t})$]. Anyway, $\partial P_{S2}/\partial x_i$ is independent of this ambiguity. From Eq. (19b), $\partial u_{iS1}/\partial \tilde{t}$ at $\tilde{t}$ is determined, irrespective of $f_2(\tilde{t})$, in such a way that $\partial(\partial u_{iS1}/\partial x_i)/\partial \tilde{t} = 0$ for the above choice of P_{S2}. Thus, the solution u_{iS1} of Eqs. (19a) and (19b) is determined by Eq. (19b) with the supplementary condition (23) instead of Eq. (19a). From Eq. (19c), $(5/2)\partial \tau_{S1}/\partial \tilde{t} - \partial P_{S1}/\partial \tilde{t}$ or $(5/2)\partial \tau_{S1}/\partial \tilde{t} - \mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ is determined, i.e.,

$$(5/2)\partial \tau_{S1}/\partial \tilde{t} - \mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t} = G(x_i, \tilde{t}), \tag{24}$$

where

$$G(x_i, \tilde{t}) = -\frac{5}{2}u_{jS1}\frac{\partial \tau_{S1}}{\partial x_j} + \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S1}}{\partial x_j^2}. \tag{25}$$

[4] The time-derivative term vanishes owing to the boundary condition mentioned in the first paragraph of Section 3.1.

Thus, τ_{S1} is determined in the unbounded-domain problem, but τ_{S1} has ambiguity owing to $f_1(\tilde{t})$ in the bounded-domain problem. The undetermined function $f_1(\tilde{t})$ is determined in the following way.

In the bounded-domain problem whose boundary is a simple boundary, the mass of the gas in the domain is invariant with respect to $\tilde{t}$. The condition at the leading order is

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \omega_{S1}\mathbf{d}\boldsymbol{x} = 0, \tag{26}$$

where V indicates the domain (or its volume in the later). With the aid of Eq. (21), we have

$$\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}V - \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} = 0. \tag{27}$$

On the other hand, from Eq. (24),

$$-\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}V + \frac{5}{2}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} = \int_V G(x_i,\tilde{t})\mathbf{d}\boldsymbol{x}. \tag{28}$$

From Eqs. (27) and (28), we obtain $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ and $\mathrm{d}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x}/\mathrm{d}\tilde{t}$ as

$$\left.\begin{aligned} \frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}} &= \frac{2}{3V}\int_V G(x_i,\tilde{t})\mathbf{d}\boldsymbol{x}, \\ \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} &= \frac{2}{3}\int_V G(x_i,\tilde{t})\mathbf{d}\boldsymbol{x}. \end{aligned}\right\} \tag{29}$$

That is, $f_1(\tilde{t})$ in the bounded-domain problem [and thus the solution τ_{S1} of Eq. (19c)] is determined.

The analysis of the higher-order equations is similar; for example, from Eqs. (20a)–(20c), u_{iS2}, τ_{S2}, and P_{S3} are determined in the unbounded-domain problem, but $f_2(\tilde{t})$, u_{iS2}, τ_{S2}, and P_{S3}, except for an additive function of $\tilde{t}$ in P_{S3}, are determined in the bounded-domain problem.[5] Let u_{iS2}, τ_{S2}, and $f_2(\tilde{t})$ be given at $\tilde{t}$ in such a way that Eq. (20a) is satisfied.[6] Taking the divergence of Eq. (20b) and using Eq. (20a) and the results obtained above, we find that P_{S3} is governed by the Poisson equation

$$\frac{\partial^2 P_{S3}}{\partial x_i^2} = \text{Inhomogeneous term}, \tag{30}$$

where the inhomogeneous term consists of u_{iS2}, P_{S2}, and the functions determined in the preceding analysis. On a simple boundary, the derivative of P_{S3} normal to it being known,[7] P_{S3} is determined by this equation, except for an additive function of $\tilde{t}$ [say, $f_3(\tilde{t})$] in the bounded-domain problem. Then, from

[5] Note that, with the aid of Eq. (21), the time-derivative term $\frac{3}{2}\partial P_{S2}/\partial\tilde{t} - \frac{5}{2}\partial\omega_{S2}/\partial\tilde{t}$ in Eq. (20c) is transformed into $\frac{5}{2}\partial\tau_{S2}/\partial\tilde{t} - \partial P_{S2}/\partial\tilde{t} + \frac{5}{2}\partial\omega_{S1}\tau_{S1}/\partial\tilde{t}$.

[6] The time derivative $\partial\omega_{S1}/\partial\tilde{t}$ is known from $\partial\tau_{S1}/\partial\tilde{t}$, $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$, and Eq. (21).

[7] Shift the discussion of the boundary condition for P_{S2} to the next order.

Eq. (20b), $\partial u_{iS2}/\partial \tilde{t}$ at $\tilde{t}$ is determined irrespective of $f_3(\tilde{t})$. From Eq. (20c), $\partial(3P_{S2} - 5\omega_{S2})/\partial \tilde{t}$ [or $\partial(5\tau_{S2} - 2P_{S2})/\partial \tilde{t}$] at $\tilde{t}$ is determined. Thus, u_{iS2} and τ_{S2} (except for the additive function $2f_2/5$ in the bounded-domain problem) [thus, ω_{S2} (except for the additive function $3f_2/5$)] are determined. In the bounded-domain problem, where the boundary is a simple boundary, the condition of invariance of the mass of the gas in the domain at the corresponding order is[8]

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_V \omega_{S2} \mathbf{d}\boldsymbol{x} = 0. \tag{31}$$

With the aid of Eq. (21), $\mathrm{d}f_2(\tilde{t})/\mathrm{d}\tilde{t}$ at $\tilde{t}$ is determined as $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ is done.

To summarize, the solution $(u_{iS1}, P_{S1}, \tau_{S1}, P_{S2})$ of the initial and boundary-value problem of Eqs. (18)–(19c) is determined, with an additive arbitrary function $f_2(\tilde{t})$ in P_{S2} in a bounded-domain problem of a simple boundary, when the initial data of u_{iS1}, P_{S1}, τ_{S1}, and P_{S2} satisfy Eqs. (19a) and (23). The additive function $f_2(\tilde{t})$ does not affect the other variables. The function $f_2(\tilde{t})$ is determined in the next-order analysis. In other words, the solution $(u_{iS1}, P_{S1}, \tau_{S1})$ of Eqs. (18)–(19c) is determined consistently by Eqs. (18), (19b), and (19c) with the supplementary condition (23), instead of Eq. (19a), when the initial data of u_{iS1}, P_{S1}, and τ_{S1} satisfy Eq. (19a). Naturally, the initial P_{S2} is required to satisfy Eq. (23). This process is natural for numerical computation.

3.1.2 Ghost-effect equations (M-3.275)–(M-3.278b):

Consider the initial and boundary-value problem of Eqs. (M-3.275)–(M-3.278b), i.e.,

$$\hat{p}_{SB0} = \hat{p}_0(\tilde{t}), \tag{32}$$

$$\hat{p}_{SB1} = \hat{p}_1(\tilde{t}), \tag{33}$$

$$\frac{\partial \hat{\rho}_{SB0}}{\partial \tilde{t}} + \frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial x_i} = 0, \tag{34a}$$

$$\begin{aligned}\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial \tilde{t}} &+ \frac{\partial \hat{\rho}_{SB0}\hat{v}_{jSB1}\hat{v}_{iSB1}}{\partial x_j} \\ &= -\frac{1}{2}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[\Gamma_1(\hat{T}_{SB0})\left(\frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\right] \\ &\quad + \frac{1}{2\hat{p}_0}\frac{\partial}{\partial x_j}\left\{\Gamma_7(\hat{T}_{SB0})\left[\frac{\partial \hat{T}_{SB0}}{\partial x_i}\frac{\partial \hat{T}_{SB0}}{\partial x_j} - \frac{1}{3}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2\delta_{ij}\right]\right\},\end{aligned} \tag{34b}$$

$$\frac{3}{2}\frac{\partial \hat{\rho}_{SB0}\hat{T}_{SB0}}{\partial \tilde{t}} + \frac{5}{2}\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}\hat{T}_{SB0}}{\partial x_i} = \frac{5}{4}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right), \tag{34c}$$

[8] The contribution of the Knudsen-layer correction to the mass in the domain is of a higher order, though it is required to ω_{S2}.

where $\hat{p}_0$ and $\hat{p}_1$ depend only on $\tilde{t}$, and

$$\left.\begin{aligned}&\hat{p}_{SB0} = \hat{\rho}_{SB0}\hat{T}_{SB0},\ \hat{p}_{SB1} = \hat{\rho}_{SB1}\hat{T}_{SB0} + \hat{\rho}_{SB0}\hat{T}_{SB1},\\ &\hat{p}_{SB2} = \hat{\rho}_{SB2}\hat{T}_{SB0} + \hat{\rho}_{SB1}\hat{T}_{SB1} + \hat{\rho}_{SB0}\hat{T}_{SB2},\end{aligned}\right\} \tag{35}$$

$$\hat{p}^*_{SB2} = \hat{p}_{SB2} + \frac{2}{3\hat{p}_0}\frac{\partial}{\partial x_k}\left(\Gamma_3(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right). \tag{36}$$

Let $\hat{\rho}$, $\hat{v}_i$, and $\hat{T}$ (thus, $\hat{p} = \hat{\rho}\hat{T}$) at time $\tilde{t}$ be given; thus, $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$ ($\hat{p}_{SB0}$), etc., including $\hat{p}_{SB2}$, are given. Then $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{\rho}_{SB0}\hat{v}_{iSB1}/\partial\tilde{t}$, and $\partial\hat{T}_{SB0}/\partial\tilde{t}$ at $\tilde{t}$ are given by Eqs. (34a)–(34c); thus, the future $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{T}_{SB0}$ (also $\hat{p}_{SB0}$) are determined. However, the future $\hat{p}_{SB0}$, as well as $\hat{p}_{SB0}$ at $\tilde{t}$, is required to be independent of x_i owing to Eq. (32). Taking this point into account, we discuss how the solution is determined. For convenience of the discussion, transform Eq. (34c) in the form

$$\frac{\partial \hat{p}_{SB0}}{\partial \tilde{t}} = \mathcal{P}, \tag{37}$$

where

$$\mathcal{P} = -\frac{5}{3}\hat{p}_{SB0}\frac{\partial \hat{v}_{iSB1}}{\partial x_i} + \frac{5}{6}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right).$$

First, consider the case where $\hat{p}$ (thus, $\hat{p}_{SB0}$, $\hat{p}_{SB1}$, etc.) is specified at some point, e.g., at infinity. Then, from Eq. (32), $\hat{p}_0(\tilde{t})$ is a given function of $\tilde{t}$, and $\hat{p}_{SB0}$ is determined. The initial value of $\hat{p}_{SB0}$ is uniform, i.e., $\hat{p}_{SB0} = \hat{p}_0(0)$. On the other hand, from Eq. (37), the variation of $\partial\hat{p}_{SB0}/\partial\tilde{t}$ is also determined by the data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$, and their space derivatives at $\tilde{t}$. This must coincide with the corresponding data given by Eq. (32), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$. Substituting this relation into Eq. (37), we have

$$\frac{\partial}{\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}}, \tag{38}$$

which requires a relation among $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ for all $\tilde{t}$, since $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is given. This condition is equivalently replaced by the following two conditions: The initial data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ are required to satisfy Eq. (38), and the time derivative of Eq. (38) has to be satisfied for all $\tilde{t}$, i.e.,

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}^2\hat{p}_0}{\mathrm{d}\tilde{t}^2}. \tag{39}$$

With the aid of Eqs. (34a)–(34c) and (37), the left-hand side of Eq. (39) is expressed in the form without the time-derivative terms, i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$, as follows:

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{1}{2}\hat{p}_{SB0}\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i}\right) + \ \mathrm{fn}_1,$$

where fn_1 is a given function of $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$, and their space derivatives. Thus, the condition (39) is reduced to an equation for $\hat{p}^*_{SB2}$, i.e.,

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i}\right) = \mathrm{Fn}, \tag{40}$$

where

$$\mathrm{Fn} = \frac{2}{\hat{p}_0}\left(\mathrm{fn}_1 + \frac{3}{5}\frac{\mathrm{d}^2\hat{p}_0}{\mathrm{d}\tilde{t}^2}\right).$$

The boundary condition for $\hat{p}^*_{SB2}$ in Eq. (40) on a simple boundary is derived by multiplying Eq. (34b) by the normal n_i to the boundary. In this process, the contribution of its time-derivative terms vanishes.[9] Thus, $\hat{p}^*_{SB2}$ (or $\hat{p}_{SB2}$) is determined in the present case, where $\hat{p}$ (thus, $\hat{p}_{SB2}$) is specified at some point. The solution $\hat{p}^*_{SB2}$ of Eq. (40) being substituted into Eq. (34b), Eqs. (34a)–(34c) with the first relation in Eq. (35) are reduced to the equations for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ which naturally determine $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$. Further, if the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ being chosen in such a way that $\hat{\rho}_{SB0}\hat{T}_{SB0}(=\hat{p}_{SB0}) = \hat{p}_0$ and that Eq. (38) is satisfied, the variation $\partial\hat{p}_{SB0}/\partial\tilde{t}$ of $\hat{p}_{SB0}(=\hat{\rho}_{SB0}\hat{T}_{SB0})$ given by these equations is consistent with Eq. (32), since Eq. (40) or (39) with the condition (38) at the initial state guarantees Eq. (38), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, for all $\tilde{t}$.

Equations (32) and (34a)–(34c) with Eqs. (35) and (36) determine $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{p}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$ consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. In the above system, $\hat{p}_{SB2}$ is determined. On the other hand, the variation $\partial\hat{p}_{SB2}/\partial\tilde{t}$ is determined independently by the counterpart of Eq. (37) at the order after next. The situation is similar to that at the leading order, where Eqs. (32), with a given $\hat{p}_0$, and (37) determine $\hat{p}_{SB0}$ independently. The analysis can be carried out in a similar way. Let $\hat{p}_{SB2}$ determined by Eq. (40) be indicated by $(\hat{p}_{SB2})_0$ and the equation for $\partial\hat{p}_{SB2}/\partial\tilde{t}$, or the counterpart of Eq. (37) at the order after next, be put in the form

$$\frac{\partial\hat{p}_{SB2}}{\partial\tilde{t}} = \mathcal{P}_2, \tag{41}$$

where $\mathcal{P}_2$ is a given function of $\hat{\rho}_{SBm}$, $\hat{v}_{iSBm+1}$ $\hat{T}_{SBm}$ $(m \le 2)$, and their space derivatives. For the consistency, $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ is substituted for $\partial\hat{p}_{SB2}/\partial\tilde{t}$ in Eq. (41), i.e.,

$$\mathcal{P}_2 = \frac{\partial(\hat{p}_{SB2})_0}{\partial\tilde{t}}, \tag{42}$$

where $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ is known. This requires a relation among $\hat{\rho}_{SBm}$, $\hat{v}_{iSBm+1}$ $\hat{T}_{SBm}$ $(m \le 2)$, and their space derivatives. This condition is equivalently replaced by the following two conditions: Equation (42) is applied only for the initial state, and the time derivative of Eq. (42), i.e.,

$$\frac{\partial\mathcal{P}_2}{\partial\tilde{t}} = \frac{\partial^2(\hat{p}_{SB2})_0}{\partial\tilde{t}^2},$$

[9]The discussion is similar to that in Footnote 4.

has to be satisfied for all $\tilde{t}$. The $\partial\hat{\rho}_{SBm}/\partial\tilde{t}$, $\partial\hat{v}_{iSBm+1}/\partial\tilde{t}$, $\partial\hat{T}_{SBm}/\partial\tilde{t}$ ($m \leq 2$) in $\partial\mathcal{P}_2/\partial\tilde{t}$ being replaced by the counterparts of Eqs. (34a)–(34c) and (37) at the corresponding order, an equation for $\hat{p}_{SB4}$ for all $\tilde{t}$ is derived.[10] The conclusion is that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced and, instead, that the condition (40) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ (thus also $\hat{p}_{SB0}$).

In this way, the solution of Eqs. (32), (34a)–(36) is determined consistently by Eqs. (34a)–(36) with the aid of the supplementary condition (40), instead of Eq. (32), when the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ satisfy Eqs. (32) and (38), where $\hat{p}_0(\tilde{t})$ is a known function of $\tilde{t}$ from the boundary condition.

Secondly, consider a bounded-domain problem of a simple boundary. In contrast to the first case, $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is unknown because no condition is imposed on $\hat{p}_{SB0}$ on a simple boundary. However, in a bounded-domain problem of a simple boundary, the mass of the gas in the domain is invariant with respect to $\tilde{t}$, i.e., at the leading order,

$$\frac{\mathrm{d}\displaystyle\int_V \hat{\rho}_{SB0}\mathbf{d}\boldsymbol{x}}{\mathrm{d}\tilde{t}} = 0, \tag{43}$$

where V indicates the domain under consideration. Using the first relation of Eq. (35), i.e., $\hat{\rho}_{SB0} = \hat{p}_0/\hat{T}_{SB0}$, in Eq. (43), we have

$$\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}}\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x} = \hat{p}_0\int_V \frac{1}{\hat{T}_{SB0}^2}\frac{\partial\hat{T}_{SB0}}{\partial\tilde{t}}\mathbf{d}\boldsymbol{x}. \tag{44}$$

Using Eq. (34c) for $\partial\hat{T}_{SB0}/\partial\tilde{t}$ in Eq. (44), we find that the variation $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is expressed with $\hat{p}_0$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ as follows:

$$\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}} = P(\tilde{t}), \tag{45}$$

where

$$P(\tilde{t}) = \hat{p}_0\int_V \frac{1}{\hat{T}_{SB0}^2}\left[\frac{5}{6\hat{\rho}_{SB0}}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right) - \frac{5}{3}\hat{v}_{iSB1}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right]\mathbf{d}\boldsymbol{x} \times\left(\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x}\right)^{-1}. \tag{46}$$

With this expression of $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, we can carry out the analysis in a similar way to that in the first case.

The variation $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ or $\partial\hat{p}_{SB0}/\partial\tilde{t}$ is also determined by Eq. (37). The two $\partial\hat{p}_{SB0}/\partial\tilde{t}$'s given by Eq. (45) with Eq. (46) and Eq. (37) have to be consistent.

[10] The conditions on the odd-order $\hat{p}_{SB2n+1}$'s are derived by the analysis starting from the condition (33) that $\hat{p}_{SB1}$ is independent of x_i.

Thus, substituting Eq. (45) with Eq. (46) into $\partial\hat{p}_{SB0}/\partial\tilde{t}$ in Eq. (37), we have

$$\frac{\partial}{\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1}-\frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right)=-\frac{3}{5}P(\tilde{t}), \tag{47}$$

where $P(\tilde{t})$ is given by Eq. (46). This must hold for all $\tilde{t}$ for consistency. This condition is equivalently replaced by the following two conditions: The initial data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$ are required to satisfy Eq. (47), and the time derivative of Eq. (47) has to be satisfied for all $\tilde{t}$, i.e.,

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1}-\frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\right)=-\frac{3}{5}\frac{\mathrm{d}P(\tilde{t})}{\mathrm{d}\tilde{t}}. \tag{48}$$

Using Eqs. (34a), (34b), and (37) for the time derivatives $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{v}_{iSB1}/\partial\tilde{t}$, and $\partial\hat{p}_{SB0}/\partial\tilde{t}$ in Eq. (48), we find that $\hat{p}^*_{SB2}$ at $\tilde{t}$ is determined by the equation

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right)+\mathcal{L}\left(\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right)=\mathrm{Fn}, \tag{49}$$

where Fn is a given functional of $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$, and their space derivatives, and $\mathcal{L}(\partial\hat{p}^*_{SB2}/\partial x_i)$ is a given linear functional of $\partial\hat{p}^*_{SB2}/\partial x_i$, i.e.,

$$\mathcal{L}\left(\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right)=-\frac{1}{\hat{p}_0}\int_V\frac{1}{\hat{T}_{SB0}}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\mathbf{d}\boldsymbol{x}\left(\int_V\frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x}\right)^{-1}.$$

On a simple boundary, the derivative of $\hat{p}^*_{SB2}$ normal to the boundary is specified. Thus, $\hat{p}^*_{SB2}$ is determined except for an additive function of $\tilde{t}$. The solution $\hat{p}^*_{SB2}$ of Eq. (49) being substituted into Eq. (34b), the result is independent of the additive function. Thus, Eqs. (34a)–(34c) with the first relation in Eq. (35) and the above $\hat{p}^*_{SB2}$ substituted are reduced to those for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$, which naturally determine $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$. Further, if the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ being chosen in such a way that $\hat{\rho}_{SB0}\hat{T}_{SB0}(=\hat{p}_{SB0})=\hat{p}_0$ and that Eq. (47) is satisfied, the variation $\partial\hat{p}_{SB0}/\partial\tilde{t}$ of $\hat{p}_{SB0}(=\hat{\rho}_{SB0}\hat{T}_{SB0})$ given by these equations is consistent with Eq. (32), since Eq. (49) or (48) with the condition (47) at the initial state guarantees Eq. (47), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t}=\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, for all $\tilde{t}$.

Equations (32) and (34a)–(34c) with Eqs. (35) and (49) determine $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{p}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$, except for an additive function of $\tilde{t}$ in $\hat{p}_{SB2}$, consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. First, the undetermined additive function in $\hat{p}_{SB2}$ is determined by the condition of invariance of the mass of the gas in the domain at the order after next as $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is determined.[11] The $\partial\hat{p}_{SB2}/\partial\tilde{t}$ or $\hat{p}_{SB2}$ determined in this way is indicated by $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ or $(\hat{p}_{SB2})_0$. On the other hand, the

[11] The Knudsen-layer correction to $\hat{\rho}_{SB1}$, already determined (see Footnote 10), contributes to the mass at this order.

variation $\partial\hat{p}_{SB2}/\partial\tilde{t}$ is determined independently by Eq. (41) or the counterpart of Eq. (37) at the order after next. The two results must coincide. The discussion from here is the same as that given from the sentence starting from Eq. (41) to the end of the paragraph. The results are that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced, and that the condition (49) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ (thus also $\hat{p}_{SB0}$).

In this way, the solution of Eqs. (32), (34a)–(34c) is determined consistently by Eqs. (34a)–(34c) with the aid of the supplementary condition (49), instead of Eq. (32), when the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ satisfy Eqs. (32) and (47).

3.2 Notes on basic equations in classical fluid dynamics

3.2.1 Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

The mass, momentum, and energy-conservation equations of fluid flow are given by

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial X_i}(\rho v_i)=0, \tag{50}$$

$$\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial}{\partial X_j}(\rho v_i v_j+p_{ij})=0, \tag{51}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{1}{2}v_i^2\right)\right]+\frac{\partial}{\partial X_j}\left[\rho v_j\left(e+\frac{1}{2}v_i^2\right)+v_ip_{ij}+q_j\right]=0, \tag{52}$$

where ρ is the density, v_i is the flow velocity, e is the internal energy per unit mass, p_{ij}, which is symmetric with respect to i and j, is the stress tenor, and q_i is the heat-flow vector. The pressure p and the internal energy e are given by the equations of state as functions of T and ρ, i.e.,

$$p=p(T,\rho),\quad e=e(T,\rho). \tag{53}$$

Especially, for a perfect gas,

$$p=R\rho T,\quad e=e(T). \tag{54}$$

Equations (51) and (52) are rewritten with the aid of Eq. (50) in the form

$$\rho\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial X_j}+\frac{\partial p_{ij}}{\partial X_j}=0, \tag{55}$$

$$\rho\frac{\partial}{\partial t}\left(e+\frac{1}{2}v_i^2\right)+\rho v_j\frac{\partial}{\partial X_j}\left(e+\frac{1}{2}v_i^2\right)+\frac{\partial}{\partial X_j}(v_ip_{ij}+q_j)=0. \tag{56}$$

The operator $\partial/\partial t + v_j \partial/\partial x_j$, which expresses the time variation along the fluid particle, is denoted by $\mathrm{D}/\mathrm{D}t$, i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}.$$

Multiplying Eq. (55) by v_i we obtain the equation for the variation of kinetic energy as

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v_i^2\right) = -v_i \frac{\partial p_{ij}}{\partial X_j}. \tag{57}$$

Another form of Eq. (52), where Eq. (57) is subtracted from Eq. (56), is given as

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p_{ij}\frac{\partial v_i}{\partial X_j} - \frac{\partial q_j}{\partial X_j}. \tag{58}$$

Noting the thermodynamic relation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = T\frac{\mathrm{D}s}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{59}$$

where s is the entropy per unit mass, and Eq. (50), Eq. (58) is rewritten as

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = -\frac{1}{T}\left[(p_{ij} - p\delta_{ij})\frac{\partial v_i}{\partial X_j} + \frac{\partial q_j}{\partial X_j}\right]. \tag{60}$$

Equation (60) expresses the variation of the entropy of a fluid particle.

Equations (50)–(53) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor p_{ij} and the heat-flow vector q_i are assumed in some ways. The *Navier–Stokes set of equations* (or the *Navier–Stokes equations*) is Eqs. (50)–(53) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right) - \mu_B \frac{\partial v_k}{\partial X_k}\delta_{ij}, \tag{61}$$

$$q_i = -\lambda \frac{\partial T}{\partial X_i}, \tag{62}$$

where μ, μ_B, and λ are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of T and ρ. The *Euler set of equations* (or the *Euler equations*) is Eqs. (50)–(53) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij}, \quad q_i = 0, \tag{63}$$

or the Navier–Stokes equations with $\mu = \mu_B = \lambda = 0$.

For the Navier–Stokes equations, in view of the relations (61) and (62), the entropy variation is expressed in the form[12]

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T}\left[\frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)^2 + \mu_B\left(\frac{\partial v_k}{\partial X_k}\right)^2 + \frac{\partial}{\partial X_i}\left(\lambda\frac{\partial T}{\partial X_i}\right)\right]. \tag{64}$$

For the Euler equations, for which p_{ij} and q_i are given by Eq.(63), the entropy of a fluid particle is invariant, i.e.,

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = 0. \tag{65}$$

For an *incompressible fluid*, the first relation of Eq. (53) is replaced by[13]

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial t} + v_j\frac{\partial\rho}{\partial X_j} = 0. \tag{66}$$

Thus, from Eqs. (50) and (66),

$$\frac{\partial v_i}{\partial X_i} = 0. \tag{67}$$

Equation (61) for the Navier–Stokes-stress tensor reduces to

$$p_{ij} = p\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right). \tag{68}$$

The first term on the right-hand side of Eq. (58) reduces to

$$\begin{aligned}-p_{ij}\frac{\partial v_i}{\partial X_j} &= -\left[p\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)\right]\frac{\partial v_i}{\partial X_j} \\ &= \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2.\end{aligned}$$

Thus, Eq. (58) reduces to

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda\frac{\partial T}{\partial X_j}\right). \tag{69}$$

[12]Note the following transformation:

$$\begin{aligned}&\frac{\partial v_i}{\partial X_j}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right) \\ &= \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij} + \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right) \\ &= \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right)^2 + \frac{1}{3}\frac{\partial v_l}{\partial X_l}\delta_{ij}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right).\end{aligned}$$

The second term in the last expression is easily seen to vanish.

[13]The density is invariant along fluid-particle paths. If ρ is of uniform value ρ_0 initially, it is a constant, i.e.,

$$\rho = \rho_0.$$

In a time-independent (or steady) problem, the density is constant along streamlines.

To summarize, the Navier–Stokes equations for incompressible fluid are

$$\frac{\partial v_i}{\partial X_i} = 0, \tag{70a}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j}\left[\mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)\right], \tag{70b}$$

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda \frac{\partial T}{\partial X_j}\right), \tag{70c}$$

with the incompressible condition (66) being supplemented, i.e.,

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{71}$$

3.2.2 Boundary condition for the Euler set

In Section M-3.5, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentum-conservation equations of the Euler set are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{72}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{73}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{74}$$

where ρ is the density, which is uniform, (u, v) is the flow velocity, and p is the pressure. Owing to Eq. (72), the stream function Ψ can be introduced as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{75}$$

Eliminating p from Eqs. (73) and (74), we have[14]

$$u\frac{\partial\Omega}{\partial x}+v\frac{\partial\Omega}{\partial y}=0, \tag{76}$$

where Ω is the vorticity, i.e.,

$$\Omega=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}. \tag{77}$$

From Eqs. (75) and (76),

$$\frac{\partial\Psi}{\partial y}\frac{\partial\Omega}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\Omega}{\partial y}=0. \tag{78}$$

This equation shows that Ω is a function of Ψ,[15] i.e.,

$$\Omega=F(\Psi). \tag{79}$$

[14]The following equation is formed from them:

$$\partial\text{Eq.}\,(73)/\partial y-\partial\text{Eq.}\,(74)/\partial x=0.$$

[15]This can be seen with the aid of theorems on implicit functions (see References M-[47, 48, 267]). The proof is outlined here. The Ω and Ψ are functions of x and y :

$$\Omega=\Omega(x,y),\quad \Psi=\Psi(x,y). \tag{$*$}$$

Solving the second equation with respect to x, we have

$$x=\hat{x}(\Psi,y). \tag{$**$}$$

With this relation into Eq. $(*)$,

$$\Omega=\Omega(\hat{x}(\Psi,y),y)=\hat{\Omega}(\Psi,y), \tag{$\sharp$a}$$

$$\Psi=\Psi(\hat{x}(\Psi,y),y)=\hat{\Psi}(\Psi,y). \tag{$\sharp$b}$$

That is, Ω is expressed as a function of Ψ and y. From Eqs. ($\sharp$a) and ($\sharp$b),

$$\frac{\partial\hat{\Omega}(\Psi,y)}{\partial y}=\frac{\partial\Omega(\hat{x}(\Psi,y),y)}{\partial y}=\frac{\partial\Omega(x,y)}{\partial x}\frac{\partial\hat{x}(\Psi,y)}{\partial y}+\frac{\partial\Omega(x,y)}{\partial y}, \tag{$\sharp\sharp$a}$$

$$\frac{\partial\hat{\Psi}(\Psi,y)}{\partial y}=0. \tag{$\sharp\sharp$b}$$

On the other hand,

$$\frac{\partial\hat{\Psi}(\Psi,y)}{\partial y}=\frac{\partial\Psi(\hat{x}(\Psi,y),y)}{\partial y}=\frac{\partial\Psi(x,y)}{\partial x}\frac{\partial\hat{x}(\Psi,y)}{\partial y}+\frac{\partial\Psi(x,y)}{\partial y}.$$

Thus,

$$\frac{\partial\Psi(x,y)}{\partial x}\frac{\partial\hat{x}(\Psi,y)}{\partial y}+\frac{\partial\Psi(x,y)}{\partial y}=0. \tag{$\ddagger$}$$

From Eqs. (78), ($\sharp\sharp$a) and ($\ddagger$), we have

$$\frac{\partial\hat{\Omega}(\Psi,y)}{\partial y}=0,\quad\text{or}\quad\Omega=\hat{\Omega}(\Psi).$$

This functional relation between Ω and Ψ is a local relation, and therefore F may be a multivalued function of Ψ. From Eqs. (77) and (79),

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=F(\Psi). \tag{80}$$

Consider a boundary-value problem in a simply-connected bounded domain, where Ψ is given on the boundary ($\Psi=\Psi_B$). Introduce a coordinate s ($0\le s<S$) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where $\partial\Psi_B/\partial s<0$, and the fluid flows out from the domain on the boundary where $\partial\Psi_B/\partial s>0$. When F is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose F on the part where $\partial\Psi_B/\partial s<0$ or $\partial\Psi_B/\partial s>0$. For example, take the case where $\partial\Psi_B/\partial s<0$ for $0<s<S_m$ and $\partial\Psi_B/\partial s>0$ for $S_m<s<S$, and choose the distribution $\Omega_B(s)$ of Ω along the boundary for the part $0<s<S_m$. By the choice of Ω_B, the function $F(\Psi)$ is determined in the following way. Inverting the relation $\Psi=\Psi_B(s)$ between Ψ and s on the part $0<s<S_m$, i.e., $s(\Psi)$, and noting the relation (79), we find that F is given by

$$F(\Psi)=\Omega_B(s(\Psi)). \tag{81}$$

Then, the boundary-value problem is fixed. That is, Eq. (80) is fixed as[16]

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=\Omega_B(s(\Psi)), \tag{82}$$

and the boundary condition is given as $\Psi=\Psi_B(s)$. This system is a standard from the point of counting of the number of boundary conditions. Obviously, from Eq. (77), the solution of the above system automatically satisfies condition $\Omega=\Omega_B(s)$ along the boundary for $0<s<S_m$. We cannot choose the distribution of Ω on the boundary for $S_m<s<S$.

The energy-conservation equation of the incompressible Euler set is given by Eq. (69) with $\mu=\lambda=0$, i.e.,

$$u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}=0,\quad\text{or}\quad \frac{\partial\Psi}{\partial y}\frac{\partial e}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial e}{\partial y}=0, \tag{83}$$

where e is the internal energy. Thus, e is a function of Ψ, i.e.,

$$e=F_1(\Psi). \tag{84}$$

In the above boundary-value problem, therefore, e can be specified on the the part ($0<s<S_m$) of the boundary, but no condition can be specified on other part ($S_m<s<S$) and vice versa.[17]

[16] There is still some ambiguity. The case where there is a region with closed stream lines $\Psi(x,y)=$const inside the domain is not excluded.

[17] From the second relation on e of Eq. (53) and the uniform-density condition, the condition on e can be replaced by the condition on the temperature T.

To summarize, we can specify three conditions for Ψ, Ω, and e on the part $\partial\Psi_B/\partial s < 0$ ($\partial\Psi_B/\partial s > 0$) of boundary but one condition for Ψ on the other part $\partial\Psi_B/\partial s > 0$ ($\partial\Psi_B/\partial s < 0$). The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

3.2.3 Ambiguity of pressure in the incompressible Navier–Stokes system

It may be better to note ambiguity of the solution of the initial and boundary-value problem of the incompressible Navier–Stokes equations in a bounded domain of simple boundaries.

Consider the Navier–Stokes equations for an incompressible fluid, i.e.,

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{85a}$$

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{85b}$$

$$\rho\frac{\partial e}{\partial t} + \rho v_j\frac{\partial e}{\partial X_j} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda\frac{\partial T}{\partial X_j}\right), \tag{85c}$$

$$\frac{\partial \rho}{\partial t} + v_i\frac{\partial \rho}{\partial x_i} = 0, \tag{85d}$$

where e, μ, and λ are functions of T and ρ.

Consider the initial and boundary-value problem of Eqs. (85a)–(85d) in a bounded domain D on the boundary ∂D of which v_i and T are specified as $v_i = v_{wi}$ and $T = T_w$ (v_{wi} and T_w are, respectively, the surface velocity and temperature of the boundary satisfying $\int_{\partial D} v_{wi}n_i \mathrm{d}S = 0$, n_i : the unit normal vector to the boundary) and no condition is imposed on ρ and p. Let $(v_i^{(s)}, \rho^{(s)}, T^{(s)}, p^{(s)})$ be a solution of the initial and boundary-value problem. Let $P^{(a)}$ be an arbitrary function of t, independent of x_i, that vanishes at initial time $t = 0$, i.e., $P^{(a)} = f(t)$ with $f(0) = 0$. Put

$$(v_i, \rho, T, p) = (v_i^{(s)}, \rho^{(s)}, T^{(s)}, p^{(s)} + P^{(a)}).$$

Then, e, μ, and λ corresponding to the new (v_i, ρ, T, p) are equal to $e^{(s)}$, $\mu^{(s)}$, and $\lambda^{(s)}$ respectively, because they are determined by ρ and T. The new (v_i, ρ, T, p) satisfy the equations (85a)–(85d) and the initial and boundary conditions.

3.2.4 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set. This statement should be made precise. The difference is briefly explained in the book "Molecular Gas Dynamics" in connection with the equations derived by the S expansion from the Boltzmann equation in Sections M-3.2.4 and M-3.7.2. Here, we explicitly show the process of analysis from the compressible Navier–Stokes set. The resulting set of equations no longer has ambiguity of pressure in contrast to the incompressible Navier–Stokes set. Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the nondimensional variables introduced by Eq. (M-1.74) in Section M-1.10 as follows:

$$\mathrm{Sh}\frac{\partial\omega}{\partial\hat{t}}+\frac{\partial(1+\omega)u_i}{\partial x_i}=0, \tag{86}$$

$$\mathrm{Sh}\frac{\partial(1+\omega)u_i}{\partial\hat{t}}+\frac{\partial}{\partial x_j}\left((1+\omega)u_iu_j+\frac{1}{2}P_{ij}\right)=0, \tag{87}$$

$$\begin{aligned}&\mathrm{Sh}\frac{\partial}{\partial\hat{t}}\left[(1+\omega)\left(\frac{3}{2}(1+\tau)+u_i^2\right)\right]\\&+\frac{\partial}{\partial x_j}\left[(1+\omega)u_j\left(\frac{3}{2}(1+\tau)+u_i^2\right)+u_i(\delta_{ij}+P_{ij})+Q_j\right]=0.\end{aligned} \tag{88}$$

The nondimensional stress tensor P_{ij}, and heat-flow vector Q_i are expressed as[18]

$$P_{ij}=P\delta_{ij}-\frac{\mu_0(2RT_0)^{1/2}}{p_0L}(1+\bar{\mu})\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \tag{89a}$$

$$Q_i=-\frac{\lambda_0T_0}{Lp_0(2RT_0)^{1/2}}(1+\bar{\lambda})\frac{\partial\tau}{\partial x_i}. \tag{89b}$$

Here, $\bar{\mu}$ and $\bar{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

$$\mu=\mu_0(1+\bar{\mu}),\quad \lambda=\lambda_0(1+\bar{\lambda}),$$

where μ_0 and λ_0 are, respectively, the values of the viscosity μ and the thermal conductivity λ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of τ and ω. The first relation of the equation of state [Eq. (54)] is expressed as

$$P=\omega+\tau+\omega\tau. \tag{90}$$

Take a small parameter ε, and consider the case where

$$u_i=O(\varepsilon),\quad \omega=O(\varepsilon),\quad \tau=O(\varepsilon),\quad \mathrm{Sh}=O(\varepsilon), \tag{91a}$$

$$\frac{\mu_0(2RT_0)^{1/2}}{p_0L}=\gamma_1\varepsilon,\quad \frac{\lambda_0T_0}{Lp_0(2RT_0)^{1/2}}=\frac{5}{4}\gamma_2\varepsilon, \tag{91b}$$

thus,

$$P=O(\varepsilon),\quad \bar{\mu}=O(\varepsilon),\quad \bar{\lambda}=O(\varepsilon).$$

[18] For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B=0$.

According to the definition of u_i in Eq. (M-1.74), ε is of the order of the Mach number. In view of this and the definition of the Prandtl number $\mathrm{Pr} = 5R\mu/2\lambda$ (see Section M-3.1.9), γ_1 and γ_2 are, respectively, of the orders of $1/\mathrm{Re}$ and $1/\mathrm{PrRe}$ (Re : the Reynolds number). According to Eq. (M-1.48a), the condition $\mathrm{Sh} = O(\varepsilon)$ in Eq. (91a) means that the time scale t_0 of the variation of variables is of the order of $L/(2RT_0)^{1/2}\varepsilon$, which is of the order of time scale of viscous diffusion. Thus, we are considering the case where the Mach number is small, the Reynolds and Prandtl numbers are of the order of unity, and the time scale of variation of the system is of the order of the time scale of viscous diffusion. We can take $t_0 = L/(2RT_0)^{1/2}\varepsilon$ without loss of generality. Then,

$$\mathrm{Sh} = \varepsilon. \tag{92}$$

Corresponding to the above situation, u_i, ω, P, and τ are expanded in power series of ε, i.e.,

$$u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots, \tag{93a}$$
$$\omega = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots, \tag{93b}$$
$$P = P_1\varepsilon + P_2\varepsilon^2 + \cdots, \tag{93c}$$
$$\tau = \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots, \tag{93d}$$
$$\bar{\mu} = \bar{\mu}_1\varepsilon + \bar{\mu}_2\varepsilon^2 + \cdots, \tag{93e}$$
$$\bar{\lambda} = \bar{\lambda}_1\varepsilon + \bar{\lambda}_2\varepsilon^2 + \cdots, \tag{93f}$$
$$P_{ij} = P_1\delta_{ij}\varepsilon + P_{ij2}\varepsilon^2 + \cdots, \tag{93g}$$
$$Q_i = Q_{i2}\varepsilon^2 + \cdots. \tag{93h}$$

Substituting Eqs. (93a)–(93h) with Eqs. (91b) and (92) into Eqs. (86)–(88) with Eqs. (89a) and (89b), and arranging the same-order terms of ε, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \quad \frac{\partial P_1}{\partial x_i} = 0, \quad \frac{\partial u_{i1}}{\partial x_i} = 0,$$

$$\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial \omega_1 u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} = 0,$$
$$\frac{\partial u_{i1}}{\partial \hat{t}} + \frac{\partial u_{i1}u_{j1}}{\partial x_j} + \frac{1}{2}\frac{\partial P_2}{\partial x_i} - \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right) = 0,$$
$$\frac{3}{2}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_j}\left(\frac{5}{2}u_{j2} + \frac{5}{2}P_1 u_{j1} - \frac{5}{4}\gamma_2\frac{\partial \tau_1}{\partial x_j}\right) = 0,$$

and so on. At the leading order, the equations derived from Eqs. (86) and (88) degenerate into the same equation $\partial u_{i1}/\partial x_i = 0$. Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{94}$$

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{95a}$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{95b}$$

$$\frac{5}{2}\frac{\partial \tau_1}{\partial \hat{t}} - \frac{\partial P_1}{\partial \hat{t}} + \frac{5}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{95c}$$

$$\frac{\partial u_{i2}}{\partial x_i} = -\frac{\partial \omega_1}{\partial \hat{t}} - \frac{\partial \omega_1 u_{i1}}{\partial x_i}, \tag{96a}$$

$$\begin{aligned}&\frac{\partial u_{i2}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i2}}{\partial x_j} + u_{j2}\frac{\partial u_{i1}}{\partial x_j}\\ &\quad= -\frac{1}{2}\left(\frac{\partial P_3}{\partial x_i} - \omega_1\frac{\partial P_2}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right)\\ &\qquad - \frac{\gamma_1\omega_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left[\bar{\mu}_1\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)\right],\end{aligned} \tag{96b}$$

$$\begin{aligned}&\frac{3}{2}\frac{\partial P_2}{\partial \hat{t}} + \frac{3}{2}u_{j1}\frac{\partial P_2}{\partial x_j} + \frac{5}{2}\left(P_1\frac{\partial u_{j2}}{\partial x_j} - \frac{\partial \omega_2}{\partial \hat{t}} - \frac{\partial(\omega_1 u_{j2} + \omega_2 u_{j1})}{\partial x_j}\right)\\ &\quad= \frac{5\gamma_2}{4}\frac{\partial}{\partial x_i}\left(\frac{\partial \tau_2}{\partial x_i} + \bar{\lambda}_1\frac{\partial \tau_1}{\partial x_i}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)^2,\end{aligned} \tag{96c}$$

where

$$P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1\tau_1. \tag{97}$$

These equations are very similar to Eqs. (M-3.265)–(M-3.268) [or Eqs. (18)–(21)] obtained by the S expansion of the Boltzmann equation in Section M-3.7.2 (or Section 3.1.1). The solution is determined in the same way as the solution of the S-expansion system is done in Section 3.1.1. What should be noted is the determination of $P_1, P_2, \cdots$ in a bounded-domain problem. They are determined by the condition of invariance of the mass of the gas in the domain with the aid of higher-order equations in the same way as $P_{S1}, P_{S2}, \cdots$ in the S-expansion system (see Section 3.1.1).

In order to compare Eqs. (95a)–(95c) and (97) with the incompressible Navier–Stokes equations (85a)–(85d), we will rewrite the latter equations for the situation where the former equations are derived. The starting equations are Eqs. (86)–(89b)[19] and the nondimensional form of Eq. (66), i.e.,

$$\mathrm{Sh}\frac{\partial \omega}{\partial \hat{t}} + u_i\frac{\partial \omega}{\partial x_i} = 0, \tag{98}$$

instead of Eq. (90).[20] The analysis is carried out in a similar way and the

[19] As the internal energy e, $3RT/2$ [$= 3RT_0(1+\tau)/2$] is chosen for consistency.

[20] From Eqs. (86) and (98), we have $\partial u_i/\partial x_i = 0$.

equations corresponding to Eqs. (95a)–(95c) are[21]

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{99a}$$

$$\frac{\partial u_{i1}}{\partial \hat{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{99b}$$

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \hat{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{99c}$$

Equations (99a) and (99b) are, respectively, of the same form as Eqs. (95a) and (95b). Equation (95c) is rewritten with the aid of Eqs. (94) and (97) as

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \hat{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} - \left(\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right) = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{100}$$

The difference of Eq. (95c) or (100) from Eq. (99c) is

$$\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i},$$

which vanishes for an incompressible fluid. The work W done on unit volume of fluid by pressure, given by $-p_0(2RT_0)^{1/2}\partial(1+P)u_i/\partial x_i$, is transformed with the aid of Eq. (96a) in the following way:

$$\begin{aligned}\frac{W}{p_0(2RT_0)^{1/2}} &= -\frac{\partial(1+P)u_i}{\partial x_i} \\ &= -\frac{\partial u_{i1}}{\partial x_i}\varepsilon - \left(P_1\frac{\partial u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i}\right)\varepsilon^2 + \cdots \\ &= -\frac{\partial u_{i2}}{\partial x_i}\varepsilon^2 + \cdots \\ &= \left(\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right)\varepsilon^2 + \cdots .\end{aligned}$$

The work vanishes up to the order considered here for an incompressible fluid, because $\partial u_i/\partial x_i = 0$ and $\partial P_1/\partial x_i = 0$ (see Footnote 21). That is, Eq. (95c) differs from Eq. (99c) by the amount of the work done by pressure.

To summarize, the mass and momentum-conservation equations of the set derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small are of the same form as the corresponding equations for the incompressible Navier–Stokes set, but the energy-conservation equation differs by the work done by pressure.[22] Incidentally, we have already

[21] We also obtain $\partial P_1/\partial x_i = 0$.

[22] When the density ρ is uniform initially, for which ρ is a constant for an incompressible fluid, the viscosity and thermal conductivity are constants, and heat production by viscosity is neglected, Eqs. (99a)–(99c) can be compared directly with Eqs. (95a)–(95c) and (97), without carrying expansion, and the same results are obtained.

seen that the pressure in the incompressible Navier–Stokes set has ambiguity of an additive function of time irrespective of the size of the parameters in contrast to the pressure in the former set derived from the compressible Navier–Stokes set.

4 Chapter M-4

4.1 Gas over a plane interface: Supplement to M-4.4

Here, the discussion of the half-space problem under the boundary condition (M-1.26) for a simple boundary in Section M-4.4 is extended to that under the boundary condition (M-1.30) or (166) for an interface of a gas and its condensed phase. That is, a plane simple boundary is replaced by a plane condensed phase of the gas, and the possible solution including the possible state at infinity is discussed in the situation when no evaporation or condensation is taking place on the condensed phase. This is the problem first discussed by Golse under the complete condensation condition (Reference M-29), which is a special case of the boundary condition (M-1.30). The analysis goes parallel to that in Section M-4.4. The full explanation is given with the difference being shown in Footnotes, though it may be redundant.

Consider a semi-infinite expanse of a gas ($X_1 > 0$) bounded by its stationary plane condensed phase with a uniform temperature T_w at $X_1 = 0$. There is no external force acting on the gas. The state of the gas is time-independent and uniform with respect to X_2 and X_3, i.e., $f = f(X_1, \boldsymbol{\xi})$, and it approaches an equilibrium state as $X_1 \to \infty$, i.e.,

$$f \to \frac{\rho_\infty}{(2\pi R T_\infty)^{3/2}} \exp\left(-\frac{(\xi_i - v_{i\infty})^2}{2RT_\infty}\right) \quad \text{as } X_1 \to \infty, \tag{101}$$

where ρ_∞, $v_{i\infty}$, and T_∞ are bounded. The boundary condition on the interface is given by Eq. (166) with the conditions (167a)–(167c) and (170), i.e.,

$$f(0, \boldsymbol{\xi}) = g_I + \int_{\xi_{1*}<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(0, \boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_* \quad (\xi_1 > 0). \tag{102}$$

Here, we are interested in the case where no evaporation or condensation is taking place on the condensed phase,[23] i.e.,

$$\rho v_1 = \int \xi_1 f \mathbf{d}\boldsymbol{\xi} = 0 \quad \text{at } X_1 = 0. \tag{103}$$

We will show that the solution of the Boltzmann equation (M-1.5), i.e.,

$$\xi_1 \frac{\partial f}{\partial X_1} = J(f, f), \tag{104}$$

[23] No mass flux across the boundary irrespective of a situation is the definition of a simple boundary.

describing the above situation exists only when

$$v_{i\infty} = 0, \quad \rho_\infty = \rho_w, \quad T_\infty = T_w,$$

where ρ_w is the saturation gas density at temperature T_w, and that the solution is uniquely given by the Maxwellian

$$f = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{\xi_i^2}{2RT_w}\right). \tag{105}$$

From the integral of the Boltzmann equation (104) over the whole space of $\boldsymbol{\xi}$ [or the conservation equation (M-1.12)], i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}X_1}\left(\int \xi_1 f \mathbf{d}\boldsymbol{\xi}\right) = 0,$$

and Eq. (103), we find that the mass flux vanishes for $X_1 \geq 0$, i.e.,

$$\int \xi_1 f \mathbf{d}\boldsymbol{\xi} = 0 \quad (0 \leq X_1 < \infty). \tag{106}$$

With this result in the condition (101) at infinity, we have

$$\int \xi_1 \xi_i^2 f \mathbf{d}\boldsymbol{\xi} = 0 \quad \text{at infinity.} \tag{107}$$

The integral of the Boltzmann equation (104) multiplied by ξ_j^2 over the whole space of $\boldsymbol{\xi}$ [or the conservation equation (M-1.14)] gives

$$\frac{\mathrm{d}}{\mathrm{d}X_1}\left(\int \xi_1 \xi_j^2 f \mathbf{d}\boldsymbol{\xi}\right) = 0. \tag{108}$$

Thus, from Eqs. (107) and (108), we have

$$\int \xi_1 \xi_j^2 f \mathbf{d}\boldsymbol{\xi} = 0 \quad (0 \leq X_1 < \infty). \tag{109}$$

For the boundary condition (166) with the conditions (167a)–(167c) and (170), the following inequality holds at $X_1 = 0$ [Eq. (189) with $\rho v_1 = 0$, $v_{wi} = 0$, $n_i = (1, 0, 0)$]:[24]

$$\int \xi_1 f \ln(f/f_w) \mathbf{d}\boldsymbol{\xi} \leq 0, \tag{110}$$

where f_w is the Maxwellian with the temperature T_w and velocity v_{wi} $(= 0)$ of the condensed phase and the saturation gas density ρ_w at temperature T_w, i.e.,

$$f_w = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{\xi_i^2}{2RT_w}\right). \tag{111}$$

[24] The same equality holds for a simple boundary except that ρ_w in f_w is a free parameter for this case (see Section M-4.4).

With the aid of Eqs. (106) and (109),

$$\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi} \le \int \xi_1 f \ln(f_w/c_0)\mathbf{d}\boldsymbol{\xi}$$
$$= -\frac{1}{2RT_w}\int \xi_1\xi_i^2 f\mathbf{d}\boldsymbol{\xi} = 0 \quad \text{at } X_1 = 0, \tag{112}$$

where c_0 is a constant to make the argument of the logarithmic function dimensionless, whose choice does not influence the result.

On the other hand, from the H theorem, i.e., Eq. (M-1.36), in a time-independent one-dimensional case,

$$-\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\Big|_{X_1=0} + \int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\Big|_{X_1=\infty} = \int_0^\infty G\mathrm{d}X_1 \le 0, \tag{113}$$

where

$$G = -\frac{1}{4m}\int (f'f'_* - ff_*)\ln\left(\frac{f'f'_*}{ff_*}\right) B\mathrm{d}\Omega\mathbf{d}\boldsymbol{\xi}_*\mathbf{d}\boldsymbol{\xi} \le 0.$$

From Eqs. (101), (106), and (107), the second term on the left-hand side of Eq. (113) vanishes, that is,

$$-\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\Big|_{X_1=0} = \int_0^\infty G\mathrm{d}X_1 \le 0. \tag{114}$$

Combining the two inequalities (112) and (114), we have

$$0 \le -\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\Big|_{X_1=0} = \int_0^\infty G\mathrm{d}X_1 \le 0.$$

Therefore, we have

$$\int_0^\infty G\mathrm{d}X_1 = 0, \qquad \text{thus,} \quad G = 0, \tag{115}$$

and

$$\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\Big|_{X_1=0} = 0.$$

From Eq. (115), f is Maxwellian in $0 < X_1 < \infty$, and Eq. (104) is reduced to $\xi_1\partial f/\partial X_1 = 0$. That is, f is a uniform Maxwellian. From the condition (101) at infinity and Eq. (106), the solution is to be in the form

$$f = \frac{\rho_\infty}{(2\pi RT_\infty)^{3/2}}\exp\left(-\frac{\xi_1^2 + (\xi_2 - v_{2\infty})^2 + (\xi_3 - v_{3\infty})^2}{2RT_\infty}\right) \quad (0 < X_1 < \infty). \tag{116}$$

From the uniqueness condition of Eq. (167c), the Maxwellian that satisfies the boundary condition (167c) is given by Eq. (111). Thus, the parameters in Eq. (116) have to be[25]

$$v_{2\infty} = v_{3\infty} = 0, \quad \rho_\infty = \rho_w, \quad T_\infty = T_w,$$

[25]For a simple boundary, we can choose ρ_∞ at our disposal, because ρ in Eq. (M-1.27c) is arbitrary.

and the solution is given by Eq. (105).

The same statement holds for the linearized Boltzmann equation with the corresponding general boundary condition (M-1.112) on an interface of the gas and its condensed phase. The temperature T_w of the condensed phase and the saturation gas density ρ_w at temperature T_w are, respectively, taken here as the reference temperature T_0 or $\tau_w = 0$ and the reference density ρ_0 or $\omega_w = 0$.[26] The linearized Boltzmann equation is given in the form

$$\zeta_1 \frac{\partial \phi}{\partial \eta} = \mathcal{L}(\phi) \quad (0 < \eta < \infty). \tag{117}$$

The boundary condition on the interface is given by Eq. (M-1.112) with the supplementary conditions (i), (ii-a), and (ii-b) as

$$E(\zeta)\phi(\eta, \boldsymbol{\zeta}) = \int_{\zeta_{1*}<0} \hat{K}_{I0}(\boldsymbol{\zeta}, \boldsymbol{\zeta}_*)\phi(\eta, \boldsymbol{\zeta}_*)E(\zeta_*)\mathbf{d}\boldsymbol{\zeta}_* \quad (\zeta_1 > 0) \ \text{ at } \eta = 0. \tag{118}$$

The condition at infinity is

$$\phi(\eta, \boldsymbol{\zeta}) \to \omega_\infty + 2\zeta_i u_{i\infty} + \left(\zeta_i^2 - \frac{3}{2}\right)\tau_\infty \quad \text{as } \eta \to \infty, \tag{119}$$

where ω_∞, $u_{i\infty}$ and τ_∞ are some constants and $\eta = x_1/k$ $(= 2X_1/\sqrt{\pi}\ell_0)$. Then, the solution of the boundary-value problem (117)–(119) exists when and only when

$$\omega_\infty = 0, \quad u_{i\infty} = 0, \quad \tau_\infty = 0, \tag{120}$$

and the unique solution is given by

$$\phi = 0. \tag{121}$$

The proof can be given in the same way as the preceding proof for the nonlinear case. From the conservation equation (M-1.99), i.e., $\partial u_1/\partial \eta = 0$, and the condition of absence of evaporation or condensation on the condensed phase ($u_1 = \int \zeta_1 \phi E \mathbf{d}\boldsymbol{\zeta} = 0$ at $\eta = 0$[27]), we have

$$u_1 = \int \zeta_1 \phi E \mathbf{d}\boldsymbol{\zeta} = 0 \qquad (0 \le \eta < \infty). \tag{122}$$

Thus,

$$u_{i\infty} = 0. \tag{123}$$

From Eqs. (119) and (123),

$$\int \zeta_1 \phi^2 E \mathbf{d}\boldsymbol{\zeta} = 0 \ \text{ at infinity.} \tag{124}$$

[26] We take the reference density ρ_w in contrast with the case of a simple boundary. This is only for convenience of explanation. For this choice, ω_w term disappears in Eq. (118) but ω_∞ term appears in Eq. (119)

[27] The boundary where this equality holds irrespective of a situation is the definition of a simple boundary.

According to the second part of Section M-A.10,[28]

$$\int \zeta_1 \phi^2 E \mathbf{d}\boldsymbol{\zeta} \leq 0 \quad \text{at } \eta = 0. \tag{125}$$

The linearized-Boltzmann-equation version of the equation for the H function given by Eq. (M-1.115) is expressed as

$$\frac{\partial}{\partial \eta} \int \zeta_1 \phi^2 E \mathbf{d}\boldsymbol{\zeta} = LG, \tag{126}$$

where

$$LG = -\frac{1}{2} \int E E_* (\phi' + \phi'_* - \phi - \phi_*)^2 \widehat{B} \mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \leq 0. \tag{127}$$

From Eqs. (124), (125), and (126) with Eq. (127), we find that LG is to be zero and that ϕ is a summational invariant or the linearized form of a Maxwellian, i.e.,

$$\phi = \omega + 2(\zeta_2 u_2 + \zeta_3 u_3) + \left(\zeta_i^2 - \frac{3}{2} \right) \tau,$$

where Eq. (122) is used. Then, Eq. (117) reduces to $\zeta_1 \partial\phi/\partial\eta = 0$, and therefore, ω, u_2, u_3, and τ are constant. In view of Eq. (119), the constants ω, u_2, u_3, τ, and ϕ are given as

$$\omega = \omega_\infty, \quad u_2 = u_{2\infty}, \quad u_3 = u_{3\infty}, \quad \tau = \tau_\infty,$$
$$\phi = \omega_\infty + 2(\zeta_2 u_{2\infty} + \zeta_3 u_{3\infty}) + \left(\zeta_i^2 - \frac{3}{2} \right) \tau_\infty.$$

Owing to the supplementary condition (ii-b) to the boundary condition (M-1.112) together with Eq. (123), we have[29]

$$\omega_\infty = 0, \quad u_{1\infty} = 0, \quad u_{2\infty} = 0, \quad u_{3\infty} = 0, \quad \tau_\infty = 0,$$
$$\phi = 0.$$

(Section 4.1: Version 5-00)

5 Chapter M-9

5.1 Processes of solution of the equations with the ghost effect of infinitesimal curvature (July 2007)

The way in which Eqs. (M-9.33)–(M-9.39b) or Eqs. (M-9.49a)–(M-9.50e), including the time-dependent case with the additional time-derivative terms given

[28]This is the linearized-Boltzmann-equation version of the inequality (189) and valid for both types of boundaries, a simple boundary and an interface. For the case of an interface, an additional condition (M-A.271), which corresponds to Eq. (170) in the nonlinear case, is imposed on the kernel $\hat{K}_{I0}$ (see also Footnote 44 in Section 6.4.2).

[29]Owing to the difference of the supplementary condition (ii-b) of Eq. (M-1.112) [or Eq. (118)] for an interface from the condition (iii) of Eq. (M-1.107) for a simple boundary, ω is determined for an interface. For a simple boundary, ω_∞ can be chosen at our disposal.

by Eq. (M-9.42) or the mathematical expressions next to Eq. (M-9.59), contain the pressure terms, $(\hat{p}_{\mathfrak{S}0},\hat{p}_{\mathfrak{S}2})$ or (P_{01}, P_{02}, P_{20}), is different from the way in which the Navier–Stokes equations (M-3.265)–(M-3.266c) do the pressure terms, (P_{S1}, P_{S2}). In Section M-9.4, we consider the time-independent solution of Eqs. (M-9.49a)–(M-9.50e) [Eqs. (M-9.56)–(M-9.57d)] that is uniform with respect to $\bar{\chi}$. Here, it may be better to explain how a solution of Eqs. (M-9.33)–(M-9.39b) or Eqs. (M-9.49a)–(M-9.50e) in a general case or a time-dependent solution that depends on χ or $\bar{\chi}$ is obtained. Incidentally, the boundary conditions for the time-dependent case are derived in the same way as in Section M-3.7.3. Naturally from the derivation of the equations, the domain of a gas is in a straight pipe or channel of infinite length whose axis is in the x or χ direction.

5.1.1 Equations (M-9.33)–(M-9.39b):

Take Eqs. (M-9.33)–(M-9.39b) with the additional time-derivative terms given by Eq. (M-9.42), i.e.,[30]

$$\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial y} = \frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial z} = 0, \tag{128}$$

$$\frac{\partial \hat{\rho}_{\mathfrak{S}0}}{\partial \hat{t}} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{z\mathfrak{S}1}}{\partial z} = 0, \tag{129}$$

$$\begin{aligned}\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \hat{t}} &+ \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) \\ &= -\frac{1}{2}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} + \frac{1}{2}\frac{\partial}{\partial y}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right),\end{aligned} \tag{130}$$

$$\begin{aligned}\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial \hat{t}} &+ \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} - \frac{1}{c^2}\hat{v}_{x\mathfrak{S}0}^2\right) \\ = &-\frac{1}{2}\frac{\partial \hat{p}_{\mathfrak{S}2}^c}{\partial y} + \frac{1}{2}\frac{\partial}{\partial \chi}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right) \\ &+ \frac{\partial}{\partial y}\left(\Gamma_1\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left[\Gamma_1\left(\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y}\right)\right] \\ &+ \frac{1}{2\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left[\Gamma_7\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)^2\right] + \frac{\partial}{\partial z}\left(\Gamma_7\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right\} \\ &+ \frac{1}{\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left[\Gamma_8\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2\right] + \frac{\partial}{\partial z}\left(\Gamma_8\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)\right\},\end{aligned} \tag{131}$$

[30]Equation (M-9.33) is replaced by its equivalent form (128).

$$
\begin{aligned}
&\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial \hat{t}} + \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right)\\
&\qquad = -\frac{1}{2}\frac{\partial \hat{p}^{c}_{\mathfrak{S}2}}{\partial z} + \frac{1}{2}\frac{\partial}{\partial \chi}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)\\
&\qquad\quad + \frac{1}{2}\frac{\partial}{\partial y}\left[\Gamma_1\left(\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(\Gamma_1\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right)\\
&\qquad\quad + \frac{1}{2\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left(\Gamma_7\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right) + \frac{\partial}{\partial z}\left[\Gamma_7\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)^2\right]\right\}\\
&\qquad\quad + \frac{1}{\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left(\Gamma_8\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) + \frac{\partial}{\partial z}\left[\Gamma_8\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]\right\}, \quad (132)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{5\hat{\rho}_{\mathfrak{S}0}}{2}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial \hat{t}} + \frac{5}{2}\hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\\
&- \frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \hat{t}} - \hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi}\\
&\quad = \frac{5}{4}\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{5}{4}\frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right) + \Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right], \quad (133)
\end{aligned}
$$

and the subsidiary relations

$$
\hat{p}_{\mathfrak{S}0}(\chi, \hat{t}) = \hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0}, \quad (134a)
$$

$$
\begin{aligned}
\hat{p}^{c}_{\mathfrak{S}2} = \hat{p}_{\mathfrak{S}2} &+ \frac{2\Gamma_1}{3}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) + \frac{\Gamma_7}{3\hat{p}_{\mathfrak{S}0}}\left[\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)^2\right]\\
&+ \frac{2}{3\hat{p}_{\mathfrak{S}0}}\left[\frac{\partial}{\partial y}\left(\Gamma_3\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_3\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]\\
&- \frac{2\Gamma_9}{3\hat{p}_{\mathfrak{S}0}}\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right], \quad (134b)
\end{aligned}
$$

where Γ_1, Γ_2, Γ_3, Γ_7, Γ_8, and Γ_9 are short forms of the functions $\Gamma_1(\hat{T}_{\mathfrak{S}0})$, $\Gamma_2(\hat{T}_{\mathfrak{S}0}), \ldots, \Gamma_9(\hat{T}_{\mathfrak{S}0})$ of $\hat{T}_{\mathfrak{S}0}$ defined in Section M-A.2.9.

Consider the solution of the initial and boundary-value problem of Eqs. (128)–(134b).

Let $\hat{\rho}$, $\hat{v}_i$, and $\hat{T}$ (thus, $\hat{p} = \hat{\rho}\hat{T}$) at time $\hat{t}$ be given; thus, $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{T}_{\mathfrak{S}0}$ ($\hat{p}_{\mathfrak{S}0}$) etc., including $\hat{p}_{\mathfrak{S}2}$, are given. Then $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{T}_{\mathfrak{S}0}/\partial\hat{t}$ at $\hat{t}$ are given by Eqs. (129)–(134b); thus, the future $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ (also $\hat{p}_{\mathfrak{S}0}$) are determined. However, the future $\hat{p}_{\mathfrak{S}0}$ is required to be independent of y and z, as well as $\hat{p}_{\mathfrak{S}0}$ at $\hat{t}$, owing to Eq. (128). Taking this into account, we will discuss how the solution is obtained by this system consistently.

First, transform Eq. (133) with the aid of Eqs. (129) and (134a) in the following form:

$$\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \hat{t}} = \mathcal{P}, \tag{135}$$

where

$$\begin{aligned} \mathcal{P} = & -\frac{5}{3}\hat{p}_{\mathfrak{S}0}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) - \hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} \\ & + \frac{5}{6}\left[\frac{\partial}{\partial y}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right] + \frac{2}{3}\Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]. \end{aligned} \tag{136}$$

For $\hat{p}_{\mathfrak{S}0}$ to be independent of y and z [see Eq. (128)], $\mathcal{P}$ as well as the initial data of $\hat{p}_{\mathfrak{S}0}$ is required to be independent of y and z. Noting that $\hat{p}_{\mathfrak{S}0}$ is independent of y and z, and taking the average of Eq. (136) over the cross section S of the pipe or channel,[31] we have another expression $\mathfrak{P}$ of $\mathcal{P}$, explicitly uniform with respect to y and z, i.e.,

$$\begin{aligned} \mathfrak{P} = & -\frac{5}{3}\frac{\partial \overline{\hat{v}_{x\mathfrak{S}0}}}{\partial \chi}\hat{p}_{\mathfrak{S}0} - \overline{\hat{v}_{x\mathfrak{S}0}}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} + \frac{5}{6}\overline{\left[\frac{\partial}{\partial y}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]} \\ & + \frac{2}{3}\overline{\Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]}, \end{aligned} \tag{137}$$

where

$$\overline{A} = \int_S A \mathrm{d}y \mathrm{d}z \bigg/ \int_S \mathrm{d}y \mathrm{d}z.$$

The expression (137) is noted to be independent of $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$. The two expressions (136) and (137) must give the same result, i.e.,

$$\mathcal{P} = \mathfrak{P},$$

or

$$\begin{aligned} & -\frac{5}{3}\hat{p}_{\mathfrak{S}0}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) - \hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} \\ & + \frac{5}{6}\left[\frac{\partial}{\partial y}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2 \frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right] + \frac{2}{3}\Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right] \\ & = \mathfrak{P}, \end{aligned} \tag{138}$$

[31](i) In a channel, where the gas extends from $z = -\infty$ to $z = \infty$, the integral $\int_S A \mathrm{d}y \mathrm{d}z$ per unit length in z, per a period in z, etc. should be considered. Otherwise, it can be infinite.

(ii) Note that $\hat{v}_{y\mathfrak{S}1} n_y + \hat{v}_{z\mathfrak{S}1} n_z = 0$ on a simple boundary where $n_i = (0, n_y, n_z)$ is the normal to the boundary.

when Eq. (128) holds, and vice versa. The condition (138) for all $\hat{t}$ is equivalently replaced by the two conditions that the initial data of $\hat{p}_{\mathfrak{S}0}$, $\hat{T}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ satisfy Eqs. (128) and (138) and that the time derivative of Eq. (138) holds for all $\hat{t}$, i.e.,

$$\frac{\partial \mathcal{P}}{\partial \hat{t}} = \frac{\partial \mathfrak{P}}{\partial \hat{t}}. \tag{139}$$

Using Eqs. (129)–(132) and (135) for $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}$ ($\hat{\rho}_{\mathfrak{S}0}\partial\hat{T}_{\mathfrak{S}0}/\partial\hat{t} = \partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t} - \hat{T}_{\mathfrak{S}0}\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$) in $\partial\mathcal{P}/\partial\hat{t}$ derived from Eq. (136), we find that $\partial\mathcal{P}/\partial\hat{t}$ is expressed with $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and $\hat{p}^{c}_{\mathfrak{S}2}$ in the form

$$\frac{\partial \mathcal{P}}{\partial \hat{t}} = \frac{5}{6}\hat{p}_{\mathfrak{S}0}\left[\frac{\partial}{\partial y}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial \hat{p}^{c}_{\mathfrak{S}2}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial \hat{p}^{c}_{\mathfrak{S}2}}{\partial z}\right)\right] + \mathrm{Fn}_1, \tag{140}$$

where Fn_1 is a given function of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives. The expression (137) of $\mathfrak{P}$ being independent of $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$, its time derivative $\partial\mathfrak{P}/\partial\hat{t}$ does not contain $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$ and $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$. Therefore, with the aid of Eqs. (129), (130), and (133), $\partial\mathfrak{P}/\partial\hat{t}$ is expressed with $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives, i.e.,

$$\frac{\partial \mathfrak{P}}{\partial \hat{t}} = \mathrm{Fn}_2(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{p}_{\mathfrak{S}0}, \text{and their space derivatives}), \tag{141}$$

where Fn_2 is a given functional of its arguments. From Eqs. (139), (140), and (141), we have

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial \hat{p}^{c}_{\mathfrak{S}2}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial \hat{p}^{c}_{\mathfrak{S}2}}{\partial z}\right) = \mathrm{Fn}, \tag{142}$$

where $\mathrm{Fn} = 6(\mathrm{Fn}_2 - \mathrm{Fn}_1)/5\hat{p}_{\mathfrak{S}0}$, and therefore, Fn is a given functional of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives. This is the equation for $\hat{p}^{c}_{\mathfrak{S}2}$ over a cross section of the pipe or channel.

The boundary condition for $\hat{p}^{c}_{\mathfrak{S}2}$ on a simple boundary is obtained by multiplying Eqs. (130)–(132) by the normal $n_i = (0, n_y, n_z)$ to the boundary; In this process, the contribution of their time-derivative terms vanishes because $\hat{v}_{y\mathfrak{S}1}n_y + \hat{v}_{z\mathfrak{S}1}n_z = 0$; Then, the $n_y\partial\hat{p}^{c}_{\mathfrak{S}2}/\partial y + n_z\partial\hat{p}^{c}_{\mathfrak{S}2}/\partial z$ is imposed as the boundary condition. Thus, $\hat{p}^{c}_{\mathfrak{S}2}$ is determined by Eq. (142) except for an additive function of $\hat{t}$ and χ. With this $\hat{p}^{c}_{\mathfrak{S}2}$ substituted into Eqs. (131) and (132), $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}$ are determined by Eqs. (129)–(134b) independently of the additive function in $\hat{p}^{c}_{\mathfrak{S}2}$ in such a way that $\partial(\partial\hat{p}_{\mathfrak{S}02}/\partial y)/\partial\hat{t} = \partial(\partial\hat{p}_{\mathfrak{S}0}/\partial z)/\partial\hat{t} = 0$ and $\partial(\partial\mathcal{P}/\partial y)/\partial\hat{t} = \partial(\partial\mathcal{P}/\partial z)/\partial\hat{t} = 0$. That is, the solution $(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{T}_{\mathfrak{S}0})$ of Eqs. (128)–(134b) is determined by Eqs. (129)–(134b) with the aid of the supplementary condition (142), instead of Eq. (128), when the initial condition for $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ is given in such a way that $\hat{p}_{\mathfrak{S}0}$ $(= \hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0})$ and $\mathcal{P}$ are independent of y and z.

Equations (128)–(134b) are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. The equation for $\partial\hat{p}_{\mathfrak{S}2}/\partial\hat{t}$, corresponding to Eq. (135), is derived at the order after next. However, owing to the consistency of $\hat{p}_{\mathfrak{S}0}$, $\hat{p}_{\mathfrak{S}2}$ is already determined by Eq. (142) except for an additive function of χ and $\hat{t}$. The situation is similar to that at the leading order. That is, $\hat{p}_{\mathfrak{S}0}$ and $\hat{p}_{\mathfrak{S}2}$ are, respectively, determined by Eqs. (128) and (142), each with an additive function of χ and $\hat{t}$ and also by Eqs. (135) and the counterpart of Eq. (135) at the order after next. Thus, the higher-order analysis can be carried out in a similar way. The results are that an additional initial condition and an equation for $\hat{p}_{\mathfrak{S}4}$, the counter part of Eq. (142), are introduced and that the condition (142) is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{\mathfrak{S}0}$, $\hat{T}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ (thus also $\hat{p}_{\mathfrak{S}0}$).

To summarize, the solution $(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{T}_{\mathfrak{S}0})$ of Eqs. (128)–(134b) is determined by Eqs. (129)–(134b) with the aid of the supplementary condition (142), instead of Eq. (128), when the initial data of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ are given in such a way that $\hat{p}_{\mathfrak{S}0}$ $(= \hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0})$ and $\mathcal{P}$ are independent of y and z.[32] The results are not affected by the higher-order analysis.

5.1.2 Equations (M-9.49a)–(M-9.50e):

Take Eqs. (M-9.49a)–(M-9.50e) with the additional time-derivative terms given in the first mathematical expressions after Eq. (M-9.59), i.e.,

$$\frac{\partial P_{01}}{\partial \bar{\chi}} = \frac{\partial P_{01}}{\partial y} = \frac{\partial P_{01}}{\partial z} = 0, \quad P_{01} = \omega + \tau, \tag{143a}$$

$$\frac{\partial P_{02}}{\partial y} = \frac{\partial P_{02}}{\partial z} = 0, \tag{143b}$$

$$\frac{\partial u_x}{\partial \bar{\chi}} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \tag{144a}$$

$$\frac{\partial u_x}{\partial \hat{t}} + u_x\frac{\partial u_x}{\partial \bar{\chi}} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} = -\frac{1}{2}\frac{\partial P_{02}}{\partial \bar{\chi}} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right), \tag{144b}$$

$$\frac{\partial u_y}{\partial \hat{t}} + u_x\frac{\partial u_y}{\partial \bar{\chi}} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z} - \frac{u_x^2}{C^2} = -\frac{1}{2}\frac{\partial P_{20}}{\partial y} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right), \tag{144c}$$

$$\frac{\partial u_z}{\partial \hat{t}} + u_x\frac{\partial u_z}{\partial \bar{\chi}} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{2}\frac{\partial P_{20}}{\partial z} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right), \tag{144d}$$

$$\frac{\partial \tau}{\partial \hat{t}} - \frac{2}{5}\frac{\partial P_{01}}{\partial \hat{t}} + u_x\frac{\partial \tau}{\partial \bar{\chi}} + u_y\frac{\partial \tau}{\partial y} + u_z\frac{\partial \tau}{\partial z} = \frac{\gamma_2}{2}\left(\frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2}\right). \tag{144e}$$

[32]If $\mathcal{P}$ is independent of y and z, $\mathcal{P} = \mathfrak{P}$ by definition.

The qualitative difference of this set of equations from the set (128)–(134b) is the absence of the time-derivative term in Eq. (144a) that corresponds to Eq. (129).

Consider the solution of the initial and boundary-value problem of Eqs. (143a)–(144e). Let u_x, u_y, u_z, and τ at $\hat{t}$ be given in such a way that Eq. (144a) is satisfied. Integrating Eq. (144a) over the cross section of the channel or pipe $\left[\int_S \text{Eq. (144a)} \mathrm{d}y\mathrm{d}z\right]$, we find that $\int_S u_x \mathrm{d}y\mathrm{d}z$ depends only on $\hat{t}$,[33] i.e.,

$$\int_S (\partial u_x/\partial \bar{\chi})\mathrm{d}y\mathrm{d}z = 0, \tag{145}$$

where S indicates the cross section. Applying Eqs. (143b), (144a), and (145) to the equation $\partial \int \text{Eq. (144b)}\mathrm{d}y\mathrm{d}z/\partial\bar{\chi}$, we have $\partial^2 P_{02}/\partial\bar{\chi}^2$ as

$$\frac{\partial^2 P_{02}}{\partial \bar{\chi}^2} = \frac{\partial}{\partial \bar{\chi}} \overline{\left[-2\frac{\partial u_x^2}{\partial \bar{\chi}} + \gamma_1\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\right]}, \tag{146}$$

where

$$\overline{A} = \int_S A\mathrm{d}y\mathrm{d}z \bigg/ \int_S \mathrm{d}y\mathrm{d}z.$$

Thus, $\partial P_{02}/\partial\bar{\chi}$ and P_{02} are determined if they are specified at a point in the gas. Here, we consider this case.[34] Using Eq. (144a) in the sum of ∂[Eq. (144b)]$/\partial\bar{\chi}$, ∂[Eq. (144c)]$/\partial y$, and ∂[Eq. (144d)]$/\partial z$, we obtain the equation for P_{20} in the form

$$\frac{\partial^2 P_{20}}{\partial y^2} + \frac{\partial^2 P_{20}}{\partial z^2} = \mathrm{Fn}(u_x, u_y, u_z, \text{and their space derivatives}), \tag{147}$$

where Fn is a given functional of the variables in the parentheses, and the time derivatives are absent owing to Eq. (144a). Thus, the right-hand side of Eq. (147) is known. This equation is the Poisson equation for P_{20} over the cross section S. Its boundary condition is obtained in a way similar to how the condition for $\hat{p}^c_{\mathfrak{S}2}$ in Eq. (142) is derived. Thus, P_{20} over each cross section is determined except for an additive function of $\hat{t}$ and $\bar{\chi}$. This ambiguity does not influence $\partial P_{20}/\partial y$ and $\partial P_{20}/\partial z$.

With P_{02} and P_{20} prepared above into Eqs. (144b)–(144e), the time derivatives $\partial u_x/\partial\hat{t}$, $\partial u_y/\partial\hat{t}$, $\partial u_z/\partial\hat{t}$, and $\partial\tau/\partial\hat{t}$ are determined in such a way that $\partial(\partial u_x/\partial\bar{\chi} + \partial u_y/\partial y + \partial u_z/\partial z)/\partial\hat{t} = 0$ owing to the above choice of P_{20}.[35] Thus, the solution (u_x, u_y, u_z, τ) of Eqs. (143b), (144a)–(144e) is determined by Eqs. (144b)–(144e) with the aid of the supplementary conditions (146) and (147) for P_{02} and P_{20}, instead of Eqs. (143b) and (144a). This process is natural for numerical computation. The undetermined additive function of $\bar{\chi}$ and $\hat{t}$ in P_{20}, which does not affect the solution (u_x, u_y, u_z, τ), is determined by the higher-order equation derived from that for $\partial\hat{v}_{x\mathfrak{S}2}/\partial\hat{t}$ (see Section 5.1.1), in a

[33]See Footnote 31, with $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$ being replaced by u_y and u_z.

[34](i) Imagine the case of the Poiseuille flow.

(ii) Here, P (thus, P_{01}) is specified at some point. Then, P_{01} is a given function of $\hat{t}$.

[35]Note that P_{01} is known (Footnote 34).

way similar to that in which P_{02} is determined by Eq. (144b). In the higher-order equation, P_{20} plays the same role as P_{02} in Eq. (144b); Equation (147) corresponds to Eq. (143b), and P_{20} and P_{02} are determined by these equations, each with an additive function of $\bar{\chi}$ and $\hat{t}$.

6 Appendix M-A

6.1 Note on the loss term of the collision integral [From Eq. (M-A.18) to Eq. (M-A.21)]

Consider the following collision term of the Boltzmann equation (M-A.18):[36]

$$\frac{d_m^2}{2m}\int_{\text{all } e,\text{ all } \boldsymbol{\xi}_*} |(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot \boldsymbol{e}|[f(\boldsymbol{\xi}')f(\boldsymbol{\xi}'_*) - f(\boldsymbol{\xi})f(\boldsymbol{\xi}_*)]\mathrm{d}\Omega(\boldsymbol{e})\mathbf{d}\boldsymbol{\xi}_*, \tag{148}$$

where

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + [\boldsymbol{\alpha}\cdot(\boldsymbol{\xi}_* - \boldsymbol{\xi})]\boldsymbol{\alpha}, \quad \boldsymbol{\xi}'_* = \boldsymbol{\xi}_* - [\boldsymbol{\alpha}\cdot(\boldsymbol{\xi}_* - \boldsymbol{\xi})]\boldsymbol{\alpha}. \tag{149}$$

The change (M-A.20) of the variable of integration from $\boldsymbol{e}$ to $\boldsymbol{\alpha}$, i.e.,

$$|(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot \boldsymbol{e}|\mathrm{d}\Omega(\boldsymbol{e}) = \frac{2}{d_m^2}B\mathrm{d}\Omega(\boldsymbol{\alpha}), \tag{150}$$

is introduced instead of expressing $\boldsymbol{\alpha}$ in Eq. (149) in terms of $\boldsymbol{e}$. The part of the integral of Eq. (148)

$$\frac{d_m^2}{2m}\int_{\text{all } e,\text{ all } \boldsymbol{\xi}_*} |(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot \boldsymbol{e}|f(\boldsymbol{\xi})f(\boldsymbol{\xi}_*)\mathrm{d}\Omega(\boldsymbol{e})\mathbf{d}\boldsymbol{\xi}_*,$$

which comes from I_- in Eq. (M-A.8) and corresponds to the loss term (see Section M-1.2) of the collision integral of the Boltzmann equation (M-1.5) or (M-A.21), does not contain $\boldsymbol{\alpha}$, and the change (150) of the variable of integration is not required.[37] Thus, the result is determined uniquely irrespective of the relation between $\boldsymbol{\alpha}$ and $\boldsymbol{e}$, that is, the loss term of the collision integral is independent of the intermolecular potential when d_m is of a finite value. That is, the loss term of the collision integral is determined only by $d_m^2/2m$ and $f(\boldsymbol{\xi})$, and is the same as that for the hard-sphere molecule with the same d_m.

(Section 6.1: Version 6-00)

6.2 Note on the loss term of the kernel representation of the linearized collision integral [Section M-A.2.10]

In Section M-A.2.10, we discussed the kernel representation of the linearized collision integral $\mathcal{L}(\phi)$ introduced in Section M-1.10, and gave its explicit form

[36] The factor $d_m^2/2m$ can be rewritten as $nd_m^2/2\rho$, where n is the number of molecules in unit volume. The numerator nd_m^2 is of the order of the inverse of the mean free path (Section M-1.5). Note Footnote M-4 in Section M-A.1.

[37] Transformation (M-A.20) or (150) is carried out to make the variable of integration to be the same. Thus, it is simply one of the changes of variable $\boldsymbol{e}$ of integration to some variable.

for a hard-sphere molecule. From the discussion in Section 6.1, the kernel representation of the loss term of the linearized collision integral for a hard-sphere molecule applies to any intermolecular potential with a finite d_m.

In Section M-A.2.10, the linearized collision integral $\mathcal{L}(\phi)$ is expressed by Eqs. (M-137a)–(M-A.139c) as

$$\begin{aligned}\mathcal{L}(\phi) &= \int E_*(\phi' + \phi'_* - \phi - \phi_*)\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_* \\ &= \mathcal{L}^G(\phi) - \mathcal{L}^{L2}(\phi) - \nu_L(\boldsymbol{\zeta})\phi \end{aligned} \tag{151}$$

where

$$\mathcal{L}^G(\phi) = \int E_*(\phi' + \phi'_*)\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_*, \tag{152a}$$

$$\begin{aligned}\mathcal{L}^{L2}(\phi) &= \int E_*\phi_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_* \\ &= \int K_2(\boldsymbol{\zeta},\boldsymbol{\zeta}_*)\phi(\boldsymbol{\zeta}_*)\mathbf{d}\boldsymbol{\zeta}_*, \end{aligned} \tag{152b}$$

$$\nu_L(\boldsymbol{\zeta}) = \int E_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_*. \tag{152c}$$

The loss term is the sum of Eqs. (152b) and (152c), i.e., $\mathcal{L}^{L2}(\phi)+\nu_L(\boldsymbol{\zeta})\phi$.[38] The kernel $K_2(\boldsymbol{\zeta},\boldsymbol{\zeta}_*)$ and the function $\nu_L(\boldsymbol{\zeta})$ for a hard-sphere molecule are given by Eqs. (M-A.149b) and (M-A.149c) as

$$K_2(\boldsymbol{\zeta},\boldsymbol{\zeta}_*) = \frac{|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|}{2\sqrt{2}\,\pi}\exp\left(-\zeta_*^2\right), \tag{153a}$$

$$\nu_L(\boldsymbol{\zeta}) = \frac{1}{2\sqrt{2}}\left[\exp(-\zeta^2) + \left(2\zeta + \frac{1}{\zeta}\right)\int_0^{\zeta}\exp(-\zeta_*^2)\mathrm{d}\zeta_*\right], \tag{153b}$$

where

$$\zeta = |\boldsymbol{\zeta}|.$$

These formulas apply to any potential with a finite d_m as well as to a hard-sphere molecule.

(Section 6.2: Version 6-00)

[38]Only the term $\nu_L(\boldsymbol{\zeta})\phi$ is often called the loss term, and the rest, i.e., $\mathcal{L}^G(\phi) - \mathcal{L}^{L2}(\phi)$, is called the gain term by misunderstanding. This is probably because the loss term of the original collision integral (148) is often written in the form $\nu_c f$, where ν_c is the collision frequency defined by Eq. (M-1.18) as

$$\nu_c = m^{-1}\int_{\text{all }\boldsymbol{\alpha},\text{ all }\boldsymbol{\xi}_*} f(\boldsymbol{\xi}_*)B\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\xi}_* = (d_m^2/2m)\int_{\text{all }\boldsymbol{e},\text{ all }\boldsymbol{\xi}_*}|(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot\boldsymbol{e}|f(\boldsymbol{\xi}_*)\mathrm{d}\Omega(\boldsymbol{e})\mathbf{d}\boldsymbol{\xi}_*.$$

Not to mention, $\mathcal{L}^{L2}(\phi)$ is derived from $\nu_c f$.

6.3 Parity of the collision integral: Supplement to Section M-A.2.7

In Section M-A.2.7, we discussed the parity of the linearized collision integral. It may be better to explain a similar property of the collision integral defined by Eq. (M-1.9), i.e.,

$$\hat{J}(\hat{f},\hat{g}) = \frac{1}{2}\int (\hat{f}'\hat{g}'_* + \hat{f}'_*\hat{g}' - \hat{f}\hat{g}_* - \hat{f}_*\hat{g})\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{\zeta}_*, \tag{154}$$

$$\widehat{B} = \widehat{B}(|\boldsymbol{\alpha}\cdot\boldsymbol{V}|/|\boldsymbol{V}|, |\boldsymbol{V}|),$$

$$\hat{f} = \hat{f}(\zeta_i),\ \hat{f}_* = \hat{f}(\zeta_{i*}),\ \hat{f}' = \hat{f}(\zeta'_i),\ \hat{f}'_* = \hat{f}(\zeta'_{i*}),$$

and a similar notation for $\hat{g},\ \hat{g}_*,\ \hat{g}',$ and $\hat{g}'_*$,

$$\zeta'_i = \zeta_i + \alpha_j V_j \alpha_i,\ \zeta'_{i*} = \zeta_{i*} - \alpha_j V_j \alpha_i,\ \zeta_{i*} = V_i + \zeta_i.$$

Here, we discuss the relation of the parity of $\hat{J}(\hat{f},\hat{g})$ with respect to a component (ζ_1, ζ_2, or ζ_3) of the variable $\boldsymbol{\zeta}$ to that of $\hat{f}$ and $\hat{g}$. Put the integral (154) in the sum

$$\hat{J}(\hat{f},\hat{g}) = \frac{1}{2}\left(IV + III - II - I\right), \tag{155}$$

where

$$I = \int \hat{f}_*\hat{g}\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{156a}$$

$$II = \int \hat{f}\hat{g}_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{156b}$$

$$III = \int \hat{f}'_*\hat{g}'\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{156c}$$

$$IV = \int \hat{f}'\hat{g}'_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{156d}$$

and discuss each term separately.[39] In Eqs. (156a)–(156d), the variable of integration is changed from $\boldsymbol{\zeta}_*$ to $\boldsymbol{V}$ $(= \boldsymbol{\zeta}_* - \boldsymbol{\zeta})$. The following change of the variables

$$\widetilde{V}_1 = -V_1,\quad \widetilde{V}_s = V_s,\quad \widetilde{\alpha}_1 = -\alpha_1,\quad \widetilde{\alpha}_s = \alpha_s \quad (s = 2, 3) \tag{157}$$

is performed in the integrals I, II, III, and IV. Noting that

$$\zeta_{i*} = V_i + \zeta_i, \qquad |\widetilde{V}_i| = |V_i|, \qquad \widetilde{\alpha}_i\widetilde{V}_i = \alpha_i V_i, \tag{158}$$

we can transform the integrals I, II, III, and IV in the following way, where the subscript s indicates $s = 2$ and 3:

$$\begin{aligned} I(\zeta_1,\zeta_s) &= \int \hat{f}(V_1+\zeta_1, V_s+\zeta_s)\hat{g}(\zeta_1,\zeta_s)\widehat{B}\,(|\alpha_i V_i|/|V_i|, |V_i|)\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V} \\ &= \int \hat{f}(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}(\zeta_1,\zeta_s)\,\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}; \end{aligned} \tag{159a}$$

[39] The separation is made only for convenience of explanation.

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in I, we have

$$\mathit{II}(\zeta_1,\zeta_s) = \int \hat{f}(\zeta_1,\zeta_s)\hat{g}(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\,\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}; \tag{159b}$$

$$\begin{aligned}\mathit{III}(\zeta_1,\zeta_s) &= \int \hat{f}(V_i+\zeta_i-\alpha_jV_j\alpha_i)\hat{g}(\zeta_i+\alpha_jV_j\alpha_i)\widehat{B}\,(|\alpha_iV_i|/|V_i|,|V_i|)\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}\\ &= \int \hat{f}(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\ &\qquad\times\hat{g}(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}};\end{aligned} \tag{159c}$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in III, we have

$$\begin{aligned}\mathit{IV}(\zeta_1,\zeta_s) = \int &\hat{f}(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\ &\times\hat{g}(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\ &\times\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}.\end{aligned} \tag{159d}$$

Now we examine the parity of the integrals I, II, III, and IV with respect to ζ_1 on the basis of Eqs. (159a)–(159d). Here, we introduce the notation: (i) the parity of $\hat{f}$ (or $\hat{g}$) is indicated by the subscript attached to it, i.e., the subscript E is attached when it is even and the subscript O when it is odd; (ii) the first subscript of I, II, III, and IV indicates the parity of $\hat{f}$ in them and the second indicates the parity of $\hat{g}$. First, when $\hat{f}$ and $\hat{g}$ are even functions of ζ_1.

$$\begin{aligned}I_{EE}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_E(\zeta_1,\zeta_s)\\ &\qquad\times\ \widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\ &= \int \hat{f}_E(\widetilde{V}_1-\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_E(-\zeta_1,\zeta_s)\\ &\qquad\times\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\ &= I_{EE}(-\zeta_1,\zeta_s),\end{aligned} \tag{160a}$$

where the last relation holds owing to the first relation of Eq. (159a); Interchanging the arguments of $\hat{f}_E$ and $\hat{g}_E$ in I_{EE}, we have

$$\mathit{II}_{EE}(\zeta_1,\zeta_s) = \mathit{II}_{EE}(-\zeta_1,\zeta_s); \tag{160b}$$

$$
\begin{aligned}
III_{EE}(\zeta_1,\zeta_s) = &\int \hat{f}_E(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\times \hat{g}_E(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
= &\int \hat{f}_E(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\times \hat{g}_E(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
= &\, III_{EE}(-\zeta_1,\zeta_s);
\end{aligned}
\tag{160c}
$$

Interchanging the arguments of $\hat{f}_E$ and $\hat{g}_E$ in III_{EE}, we have

$$
IV_{EE}(\zeta_1,\zeta_s) = IV_{EE}(-\zeta_1,\zeta_s). \tag{160d}
$$

When both $\hat{f}$ and $\hat{g}$ are odd with respect to ζ_1,

$$
\begin{aligned}
I_{OO}(\zeta_1,\zeta_s) &= \int \hat{f}_O(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}_O(\zeta_1,\zeta_s)\,\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= \int \hat{f}_O(\widetilde{V}_1-\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}_O(-\zeta_1,\zeta_s)\,\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= I_{OO}(-\zeta_1,\zeta_s);
\end{aligned}
\tag{161a}
$$

Interchanging the arguments of $\hat{f}_O$ and $\hat{g}_O$ in II_{OO}, we have

$$
II_{OO}(\zeta_1,\zeta_s) = II_{OO}(-\zeta_1,\zeta_s); \tag{161b}
$$

$$
\begin{aligned}
III_{OO}(\zeta_1,\zeta_s) = &\int \hat{f}_O(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\times \hat{g}_O(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
= &\int \hat{f}_O(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\times \hat{g}_O(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
= &\, III_{OO}(-\zeta_1,\zeta_s);
\end{aligned}
\tag{161c}
$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in III_{OO}, we have

$$
IV_{OO}(\zeta_1,\zeta_s) = IV_{OO}(-\zeta_1,\zeta_s). \tag{161d}
$$

When $\hat{f}$ is even and $\hat{g}$ is odd with respect to ζ_1,

$$
\begin{aligned}
I_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}_O(\zeta_1,\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= -\int \hat{f}_E(\widetilde{V}_1-\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}_O(-\zeta_1,\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= -I_{EO}(-\zeta_1,\zeta_s);
\end{aligned}
\tag{162a}
$$

$$
\begin{aligned}
\mathit{II}_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(\zeta_1,\zeta_s)\hat{g}_O(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(-\zeta_1,\zeta_s)\hat{g}_O(\widetilde{V}_1-\zeta_1,\widetilde{V}_s+\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\mathit{II}_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{162b}
$$

$$
\begin{aligned}
\mathit{III}_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\mathit{III}_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{162c}
$$

$$
\begin{aligned}
\mathit{IV}_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\int \hat{f}_E(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \hat{g}_O(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= -\mathit{IV}_{EO}(-\zeta_1,\zeta_s).
\end{aligned} \tag{162d}
$$

For I_{OE}, II_{OE}, III_{OE}, and IV_{OE}, interchanging the role of $\hat{f}$ and $\hat{g}$, respectively, in II_{EO}, I_{EO}, IV_{EO}, and III_{EO}, we have

$$ I_{OE}(\zeta_1,\zeta_s) = -I_{OE}(\zeta_1,\zeta_s), \tag{163a} $$

$$ \mathit{II}_{OE}(\zeta_1,\zeta_s) = -\mathit{II}_{OE}(\zeta_1,\zeta_s), \tag{163b} $$

$$ \mathit{III}_{OE}(\zeta_1,\zeta_s) = -\mathit{III}_{OE}(\zeta_1,\zeta_s), \tag{163c} $$

$$ \mathit{IV}_{OE}(\zeta_1,\zeta_s) = -\mathit{IV}_{OE}(\zeta_1,\zeta_s). \tag{163d} $$

The parity is common to I, II, III, and IV. Therefore, the parity of $\hat{J}(\hat{f},\hat{g})$ is the same as I, i.e.,

$$ \hat{J}(\hat{f}_E,\hat{g}_E)(\zeta_1,\zeta_s) = \hat{J}(\hat{f}_E,\hat{g}_E)(-\zeta_1,\zeta_s), \tag{164a} $$

$$ \hat{J}(\hat{f}_O,\hat{g}_O)(\zeta_1,\zeta_s) = \hat{J}(\hat{f}_O,\hat{g}_O)(-\zeta_1,\zeta_s), \tag{164b} $$

$$ \hat{J}(\hat{f}_E,\hat{g}_O)(\zeta_1,\zeta_s) = -\hat{J}(\hat{f}_E,\hat{g}_O)(-\zeta_1,\zeta_s), \tag{164c} $$

$$ \hat{J}(\hat{f}_O,\hat{g}_E)(\zeta_1,\zeta_s) = -\hat{J}(\hat{f}_O,\hat{g}_E)(-\zeta_1,\zeta_s). \tag{164d} $$

Obviously, the same parity holds for the other components, i.e., ζ_2, ζ_3, of $\boldsymbol{\zeta}$.
(Section 6.3: Version 4-00)

6.4 Supplement to Section M-A.10

6.4.1 On the equality condition of Eq. (M-A.266)

Here we will discuss the equality condition in the Darrozes–Guiraud inequality in Section M-A.10 in more detail. The equality in the Jensen inequality (M-A.265) is proved to hold when and only when ϕ is independent of $\boldsymbol{\xi}$ (see, e.g., Reference M-129). It should be noted that the uniqueness condition of the equality applies only to the region of $\boldsymbol{\xi}$ where $\psi > 0$ and that no condition is required of ϕ where $\psi = 0$. Choose a $\boldsymbol{\xi}$ in $(\xi_i - v_{wi})n_i > 0$, and consider the condition for equality in Eq. (M-A.266). According to the above note, the equality holds only when $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is a constant (say, c_1) in the region D_1 of $\boldsymbol{\xi}_*$, joint or disjoint, where $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$. If we choose another $\boldsymbol{\xi}$, $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ in a different range D_2 of $\boldsymbol{\xi}_*$, and $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*) = c_2$ (c_2 : const) is required in D_2. The constants c_1 and c_2 may be different if D_1 and D_2 are disjoint. The two constants are required to be the same ($c_1 = c_2$), if D_1 and D_2 overlap for some range of $\boldsymbol{\xi}_*$ (their intersection is neither empty nor measure zero).[40] From the condition (M-1.27b), there is a region of $\boldsymbol{\xi}$ where $K_B > 0$ for any $\boldsymbol{\xi}_*$ in $(\xi_{i*} - v_{wi})n_i < 0$. Thus, the collection of the regions of $\boldsymbol{\xi}_*$ where $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ with respect to all $\boldsymbol{\xi}$ in $(\xi_i - v_{wi})n_i > 0$ covers $(\xi_{i*} - v_{wi})n_i < 0$. If K_B is such a kernel that the series of the ranges $\boldsymbol{\xi}_*$ of different $\boldsymbol{\xi}$ constituting the above collection overlap with nonzero measure at the intersecting points, the constant is unique over $(\xi_{i*} - v_{wi})n_i < 0$, i.e., $f(\boldsymbol{\xi}_*) = c_0 f_0(\boldsymbol{\xi}_*)$ (c_0 : a constant) in $(\xi_{i*} - v_{wi})n_i < 0$ (see Fig. 1).[41] Then, from the condition (M-1.27c),

$$f(\boldsymbol{\xi}) = c_0 f_0(\boldsymbol{\xi}) \quad \text{for all } \boldsymbol{\xi}. \tag{165}$$

Incidentally, the kernel K_B that is positive almost everywhere (Footnote M-5 in Section M-1.2) is classified as positive, and Eq. (165) holds almost everywhere of $\boldsymbol{\xi}$. When the overlap-covering condition is not satisfied, the above Maxwellian is not necessarily required for the equality.[42]

The equality condition of Eq. (M-A.267) is seen to be the same as that of Eq. (M-A.266) in the following way. Obviously, $B = A \Leftrightarrow \int_V a(\boldsymbol{\xi})[B(\boldsymbol{\xi}) - A(\boldsymbol{\xi})]\mathbf{d}\boldsymbol{\xi} = 0$ if $A(\boldsymbol{\xi}) \le B(\boldsymbol{\xi})$ and $a(\boldsymbol{\xi}) > 0$. Taking

$$A(\boldsymbol{\xi}) = F\left(\frac{f(\boldsymbol{\xi})}{f_0(\boldsymbol{\xi})}\right), \qquad B(\boldsymbol{\xi}) = \int_{(\xi_{i*} - v_{wi})n_i < 0} \frac{K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f_0(\boldsymbol{\xi}_*)}{f_0(\boldsymbol{\xi})} F\left(\frac{f(\boldsymbol{\xi}_*)}{f_0(\boldsymbol{\xi}_*)}\right) \mathbf{d}\boldsymbol{\xi}_*,$$

[40](i) In the common region, $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ cannot take two values. On a set with measure zero, whether $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is determined or not can be ignored. (See Footnote M-5 in Section M-1.2 for the set with measure zero.)

(ii) If the intersection is empty or measure zero, the integrations with respect to $\boldsymbol{\xi}_*$ at different $\boldsymbol{\xi}$'s, are not influenced by the $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ determined by the other $\boldsymbol{\xi}$.

(iii) The equality only on a set of $\boldsymbol{\xi}$ with measure zero is ignored. Thus, the above set of $\boldsymbol{\xi}_*$ where $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is constant is required to have some extent with measure nonzero with respect to $\boldsymbol{\xi}$ including the intersections.

[41]The collection has to have some extent mentioned in Footnote 40 (iii).

[42]In fact, Takata (private communication) constructed a kernel K_B, which is zero in $[(\xi_i - v_{wi})n_i - C][(\xi_{i*} - v_{wi})n_i + C_*] > 0$ (C and C_*: some positive constants) and satisfies the conditions (M-1.27a)–(M-1.27c), for which the equality holds for another function.

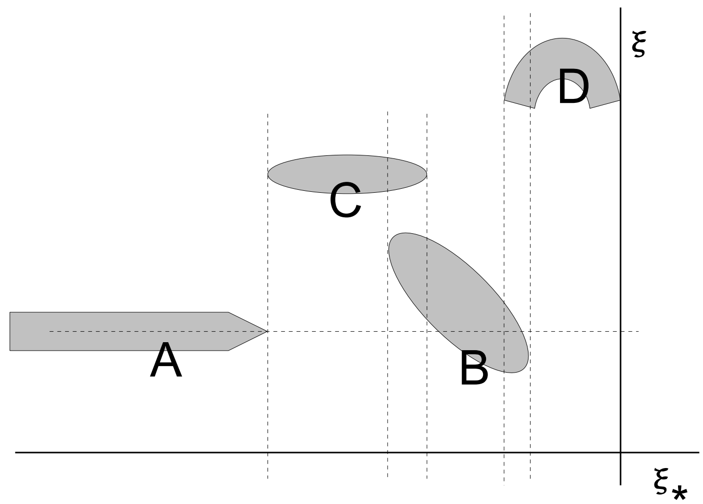

Figure 1: Kernel $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ that requires $f(\boldsymbol{\xi}) = c_0 f_0(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$. The quarter in the figure is the range $(\xi_{i*} - v_{wi})n_i < 0$ and $(\xi_i - v_{wi})n_i > 0$ in the space $(\boldsymbol{\xi_*}, \boldsymbol{\xi})$. Let $K_B > 0$ in the regions A, B, C, and D at least, and their ranges of $\boldsymbol{\xi}_*$ cover $(\xi_{i*} - v_{wi})n_i < 0$. Then, $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is constant in each of A, B, C, and D (say, a in A, b in B, c in C, and d in D). Some ranges in A and B being on a common $\boldsymbol{\xi}$ having some extent, $a = b$. In view of the intersection of the ranges of $\boldsymbol{\xi}_*$ of B and C and that of B and D, $c = b\,(= a)$, and $d = b\,(= a)$. Thus, $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*) = a$ in $(\xi_{i*} - v_{wi})n_i < 0$. It may be noted that the regions of $\boldsymbol{\xi}_*$ of A and C are required to be only in contact with each other because the intersection of the ranges of $\boldsymbol{\xi}_*$ of C and B is not measure zero.

and $(\xi_i - v_{wi})n_i > 0$ as the domain V of integration, and comparing Eq. (M-A.266) and its next equation without number, we find the equivalence of the equality conditions of Eqs. (M-A.266) and (M-A.267). The above discussion being common for a strictly convex function F, the equality condition applies to the Darrozes–Guiraud inequality (M-A.262) and Eq. (M-A.268).

(Section 6.4.1: Version 5-00)

6.4.2 Extension of the Darrozes–Guiraud inequality to an interface

Darrozes–Guiraud inequality (M-A.262) or (M-A.267) is proved for a function f satisfying the boundary condition (M-1.26) on a simple boundary (Reference M-70). Here, we discuss its extension to f that satisfies the boundary condition (M-1.30) on an interface of a gas and its condensed phase.

The boundary condition on the interface is given as[43]

$$f(\boldsymbol{\xi}) = g_I(\boldsymbol{\xi}) + \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(\boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0], \tag{166}$$

where g_I and K_I are independent of f. Further, g_I and K_I satisfy the following conditions [see Eqs. (M-1.31a)–(M-1.31c)]:

(i) Nonnegativity of g_I

$$g_I(\boldsymbol{\xi}) \geq 0 \qquad [(\xi_i - v_{wi})n_i > 0]. \tag{167a}$$

(ii) Nonnegativity of K_I

$$K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) \geq 0 \qquad [(\xi_i - v_{wi})n_i > 0, \quad (\xi_{i*} - v_{wi})n_i < 0]. \tag{167b}$$

(iii) Condition of establishment of the equilibrium state

$$f_w(\boldsymbol{\xi}) = g_I(\boldsymbol{\xi}) + \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f_w(\boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0], \tag{167c}$$

where f_w is the Maxwellian determined by the temperature T_w and velocity v_{wi} of the interface and the saturation gas density ρ_w at temperature T_w i.e.,

$$f_w(\boldsymbol{\xi}) = \frac{\rho_w}{(2\pi R T_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_{wi})^2}{2RT_w}\right). \tag{168}$$

It is also required here that if $f(\boldsymbol{\xi}_*)$ for $(\xi_{i*} - v_{wi})n_i < 0$ is the corresponding part of another Maxwellian [say, $f_e(\boldsymbol{\xi})$], it does not give $f_e(\boldsymbol{\xi})$ for $(\xi_i - v_{wi})n_i > 0$, which will be called the uniqueness condition of Eq. (167c) for shortness.
In the following discussion, we impose another condition in addition to Eqs. (167a)–(167c), i.e., putting

$$\alpha(\boldsymbol{\xi}_*) = -\int_{(\xi_i-v_{wi})n_i>0} \frac{(\xi_i - v_{wi})n_i}{(\xi_{j*} - v_{wj})n_j} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi} \quad [(\xi_{j*} - v_{wj})n_j < 0], \tag{169}$$

[43] The variables $\boldsymbol{X}$ and t are not shown here because they are not important in the present discussion [see Footnote M-10 (ii) in Section M-1.5].

we assume[44] that

$$0 \le \alpha(\boldsymbol{\xi}_*) \le 1 \qquad [(\xi_{i*} - v_{wi})n_i < 0]. \tag{170}$$

Incidentally, from Eqs. (167a)–(167c),

$$f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) \ge 0. \tag{171}$$

We will show that the inequality (M-A.267) with f_0 being replaced by f_w, i.e.,

$$\int_{\text{all }\boldsymbol{\xi}} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) F[f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]\mathbf{d}\boldsymbol{\xi} \le 0, \tag{172}$$

holds when $F(x)$ is such a strictly convex function (see Footnote M-52 in Section M-A.10) that

$$F(x) \ge 0 \text{ and } F(1) = 0.$$

The equality of the relation (172) holds when $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$, and this relation is required except for some boundary conditions shown later. The inequality is proved with the aid of the Jensen inequality [see Eq. (M-A.265) or References M-110, M-129, M-158, or M-171]

$$F\bigg(\int \phi\psi \mathbf{d}\boldsymbol{\xi} \bigg/ \int \psi \mathbf{d}\boldsymbol{\xi}\bigg) \le \int \psi F(\phi)\mathbf{d}\boldsymbol{\xi} \bigg/ \int \psi \mathbf{d}\boldsymbol{\xi} \qquad (\psi \ge 0), \tag{173}$$

where $F(x)$ is a strictly convex function, and ϕ and ψ ($\psi \ge 0$) are arbitrary functions of $\boldsymbol{\xi}$. The equality sign holds when ϕ is independent of $\boldsymbol{\xi}$; it is also required where $\psi > 0$ for the equality.

[44](i) This condition corresponds to Eq. (M-1.27b) for a simple boundary. The simple boundary consists of molecules of different kinds from the gas molecules, and they stay there forever. The gas molecules impinging on the boundary are reflected without time delay (in the time scale of our interest), and there is no net mass flux to the boundary in this process. The condition (M-1.27b) is derived from this situation, as explained in Footnote M-13 in Section M-1.6.1. In the case of an interface, the condition (170) is derived similarly, if we consider that some of the molecules impinging on the interface do not reflect and stay there. However, the interface is the condensed phase of the gas and consists of the same kind of molecules as the gas. On the interface, molecules leave it depending on the condition of the interface even if there is no impinging molecules; this is the g_I part in Eq. (166). When a molecule impinges on the interface, it interacts with molecules of the interface, and some molecules leave the interface. Whether the impinging molecule is reflected or kicks out another molecule has no difference. Further, depending on the condition (e.g., speed or direction) of the impinging molecule and that of the interface, more than one molecule may be kicked out or no molecule may be kicked out or reflected. Thus, it is not clear that the condition (170) holds or not. However, it is sure that the size of the kernel K_I is limited owing to the conditions (167a)–(167c), e.g., $K_I = 0$ if $g_I = f_w$ (the complete condensation). See also Footnote 47 in Section 6.4.2.

(ii-a) The case $\alpha(\boldsymbol{\xi}_*) = 1$ for $(\xi_{j*} - v_{wj})n_j < 0$ is excluded by the uniqueness condition of Eq. (167c). In fact, multiplying Eq. (166) by $(\xi_j - v_{wj})n_j$ and integrating with respect to $\boldsymbol{\xi}$ over $(\xi_j - v_{wj})n_j > 0$, we obtain $g_I(\xi) = 0$. Thus, Cf_w (C : a constant) also satisfies Eq. (166).

(ii-b) When $\alpha(\boldsymbol{\xi}_*) = 0$ for $(\xi_{j*} - v_{wj})n_j < 0$, the kernel $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ degenerates, i.e., $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) = 0$ for $(\xi_j - v_{wj})n_j > 0$. This is the case of the complete condensation.

Let $F(x)$ be a nonnegative strictly convex function that takes value zero at $x = 1$,[45] i.e.,

$$F(x) \geq 0, \quad F(1) = 0. \tag{174}$$

Consider the function $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$, where $f_w(\boldsymbol{\xi})$ is given by Eq. (168). The function $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$ for $(\xi_i - v_{wi})n_i > 0$ is bounded by an integral of $f(\boldsymbol{\xi})$ for $(\xi_i - v_{wi})n_i < 0$ with the aid of Eq. (166) in the following way:

$$\begin{aligned}
F\left(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right) &= F\left(\frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} + \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} f(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_*\right) \\
&= F\left[\frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} + \left(1 - \frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)\int_{<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1 - g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\right] \\
&\leq \frac{g_I}{f_w}F(1) + \left(1 - \frac{g_I}{f_w}\right)F\left(\int_{<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1 - g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\right) \\
&= \left(1 - \frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)F\left(\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1 - g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\right) \\
&\qquad [(\xi_i - v_{wi})n_i > 0].
\end{aligned} \tag{175}$$

Here, we, for a moment, consider the point of $\boldsymbol{\xi}$ $[(\xi_i - v_{wi})n_i > 0]$ where

$$f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) > 0,$$

for which

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1 - g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})}\mathbf{d}\boldsymbol{\xi}_* = 1 \qquad [(\xi_i - v_{wi})n_i > 0],$$

because of Eq. (167c); in the second and third lines, the simple $<$ sign of the subscript of the integral sign $\int$ indicates $(\xi_{i*} - v_{wi})n_i < 0$; the convex property of $F(x)$ is used from the second line to the third, and $F(1) = 0$ is used from the third to the fourth.

Now, we apply the Jensen inequality (173) to the function F on the fourth line in Eq. (175). Here, we choose $\phi(\boldsymbol{\xi}_*)$ and $\psi(\boldsymbol{\xi}_*)$ as

$$\begin{aligned}
\phi(\boldsymbol{\xi}_*) &= \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}, \\
\psi(\boldsymbol{\xi}_*) &= \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1 - g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \geq 0 \quad [(\xi_i - v_{wi})n_i > 0, \quad (\xi_{i*} - v_{wi})n_i < 0].
\end{aligned}$$

It should be noted that $\phi(\boldsymbol{\xi}_*)$ is defined for the whole range of $\boldsymbol{\xi}_*$ and that $\psi(\boldsymbol{\xi}_*)$ depends also on $\boldsymbol{\xi}$ and satisfies the relation, irrespective of $\boldsymbol{\xi}$,

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \psi(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* = 1 \qquad [(\xi_i - v_{wi})n_i > 0].$$

[45] Note that $x = 1$ is the unique zero point of $F(x)$.

Then, $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$ for $(\xi_i - v_{wi})n_i > 0$ is bounded as

$$F\bigg(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\bigg) \leq \bigg(1 - \frac{g_I}{f_w}\bigg) F\bigg(\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\bigg)$$
$$\leq \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} F\bigg(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\bigg)\mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0]. \tag{176}$$

Up to this point, we limited our discussion to the point of $\boldsymbol{\xi}$ $[(\xi_i - v_{wi})n_i > 0]$ where

$$f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) > 0.$$

If it vanishes at some $\boldsymbol{\xi}_A$ $[(\xi_{iA} - v_{wi})n_i > 0]$, i.e.,

$$f_w(\boldsymbol{\xi}_A) - g_I(\boldsymbol{\xi}_A) = 0, \tag{177}$$

the integral $\int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_*$ vanishes there, i.e.,

$$\int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* = 0,$$

because of the condition (167c). The function $f_w(\boldsymbol{\xi}_*)$ being positive for all $\boldsymbol{\xi}_*$, the kernel $K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*)$ must vanish for $(\xi_{i*} - v_{wi})n_i < 0$, i.e.,

$$K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*) = 0 \qquad [(\xi_{i*} - v_{wi})n_i < 0]. \tag{178}$$

Thus, from the boundary condition (166),

$$f(\boldsymbol{\xi}_A) = g_I(\boldsymbol{\xi}_A) = f_w(\boldsymbol{\xi}_A).$$

Therefore, the function $F(f(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A))$ vanishes, i.e.,

$$F(f(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A)) = F(1) = 0. \tag{179}$$

From Eqs. (178) and (179), the equality holds between the left-most side and the right-most of Eq. (176) at $\boldsymbol{\xi} = \boldsymbol{\xi}_A$. In conclusion, the inequality

$$F\bigg(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\bigg) \leq \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} F\bigg(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\bigg)\mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0], \tag{180}$$

holds without the assumption of $f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) > 0$.

When $f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}) = 1$ for all $\boldsymbol{\xi}$, $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$ vanishes in Eq. (180), and the equality holds there. We look for the other possibilities of the equality. The first inequality in Eq. (176) comes from that of Eq. (175), for which the equality holds at $\boldsymbol{\xi} = \boldsymbol{\xi}_A$ when (i) $g_I(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A) = 0$ or (ii) $g_I(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A) = 1$, or (iii) the arguments of two F's on the third line of Eq. (175) are equal, i.e.,

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A)]f_w(\boldsymbol{\xi}_A)} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_* = 1, \tag{181}$$

for some $f(\boldsymbol{\xi}_*)$. In the third case, the equality relation being imposed between the first and the second line on the right-hand side of Eq. (176) under the condition (181), we find that

$$f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*) \text{ in } \mathrm{B_A}(\boldsymbol{\xi}_*),$$

where $\mathrm{B}_A(\boldsymbol{\xi}_*)$ is the region of $\boldsymbol{\xi}_*$ in which $K_I(\boldsymbol{\xi}_A, \boldsymbol{\xi}_*) > 0$.

If $g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}) = 0$ for $(\xi_i - v_{wi})n_i > 0$, the boundary condition (166) reduces to

$$f(\boldsymbol{\xi}) = \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(\boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_*. \tag{182}$$

Then, the Maxwellian $a_0 f_w(\boldsymbol{\xi})$ (a_0 : a constant) also satisfies the boundary condition (166), which is not allowed by the uniqueness condition of Eq. (167c). Thus, this case is excluded. If $g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}) = 1$ for $(\xi_i - v_{wi})n_i > 0$, the kernel $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ vanishes for $(\xi_i - v_{wi})n_i > 0$ and $(\xi_{i*} - v_{wi})n_i < 0$ from the discussion in the preceding paragraph. That is, $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ in $(\xi_i - v_{wi})n_i > 0$ irrespective of $f(\boldsymbol{\xi})$ in $(\xi_i - v_{wi})n_i < 0$ (this is the case of the complete condensation condition). For this case the equality holds in Eq. (180). If the third condition holds for $(\xi_i - v_{wi})n_i > 0$, we have

$$f_w(\boldsymbol{\xi}) = g_I(\boldsymbol{\xi}) + \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(\boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0]. \tag{183}$$

From the discussion of the preceding paragraph,

$$f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*) \text{ in } \mathrm{B}(\boldsymbol{\xi}_*), \tag{184}$$

where $\mathrm{B}(\boldsymbol{\xi}_*)$ is the region of $\boldsymbol{\xi}_*$ in which $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ for some $\boldsymbol{\xi}$. This condition is paraphrased as

$$f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*) \text{ except in the region } \alpha(\boldsymbol{\xi}_*) = 0. \tag{185}$$

Whether $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ or $\alpha(\boldsymbol{\xi}_*) = 0$ in $(\xi_{i*} - v_{wi})n_i < 0$,

$$f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi}) \;\; [(\xi_i - v_{wi})n_i > 0].$$

Let us consider the case where the three situations (i), (ii), and (iii) listed just before Eq. (181) take place for different $\boldsymbol{\xi}$, say, (i) for $\boldsymbol{\xi}$ in A_1, (ii) for $\boldsymbol{\xi}$ in A_2, and (iii) for $\boldsymbol{\xi}$ in A_3. The A_2 part does not contribute to the restriction on $f(\boldsymbol{\xi}_*)$. When A_1 is empty, the condition is the same as for the case of Eq. (183), i.e., Eq. (184) or (185). When A_1 is not empty, from the discussion for $\boldsymbol{\xi}$ in A_3, $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ in the region of $\boldsymbol{\xi}_*$ where $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ for some $\boldsymbol{\xi}$ in A_3 [say, $\mathrm{B}_3(\boldsymbol{\xi}_*)$], and the condition for the remaining $\boldsymbol{\xi}_*$ is determined only by the behavior of K_I for $\boldsymbol{\xi}$ in A_1, that is, the region $f(\boldsymbol{\xi}_*)/f_w(\boldsymbol{\xi}_*) = \text{const}$ [say, $\mathrm{B}_1(\boldsymbol{\xi}_*)$] is looked for in the range $(\xi_{i*} - v_{wi})n_i < 0$ in the same way as in Section 6.4.1 and if B_1 has a common region with B_3, $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ in B_1. In the region of the remaining $\boldsymbol{\xi}_*$ [say, $\mathrm{R}(\boldsymbol{\xi}_*)$], $f(\boldsymbol{\xi}_*)$ other than $f_w(\boldsymbol{\xi}_*)$ can exist. The region $\alpha(\boldsymbol{\xi}_*) = 1$ in $\mathrm{R}(\boldsymbol{\xi}_*)$ is denoted by $\mathrm{R}_{\alpha=1}$ for the convenience in the later citation.

When A_3 is empty, the boundary condition (166) is expressed as

$$f(\boldsymbol{\xi}) = \begin{pmatrix} 0 \\ f_w(\boldsymbol{\xi}) \end{pmatrix} + \int_{(\xi_{i*}-v_{wi})n_i<0} \begin{pmatrix} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*) \\ 0 \end{pmatrix} f(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* \quad \begin{matrix} [\boldsymbol{\xi} \text{ in } A_1] \\ [\boldsymbol{\xi} \text{ in } A_2] \end{matrix}, \tag{186}$$

where

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})}\mathbf{d}\boldsymbol{\xi}_* = 1 \qquad [(\xi_i - v_{wi})n_i > 0 \text{ and } \boldsymbol{\xi} \text{ in } A_1].$$

The boundary condition (186) obviously satisfies the conditions (167a)–(167c).[46] In this case, the restriction on $f(\boldsymbol{\xi}_*)$ is determined by K_I in A_1. Substituting $f(\boldsymbol{\xi}_*) = C_D f_w(\boldsymbol{\xi}_*)$ [$(\xi_{i*} - v_{wi})n_i < 0$, C_D : independent of $\boldsymbol{\xi}_*$], which is the strongest restriction on $f(\boldsymbol{\xi}_*)$, into Eq. (186), we have $f(\boldsymbol{\xi}) = C_D f_w(\boldsymbol{\xi})$ [in A_1] and $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ [in A_2] for $(\xi_i - v_{wi})n_i > 0$. For this $f(\boldsymbol{\xi})$, the equality holds in Eq. (180). Thus, for the boundary condition (186) as well as the complete condensation condition, the equality in Eq. (180) holds for $f(\boldsymbol{\xi})$ other than $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ [$f(\boldsymbol{\xi}_*) = C_D f_w(\boldsymbol{\xi}_*)$ for $(\xi_{i*} - v_{wi})n_i < 0$ for Eq. (186), and $f(\boldsymbol{\xi}_*)$ is arbitrary for $(\xi_{i*} - v_{wi})n_i < 0$ for the complete condensation]. This is an example of $f(\boldsymbol{\xi}_*)$ that satisfies the equality in Eq. (180).

With the aid of the inequality (180) and Eq. (169), we have

$$\begin{aligned}
&\int_{(\xi_i-v_{wi})n_i>0} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) F\left(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)\mathbf{d}\boldsymbol{\xi} \\
&\leq \int_{(\xi_i-v_{wi})n_i>0} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right)\mathbf{d}\boldsymbol{\xi}_*\mathbf{d}\boldsymbol{\xi} \\
&= \int_{(\xi_{i*}-v_{wi})n_i<0} f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right) \int_{(\xi_i-v_{wi})n_i>0} (\xi_i - v_{wi})n_i K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}\mathbf{d}\boldsymbol{\xi}_* \\
&= -\int_{(\xi_{i*}-v_{wi})n_i<0} \alpha(\boldsymbol{\xi}_*)(\xi_{i*} - v_{wi})n_i f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right)\mathbf{d}\boldsymbol{\xi}_*,
\end{aligned} \tag{187}$$

where $0 \leq \alpha(\boldsymbol{\xi}_*) \leq 1$ [the assumption (170)]. Thus, we obtain the extension of Eq. (M-A.267) to the case of an interface as follows:

$$\begin{aligned}
&\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) F\left(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)\mathbf{d}\boldsymbol{\xi} \\
&\qquad \leq \int_{(\xi_{i*}-v_{wi})n_i<0} [1-\alpha(\boldsymbol{\xi}_*)](\xi_{i*} - v_{wi})n_i f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right)\mathbf{d}\boldsymbol{\xi}_* \leq 0.
\end{aligned} \tag{188}$$

Obviously, the equal sign holds in the two inequalities of Eq. (188) when $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$. Conversely, it is required for the equal sign to hold in the inequalities

[46]To confirm the uniqueness condition of Eq. (167c) is simple. Note $f(\boldsymbol{\xi})$ [$(\xi_i - v_{wi})n_i > 0$] for $\boldsymbol{\xi}$ in A_2.

that $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$ when $\mathrm{R}_{\alpha=1}$ is empty.[47] It should be noted that $F(x)$ is required to satisfy that $F(x) \geq 0$ and $F(1) = 0$ in addition to convexity. Here, we take

$$F(x) = x(\ln x - 1) + 1,$$

which is strictly convex, nonnegative, and zero at $x = 1$. Then,

$$\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i \left[f(\boldsymbol{\xi}) \left(\ln \frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} - 1 \right) + f_w(\boldsymbol{\xi}) \right] \mathbf{d}\boldsymbol{\xi} \leq 0,$$

or

$$\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln \frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} \mathbf{d}\boldsymbol{\xi} \leq \rho(v_i - v_{wi})n_i. \tag{189}$$

This is the extension of Eq. (M-A.262) for a simple boundary to an interface.

We try to express the inequality (189) in terms of macroscopic variables. It is simply transformed in the following form:

$$\begin{aligned}
&\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln \frac{f(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi} \\
&\leq \int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln \frac{f_w(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi} + \rho(v_i - v_{wi})n_i \\
&= -\frac{1}{RT_w} \left[q_i n_i + (v_j - v_{wj})\tilde{p}_{ij} n_i + \rho(v_i - v_{wi})n_i \left(\frac{5}{2}RT + \frac{1}{2}(v_j - v_{wj})^2 \right) \right] \\
&\quad + \rho(v_i - v_{wi})n_i \left(\ln \frac{\rho_w}{(2\pi RT_w)^{3/2} c_0} + 1 \right),
\end{aligned}$$

where c_0 is a constant to make the argument of the logarithmic function dimensionless, and

$$\tilde{p}_{ij} = p_{ij} - p\delta_{ij}, \tag{190}$$

[47](i) The integration of a nonnegative function multiplied by a positive function does not change the equality condition. Thus, the equality condition of the inequality of Eq. (187) is the same as that of Eq. (180) $[B = A \Leftrightarrow \int a(\boldsymbol{\xi})[B(\boldsymbol{\xi}) - A(\boldsymbol{\xi})]\mathbf{d}\boldsymbol{\xi} = 0$ if $A(\boldsymbol{\xi}) \leq B(\boldsymbol{\xi})$ and $a(\boldsymbol{\xi}) > 0]$. Thus, the range where $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ is required is outside R. For the equality of the Darrozes–Guiraud inequality, we have to examine the equality of the second inequality in Eq. (188). The second equal sign holds only when $F(f(\boldsymbol{\xi}_*)/f_w(\boldsymbol{\xi}_*)) = 0$ in R outside $\mathrm{R}_{\alpha=1}$ because $f_w(\boldsymbol{\xi}_*) > 0$ and $1 - \alpha(\boldsymbol{\xi}_*) > 0$ there. Thus, $f(\boldsymbol{\xi}_*)/f_w(\boldsymbol{\xi}_*) = 1$ outside $\mathrm{R}_{\alpha=1}$ in $(\xi_{i*} - v_{wi})n_i < 0$ (see Footnote 45 in Section 6.4.2). When $\mathrm{R}_{\alpha=1}$ is empty, the integral $\int_{\text{all } \boldsymbol{\xi}}$ on the left-most side reduces to $\int_{(\xi_i - v_{wi})n_i > 0}$. This vanishes only when $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})) = 0$, i.e., $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for $(\xi_i - v_{wi})n_i > 0$. Thus, $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$ when $\mathrm{R}_{\alpha=1}$ is empty. It may be noted that when A_3 is empty [or for the boundary condition (186)], $\mathrm{R}_{\alpha=1}$ is the range of $\boldsymbol{\xi}_*$ where $\alpha(\boldsymbol{\xi}_*) = 1$ in $(\xi_{i*} - v_{wi})n_i < 0$. Incidentally, $g_I(\boldsymbol{\xi})$ that is positive almost everywhere (Footnote M-5 in Section M-1.2) is classified positive, for which A_1 in the paragraph following to that of Eq. (185) is empty and Eq. (185) holds (that is, $\mathrm{R}_{\alpha=1}$ is empty), and therefore the equal signs hold in Eq. (188) only when $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$.

(ii) If $\alpha(\boldsymbol{\xi}_*)$ exceeds unity for some range of $\boldsymbol{\xi}_*$ in $(\xi_{i*} - v_{wi})n_i < 0$ and the assumption (170) is violated, but the integral

$$\int_{(\xi_{i*} - v_{wi})n_i < 0} [1 - \alpha(\boldsymbol{\xi}_*)](\xi_{i*} - v_{wi})n_i f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)} \right) \mathbf{d}\boldsymbol{\xi}_*$$

is nonpositive, the inequality holds.

The $\tilde{p}_{ij}$ is the part of stress tensor with the pressure contribution subtracted. Only the tangential component of the stress $\tilde{p}_{ij}n_i$ contributes to $(v_j - v_{wj})\tilde{p}_{ij}n_i$ when no flow to the boundary. Further, $\ln \rho_w/(2\pi RT_w)^{3/2}c_0$ is related to the H function H_w for $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ as

$$\frac{H_w}{\rho_w} = \ln \frac{\rho_w}{(2\pi RT_w)^{3/2}c_0} - \frac{3}{2}, \tag{191}$$

which is independent of v_{wi}. That is,

$$H_w = \int_{\text{all }\boldsymbol{\xi}} f_w(\boldsymbol{\xi}) \ln \frac{f_w(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi} = \int_{\text{all }\boldsymbol{\xi}} f_w^{(v)}(\boldsymbol{\xi}) \ln \frac{f_w^{(v)}(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi},$$

where

$$f_w^{(v)}(\boldsymbol{\xi}) = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_i)^2}{2RT_w}\right).$$

On the other hand, by definition (see Section M-1.7),

$$\int_{\text{all }\boldsymbol{\xi}} (\xi_i - v_{wi}) n_i f(\boldsymbol{\xi}) \ln[f(\boldsymbol{\xi})/c_0] \mathbf{d}\boldsymbol{\xi} = (H_i - H v_{wi}) n_i.$$

Therefore,

$$\begin{aligned}
&(H_i - H v_{wi}) n_i \\
&\quad \leq -\frac{1}{RT_w} \left[q_i n_i + (v_j - v_{wj})\tilde{p}_{ij} n_i\right] \\
&\qquad + \rho(v_i - v_{wi}) n_i \left[\frac{H_w}{\rho_w} - \frac{1}{RT_w}\left(\frac{5}{2}R(T - T_w) + \frac{1}{2}(v_j - v_{wj})^2\right)\right].
\end{aligned} \tag{192}$$

When $f = f_w$, both sides of the inequality vanish and the equal sign holds. Conversely, for the kernel K_I with $R_{\alpha=1}$ empty, e.g., g_I that is positive almost everywhere, the equal sign holds only when $f = f_w$.

Finally, we consider the variation of the integral $\overline{H}$ of H over the domain D. According to Eq. (M-1.36),

$$\frac{\mathrm{d}\overline{H}}{\mathrm{d}t} = \int_{\partial D} (H_i - H v_{wi}) n_i + \int_D G \boldsymbol{dX},$$

where

$$\overline{H} = \int_D H \boldsymbol{dX}.$$

With the aid of Eq. (192), the variation is bounded as

$$\begin{aligned}
\frac{\mathrm{d}\overline{H}}{\mathrm{d}t} &\leq -\frac{1}{RT_w} \left[q_i n_i + (v_j - v_{wj})\tilde{p}_{ij} n_i\right] \\
&\quad + \rho(v_i - v_{wi}) n_i \left[\frac{H_w}{\rho_w} - \frac{1}{RT_w}\left(\frac{5}{2}R(T - T_w) + \frac{1}{2}(v_j - v_{wj})^2\right)\right],
\end{aligned} \tag{193}$$

because $\int_D G \boldsymbol{dX} \leq 0$ [see Eq. (M-1.34b)].

(Section 6.4.2: Version 5-00)