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Molecular Gas Dynamics
Yoshio Sone
(Birkhäuser, Boston, 2007)

Yoshio Sone

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Bibliography

Update of bibliography

• [19] Arkeryd, L. and A. Nouri [2006]:
  (to be published) → 401–443
• [29] Bardos, C., F. Golse, and Y. Sone [2006]:
  (to be published) → 275–300
  [2006] → [2007]
  59, (to be published) → 60, 147–163
  [2006a] → [2006]
• [137] Liu, T.-P. and S.-H. Yu [2006b]:
  [2006b] → [2007]
  59, (to be published) → 60, 295–356

The corresponding corrections in the text

• p. 166, the first line in the third paragraph:
  [2004b, 2006a] → [2004b, 2006]
• p. 166, the last line in the third paragraph:
  [2006b] → [2007]
• p. 183, the second line:
  [2006] → [2007]
Errata

• p. 9, the 7th line:  
  specular condition  →  specular reflection

• p. 27, the 3rd line of Footnote 26:  
  Eq. (1.99)  →  a linear combination of Eqs. (1.99) and (1.101)

• p. 27, the 6th line of Footnote 26:  
  except for a common constant factor  →  except for a common constant factor and additive functions
  (say, $f_a$ in $\tilde{H}$ and $f_{b_i}$ in $\tilde{H}_i$ in their second order) satisfying
  $\mathbf{Sh}\partial f_a/\partial t + \partial f_{b_i}/\partial x_i = 0$

• p. 48, the 21st line, p. 49, the 3rd line from below, and p. 488, the 4th line:  
  solid angle element  →  solid-angle element

• p. 81, the 4th line in Footnote 7:  
  $u_{iGm}  \rightarrow  u_{iGm} - u_{jGm}n_jn_i$
  or $\phi_{eGm}  \rightarrow  \phi_{eGm}$

• p. 83, the first line in Footnote 14:  
  $u_{iGm}  \rightarrow  u_{iGm} - u_{jGm}n_jn_i$

• p. 503, the 13th line from below:  
  solid angle elements  →  solid-angle elements

• p. 504, the first line in Footnote 24:  
  domain  →  domain

• p. 505, Eq. (A.60):  
  \[
  \frac{1}{\sin \theta_c} \frac{\partial}{\partial \theta_c} \sin^2 \theta_c \rightarrow \frac{1}{\sin \theta_c} \frac{d}{d \theta_c} \sin^2 \theta_c
  \]
  \[
  \frac{1}{\sin \theta_c} \frac{\partial b^2}{\partial \theta_c} \rightarrow \frac{1}{\sin \theta_c} \frac{db^2}{d \theta_c}
  \]

• p. 506, the 13th line [The line next to Eq. (M-A.63)]:  
  with respect to $\theta_c$  →  with respect to $\theta_a$

• p. 617, the right-hand side of Eq. (C.2b):  
  In order to avoid misunderstanding, $\frac{2(n+1)!}{\beta n+2} \pi$ is better expressed as
  $$\frac{2\pi (n+1)!}{\beta n+2}$$

• p. 628, Reference [110]:  

Reference [110] should be placed after Reference [112].

- p. 639, the 3rd line in Reference [262]:
  $gs \rightarrow gas$

Supplementary Notes

In the present supplementary notes, the letter M is attached to the labels of sections, equations, etc. in the book Molecular Gas Dynamics and the letter K is attached to those in Y. Sone, Kinetic Theory and Fluid Dynamics (Birkhäuser, Boston, 2002) to avoid confusion. The two books, Molecular Gas Dynamics and Kinetic Theory and Fluid Dynamics, themselves are, respectively, referred to as MGD and KF.

1 Chapter M-1

1.1 Supplement to Footnote M-9 in Section M-1.3

We will explicitly show the process of derivation of the conservation equations (M-1.12)–(M-1.14) by taking into account the discontinuity of the velocity distribution function $f(X, \xi, t)$ for a typical case.

Let $S(X)$ be a continuous and sectionally smooth function of $X$, and let the surface in the $X$ space consisting of the points $X_0$ that satisfy $S(X_0) = 0$ be indicated by $S_0$. The surface $S_0$ may be an infinite surface or a bounded surface separating the space $X$ into two regions. The velocity distribution function $f$ at time $t_0$ is assumed to be discontinuous across the surface $S_0$ and to be smooth except on $S_0$. The discontinuity propagates along the characteristics of the Boltzmann equation (M-1.5), i.e., $X_i - \xi_i(t - t_0) = X_{0i}$, for each $\xi$. Take a point $(X, t)$ in the space and time, where $t > t_0$. At this point or at $(X, t)$, the discontinuity of $f$ lies on the surface $S^{(\xi)}(X, t)$ in the $\xi$ space that consists of the points $\xi_D$ satisfying

$$S(X_i - \xi_{Di}(t - t_0)) = 0, \text{ or } X_i - \xi_{Di}(t - t_0) = X_{0i}. \quad (1)$$

The point $\xi_D$ is determined by $X$, $t$, and $X_0$, i.e., $\xi_D(X, t; X_0)$. Let the side of the domain in the $\xi$ space that satisfies $S(X_i - \xi_i(t - t_0)) > 0$ be indicated by $V_+$, and the other side of the domain by $V_-$; let the outward unit normal to the surface $S^{(\xi)}(X, t)$ with respect to $V_+$ be indicated by $n_{Di}(\xi_D; X, t)$. Then,

$$n_{Di}(\xi_D; X, t) = -\frac{\partial S(X - \xi(t - t_0))/\partial \xi_i}{\partial S(X - \xi(t - t_0))/\partial \xi_j} \bigg|_{\xi = \xi_D} = \frac{\partial S(Y)/\partial Y_i}{\partial S(Y)/\partial Y_j} \bigg|_{\xi_D}, \quad (2)$$

1It is assumed that $(\partial S/\partial X_i)^2 \neq 0$ on $S_0$. The normal to the surface $S_0$ is defined except at special points.

2For simplicity of explanation, we consider the case where $F_i = 0$ here.
where \(|a_i| = (a_i^2)^{1/2}\) and the subscript \(D\) to \(\partial S(Y)/\partial Y_j\) indicates \(Y = X - \xi_D(t - t_0)\). The variations of \(\xi_D\) with respect to \(X\) or \(t\) for a given \(X_0\), i.e., \(\partial \xi_D/\partial X_i\) and \(\partial \xi_D/\partial t\), are determined from Eq. (1) as

\[
\frac{\partial S(Y)}{\partial Y_j}\bigg|_D \left( \delta_{ij} - \frac{\partial \xi_{Dj}}{\partial X_i}(t - t_0) \right) = 0, \quad \frac{\partial S(Y)}{\partial Y_j}\bigg|_D \left( \frac{\partial \xi_{Dj}}{\partial t}(t - t_0) + \xi_{Dj} \right) = 0.
\]

Thus, with the aid of Eq. (2),

\[
n_{Dj} \frac{\partial \xi_{Dj}}{\partial X_i} = \frac{n_{Di}}{t - t_0}, \quad n_{Dj}\frac{\partial \xi_{Dj}}{\partial t} = -\frac{n_{Dj} \xi_{Dj}}{t - t_0}. \quad (3)
\]

The integral of such a discontinuous function with respect to \(\xi\) over its whole space is split into two parts as

\[
\int \psi(\xi) f d\xi = \int_{V_+} \psi(\xi) f d\xi + \int_{V_-} \psi(\xi) f d\xi,
\]

where \(\psi(\xi)\) is a smooth function of \(\xi\). Then, the integrand is smooth in each of \(V_+\) and \(V_-\). According to Lemma in page M-492, the following derivatives of integrals over the domain \(V_+\) are transformed as\(^3\)

\[
\frac{\partial}{\partial t} \int_{V_+} \psi(\xi) f d\xi = \int_{V_+} \psi(\xi) \frac{\partial f}{\partial t} d\xi + \int_{S(\xi)} \psi(\xi) f \frac{\partial \xi_{Dj}}{\partial t} n_{Dj} d^2\xi,
\]

\[
\frac{\partial}{\partial X_i} \int_{V_+} \xi_i \psi(\xi) f d\xi = \int_{V_+} \xi_i \psi(\xi) \frac{\partial f}{\partial X_i} d\xi + \int_{S(\xi)} \xi_i \psi(\xi) f \frac{\partial \xi_{Dj}}{\partial X_i} n_{Dj} d^2\xi,
\]

where the integral over the surface \(S(\xi)\) of the second term on the right-hand side of each equation is due to the variation of the domain \(V_+\) with \(t\) or \(X_i\). Summing the above two derivatives and noting Eq (3), we have

\[
\frac{\partial}{\partial t} \int_{V_+} \psi(\xi) f d\xi + \frac{\partial}{\partial X_i} \int_{V_+} \xi_i \psi(\xi) f d\xi = \int_{V_+} \psi(\xi) \frac{\partial f}{\partial t} d\xi + \int_{V_+} \xi_i \psi(\xi) \frac{\partial f}{\partial X_i} d\xi,
\]

where the surface integrals over \(S(\xi)\) are cancelled. Similarly,

\[
\frac{\partial}{\partial t} \int_{V_-} \psi(\xi) f d\xi + \frac{\partial}{\partial X_i} \int_{V_-} \xi_i \psi(\xi) f d\xi = \int_{V_-} \psi(\xi) \frac{\partial f}{\partial t} d\xi + \int_{V_-} \xi_i \psi(\xi) \frac{\partial f}{\partial X_i} d\xi.
\]

Thus, we have

\[
\frac{\partial}{\partial t} \int \psi(\xi) f d\xi + \frac{\partial}{\partial X_i} \int \xi_i \psi(\xi) f d\xi = \int \psi(\xi) \frac{\partial f}{\partial t} d\xi + \int \xi_i \psi(\xi) \frac{\partial f}{\partial X_i} d\xi. \quad (4)
\]

It may be noted that the interchange of differentiation and integration is possible only for the above combination of the integrals. With this formula, the conservation equations are derived by choosing \(1, \xi_i, \text{ and } \xi_i^2\) as \(\psi(\xi)\).
When the surface $S_0$, i.e., $S(X) = 0$, is a finite surface or semi-infinite surface which does not divide the $\xi$ space into $V_+$ and $V_-$, we can take it as a special case where some part of $S_0$ joins to its other part and $V_-$ degenerates empty. When there is a body in a gas, the discontinuity as shown in Section M-3.1.6 generally exists. The analysis can be carried out in a similar way; that is, determine the position of the discontinuity in the $\xi$ space first, carry out the differentiations in each region where the velocity distribution function is smooth with the aid of the lemma in page M-492, and sum up the results.

(Section 1.1: Version 6-00)

1.2 Bulk viscosity (Section M-1.3)

The assumptions (M-1.15) and (M-1.16) for the stress tensor and heat-flow vector in classical gas dynamics are what is to be studied by kinetic theory (see Chapter M-3). For a monatomic gas, consisting of identical molecules whose intermolecular potential is spherically symmetric, which is discussed in MGD, the bulk viscosity is easily seen to vanish. From Eqs. (M-1.2d) and (M-1.2f),

$$p_{ii} = 3p. \quad (5)$$

On the other hand, the trace of the first relation of Eq. (M-1.16) is

$$p_{ii} = 3p - 3\mu_B \frac{\partial v_i}{\partial X_i}. \quad (6)$$

Thus, from the two relations, we have

$$\mu_B = 0. \quad (6)$$

(Section 1.2: Version 7-00)

1.3 Note on the equality condition of Eq. (M-1.38)

The statement of the equality condition of Eq. (M-1.38), i.e., "The equality in Eq. (1.38) holds when and only when $f$ is the Maxwellian that satisfies the

---

Footnote: For molecules with internal degree of freedom (e.g., rotational and vibrational freedoms), this freedom contributes to the integrands of Eqs. (M-1.26)-(M-1.2g). Thus, Eq. (5) does not generally hold. More precisely, the velocity distribution function $f$ depends also on the variables of the internal degree of freedom of a molecule. The integration with respect to these variables in Eqs. (M-1.2a)-(M-1.2g) has to be carried out. The angular momentum due to the rotation of molecules of infinitesimal size per unit mass is negligible even when the energy of rotation is not negligible. The density $\rho$ and the specific internal energy $e$ can be clearly defined whether the gas is in an equilibrium state or not. The specific internal energy $e/i_f$ per unit freedom of a molecule is taken as $RT/2$, i.e., $e = i_f RT/2$, where $i_f$ is the degree of freedom of a molecule; thus, the relation between $e$ and $T$ is independent of the state of the gas (equilibrium or nonequilibrium). The pressure is defined by the equation of state, i.e., the perfect gas relation $p = \rho RT$; thus, except for a monatomic gas without internal degree of freedom, the pressure differs generally from the isotropic part of stress tensor in a nonequilibrium state.
boundary condition (1.26)...", needs supplementary explanation. Some condition is required of the scattering kernel $K_B$ in the boundary condition (M-1.26) for $f$ to be limited to the Maxwellian. For some $K_B$, the equality holds in Eq. (M-1.38) for $f$ other than the Maxwellian. See Section 6.4.1 for more detailed discussion.

(Section 1.3: Version 5-00)

1.4 Supplement to Footnote M-26 in Chapter M-1

Footnote M-26 is supplemented with more explicit mathematical expressions for the process given there. Take the non-dimensional form of the equation for the $H$ function, i.e., Eq. (M-1.72):

$$\frac{\partial \hat{H}}{\partial t} + \frac{\partial \hat{H}_i}{\partial x_i} = \frac{1}{k} \hat{G},$$

where

$$\dot{H}(x_i, t) = \int \hat{f} \ln(\hat{f}/\hat{c}_0) d\zeta, \quad \dot{H}_i(x_i, t) = \int \zeta_i \hat{f} \ln(\hat{f}/\hat{c}_0) d\zeta,$$

$$\dot{G} = -\frac{1}{4} \int (\hat{f}_\ell \hat{f}' \ell - \hat{f} \hat{f}_\ell) \ln \left( \frac{\hat{f}' \ell \hat{f}_\ell}{\hat{f} \hat{f}_\ell} \right) B d\Omega d\zeta, d\zeta \leq 0,$$

(8)

$$\hat{c}_0 = c_0 (2RT_0)^{3/2}/\rho_0.$$ The perturbed form of the velocity distribution function $\hat{f}$ is defined by

$$\hat{f} = E (1 + \phi),$$

where

$$E = \frac{1}{\pi^{3/2}} \exp(-\zeta^2).$$

Let $\varepsilon$ be a small quantity. Here, we take the case in which $\phi$ is of the order of $\varepsilon$, and examine the terms of the order of $\varepsilon^2$ of Eq. (7). The perturbed function $\phi$ is expressed as

$$\phi = \phi_1 \varepsilon + \phi_2 \varepsilon^2 + \cdots.$$ (10)

Corresponding to the expansion, the macroscopic variables, i.e., $\omega$, $u_i$, $P$, etc., $\hat{H}$, $\hat{H}_i$, and $\hat{G}$ are also expressed as

$$h = h_1 \varepsilon + h_2 \varepsilon^2 + \cdots,$$

$$\dot{H} = \dot{H}_0 + \dot{H}_1 \varepsilon + \dot{H}_2 \varepsilon^2 + \cdots,$$ (11a)

$$\dot{H}_i = \dot{H}_{i0} + \dot{H}_{i1} \varepsilon + \dot{H}_{i2} \varepsilon^2 + \cdots,$$ (11b)

$$\dot{G} = \dot{G}_0 + \dot{G}_1 \varepsilon + \dot{G}_2 \varepsilon^2 + \cdots,$$ (11c)

$$\dot{H} = \dot{H}_0 + \dot{H}_1 \varepsilon + \dot{H}_2 \varepsilon^2 + \cdots,$$ (11d)

where $h$ represents the perturbed macroscopic variables, $\omega$, $u_i$, $P$, etc., and the quantities $\phi_n$, $h_n$, $\dot{H}_n$, $\dot{H}_{in}$, and $\dot{G}_n$ are of the order of unity. Then, with the
aid of the expanded forms of Eqs. (M-1.78a)-(M-1.78f), \( \hat{H}_n, \hat{H}_{in}, \) and \( \hat{G}_n \) are expressed as

\[
\hat{H}_0 = -\frac{3}{2} - \ln \pi^{3/2} \hat{c}_0, \\
\hat{H}_1 = (1 - \ln \pi^{3/2} \hat{c}_0) \int E \phi_1 d\zeta - \int \zeta^2 E \phi_1 d\zeta \\
= (1 - \ln \pi^{3/2} \hat{c}_0) \omega_1 - \frac{3}{2} P_1, \\
\hat{H}_2 = (1 - \ln \pi^{3/2} \hat{c}_0) \int E \phi_2 d\zeta - \int \zeta^2 E \phi_2 d\zeta + \frac{1}{2} \int \phi_2^2 d\zeta \\
= (1 - \ln \pi^{3/2} \hat{c}_0) \omega_2 - \left( \frac{3}{2} P_2 + u_{i1}^2 \right) + \frac{1}{2} \int \phi_2^2 d\zeta,
\]

\[
\hat{H}_{i0} = 0, \\
\hat{H}_{i1} = (1 - \ln \pi^{3/2} \hat{c}_0) \int \zeta_i E \phi_1 d\zeta - \int \zeta_i^2 E \phi_1 d\zeta \\
= (1 - \ln \pi^{3/2} \hat{c}_0) u_{i1} - \left( Q_{i1} + \frac{5}{2} u_{i1} \right), \\
\hat{H}_{i2} = (1 - \ln \pi^{3/2} \hat{c}_0) \int \zeta_i E \phi_2 d\zeta - \int \zeta_i^2 E \phi_2 d\zeta + \frac{1}{2} \int \zeta_i E \phi_2^2 d\zeta \\
= (1 - \ln \pi^{3/2} \hat{c}_0) (u_{i2} + \omega_1 u_{i1}) - \left( Q_{i2} + \frac{5}{2} u_{i2} + u_{j1} P_{ij1} + \frac{3}{2} u_{i1} P_{i1} \right) \\
+ \frac{1}{2} \int \zeta_i E \phi_2^2 d\zeta,
\]

\[
\hat{G}_0 = 0, \\
\hat{G}_1 = 0, \\
\hat{G}_2 = -\frac{1}{4} \int EE_s (\phi_1 + \phi_{1s} - \phi_1 - \phi_{1s})^2 \hat{B} d\Omega d\zeta, d\zeta \leq 0.
\]

With the aid of these expressions, the \( \varepsilon \) and \( \varepsilon^2 \)-order expressions of Eq (7) are
given as
\[
\begin{align*}
\text{Sh} \frac{\partial \hat{H}_1}{\partial t} + \frac{\partial \hat{H}_{1i}}{\partial x_i} &= (1 - \ln \pi^{3/2} \hat{c}_0) \left( \text{Sh} \frac{\partial \omega_1}{\partial t} + \frac{\partial u_{1i}}{\partial x_i} \right) \\
&- \left[ \frac{3}{2} \text{Sh} \frac{\partial P_1}{\partial t} + \frac{\partial}{\partial x_i} \left( \frac{5}{2} u_{1i} + Q_{1i} \right) \right], \\
(15a)
\end{align*}
\]

\[
\begin{align*}
\text{Sh} \frac{\partial \hat{H}_2}{\partial t} + \frac{\partial \hat{H}_{2i}}{\partial x_i} &= (1 - \ln \pi^{3/2} \hat{c}_0) \left( \text{Sh} \frac{\partial \omega_2}{\partial t} + \frac{\partial (u_{2i} + \omega_1 u_{1i})}{\partial x_i} \right) \\
&- \text{Sh} \frac{\partial}{\partial t} \left( \frac{3}{2} P_2 + u_{2i}^2 \right) - \frac{\partial}{\partial x_i} \left( Q_{2i} + \frac{5}{2} u_{2i} + u_{j1} P_{ij1} + \frac{3}{2} u_{1i} P_1 \right) \\
&+ \frac{1}{2} \left( \text{Sh} \frac{\partial}{\partial t} \int E \phi_1^2 d\zeta + \frac{\partial}{\partial x_i} \int \zeta_i E \phi_1^2 d\zeta \right). \\
(15b)
\end{align*}
\]

Substituting the series expansion (11a) into the conservation equation (M-1.87), we have
\[
\begin{align*}
\text{Sh} \frac{\partial \omega_1}{\partial t} + \frac{\partial u_{1i}}{\partial x_i} &= 0, \\
(16a)
\end{align*}
\]

\[
\begin{align*}
\text{Sh} \frac{\partial \omega_2}{\partial t} + \frac{\partial (u_{2i} + \omega_1 u_{1i})}{\partial x_i} &= 0. \\
(16b)
\end{align*}
\]

Similarly, from the conservation equation (M-1.89), we have
\[
\begin{align*}
\frac{3}{2} \text{Sh} \frac{\partial P_1}{\partial t} + \frac{\partial}{\partial x_i} \left( \frac{5}{2} u_{1i} + Q_{1i} \right) &= 0, \\
(17a)
\end{align*}
\]

\[
\begin{align*}
\text{Sh} \frac{\partial}{\partial t} \left( \frac{3}{2} P_2 + u_{2i}^2 \right) + \frac{\partial}{\partial x_i} \left( Q_{2i} + \frac{5}{2} u_{2i} + u_{j1} P_{ij1} + \frac{3}{2} u_{1i} P_1 \right) &= 0. \\
(17b)
\end{align*}
\]

With the aid of the expanded forms (16a)–(17b) of the conservation equations (M-1.87) and (M-1.89), Eqs. (15a) and (15b) are reduced to, for the solution of the Boltzmann equation (M-1.47) or (M-1.75a),
\[
\begin{align*}
\text{Sh} \frac{\partial \hat{H}_1}{\partial t} + \frac{\partial \hat{H}_{1i}}{\partial x_i} &= 0, \\
(18a)
\end{align*}
\]

\[
\begin{align*}
\text{Sh} \frac{\partial \hat{H}_2}{\partial t} + \frac{\partial \hat{H}_{2i}}{\partial x_i} &= \frac{1}{2} \left( \text{Sh} \frac{\partial}{\partial t} \int E \phi_1^2 d\zeta + \frac{\partial}{\partial x_i} \int \zeta_i E \phi_1^2 d\zeta \right). \\
(18b)
\end{align*}
\]

Thus, the \(o(\varepsilon^2)\) terms being neglected in Eq. (7), it is reduced to
\[
\begin{align*}
\text{Sh} \frac{\partial}{\partial t} \int E \phi_1^2 d\zeta + \frac{\partial}{\partial x_i} \int \zeta_i E \phi_1^2 d\zeta \\
&= - \frac{1}{2k} \int E E_\ast (\phi'_1 + \phi'_{1\ast} - \phi_1 - \phi_{1\ast})^2 B d\Omega d\zeta \cdot d\zeta \leq 0. \\
(19)
\end{align*}
\]

This expression does not contain \(\phi_2\).
2 Chapter M-2

2.1 Section M-2.5

2.1.1 Section M-2.5.1

The following form:

$$\sigma = -\frac{2}{\pi} \int_{0<\xi<\infty, i, n, <0} \xi^3 l_j n_j f(X, \xi) d\xi d\Omega(l),$$

is more appropriate as Eq. (M-2.39b) than the one in the book. Then, the explanation of $d\Omega(l)$, i.e.,

$$d\Omega(l) \text{ is the solid-angle element in the direction of } l,$$

has to be inserted between 'where' and ' $T_w$ ' just after Eq. (M-2.39c).

(Section 2.1.1: Version 6-00)

3 Chapter M-3

3.1 Processes of solution of the systems in Section M-3.7.2

(July 2007)

The processes of solutions of the fluid-dynamic-type equations derived in Section M-3.7.1 are straightforward and may not need explanation. For the equations in Section M-3.7.2, some explanation may be better to be given. The discussion will be made on the basis of the boundary conditions in Section M-3.7.3 for a simple boundary where the shape of the boundary is invariant and its velocity component normal to it is zero.

3.1.1 “Incompressible Navier–Stokes set”

Consider the initial and boundary-value problem of Eqs. (M-3.265)–(M-3.268), i.e.,

$$\frac{\partial P_{S_1}}{\partial x_i} = 0,$$  \hspace{1cm} (20)

$$\frac{\partial u_{iS_1}}{\partial x_i} = 0,$$  \hspace{1cm} (21a)

$$\frac{\partial u_{iS_1}}{\partial t} + u_j S_1 \frac{\partial u_{iS_1}}{\partial x_j} = -\frac{1}{2} \frac{\partial P_{S_2}}{\partial x_i} + \frac{\gamma_1}{2} \frac{\partial^2 u_{iS_1}}{\partial x_j^2},$$  \hspace{1cm} (21b)

$$\frac{5}{2} \frac{\partial \tau_{S_1}}{\partial t} - \left( \frac{5}{2} \frac{\partial P_{S_1}}{\partial t} + \frac{5}{2} u_j S_1 \frac{\partial \tau_{S_1}}{\partial x_j} \right) = \frac{5\gamma_2}{4} \frac{\partial^2 \tau_{S_1}}{\partial x_j^2},$$  \hspace{1cm} (21c)
\[
\begin{align*}
\frac{\partial u_{i,S_2}}{\partial x_i} &= \frac{\partial \omega_{S_1}}{\partial t} - \frac{\partial \omega_{S_1} u_{i,S_1}}{\partial x_i}, \\
\frac{\partial u_{i,S_2}}{\partial t} + u_{j,S_1} \frac{\partial u_{i,S_2}}{\partial x_j} + u_{j,S_2} \frac{\partial u_{i,S_1}}{\partial x_j} &= -\frac{1}{2} \left( \frac{\partial P_{S_3}}{\partial x_i} - \omega_{S_1} \frac{\partial P_{S_2}}{\partial x_i} \right) + \frac{\gamma_1}{2} \frac{\partial}{\partial x_j} \left( \frac{\partial u_{i,S_2}}{\partial x_j} + \frac{\partial u_{j,S_1}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{k,S_2}}{\partial x_k} \delta_{ij} \right) \\
&\quad - \frac{\gamma_1 \omega_{S_1}}{2} \frac{\partial^2 u_{i,S_1}}{\partial x_j^2} + \frac{\gamma_4}{2} \frac{\partial}{\partial x_j} \left[ \tau_{S_1} \left( \frac{\partial u_{i,S_1}}{\partial x_j} + \frac{\partial u_{j,S_1}}{\partial x_i} \right) \right] - \frac{\gamma_3}{3} \frac{\partial}{\partial x_i} \frac{\partial^2 \tau_{S_1}}{\partial x_j^2}, \\
\frac{3}{2} \frac{\partial P_{S_2}}{\partial t} + 3 \frac{u_{j,S_1}}{2} \frac{\partial P_{S_2}}{\partial x_j} + \frac{5}{2} \left( \frac{\partial P_{S_1} u_{j,S_1}}{\partial x_j} - \frac{\partial \omega_{S_2}}{\partial t} - \frac{\partial (\omega_{S_2} u_{j,S_1} + \omega_{S_1} u_{j,S_2})}{\partial x_j} \right) &= \frac{5 \gamma_2}{4} \frac{\partial^2 \tau_{S_2}}{\partial x_j^2} + \frac{5 \gamma_5}{4} \frac{\partial}{\partial x_j} \left( \tau_{S_1} \frac{\partial \tau_{S_1}}{\partial x_j} \right) + \frac{\gamma_1}{2} \left( \frac{\partial u_{i,S_1}}{\partial x_j} + \frac{\partial u_{j,S_1}}{\partial x_i} \right) \frac{\partial}{\partial x_j} \left( \tau_{S_1} \frac{\partial \tau_{S_1}}{\partial x_j} \right), \quad (22b)
\end{align*}
\]

where
\[
P_{S_1} = \omega_{S_1} + \tau_{S_1}, \quad P_{S_2} = \omega_{S_2} + \omega_{S_1} \tau_{S_1} + \tau_{S_2}. \quad (23)
\]

From Eq. (20), \(P_{S_1}\) is a function of \(t\), i.e.,
\[
P_{S_1} = f_1(t).
\]

In an unbounded-domain problem where the pressure at infinity is specified (or the pressure is specified at some point), \(P_{S_1} = f_1(t)\) is known, but in a bounded-domain problem of a simple boundary, \(f_1(t)\) is unknown at this moment and is determined later. Let \(u_{i,S_1}\) and \(\tau_{S_1}\) as well as \(f_1(t)\) be given at time \(t\) in such a way that \(u_{i,S_1}\) satisfies Eq. (21a). Taking the divergence of Eq. (21b) and using Eq. (21a), we have
\[
\frac{\partial^2 P_{S_2}}{\partial x_j^2} = -2 \frac{\partial u_{i,S_1}}{\partial x_i} \frac{\partial u_{i,S_1}}{\partial x_j} \quad (25)
\]

On a simple boundary, the derivative of \(P_{S_2}\) normal to it is found to be expressed with \(u_{i,S_1}\) and its space derivatives by multiplying Eq. (21b) by the normal vector to the boundary.\(^5\) In the unbounded-domain problem, where \(f_1(t)\) is known, \(P_{S_2}\) is determined by Eq. (25). In the bounded-domain problem, \(P_{S_2}\) is determined by Eq. (25) except for an additive function of \(t\) [say, \(f_2(t)\)]. Anyway, \(\frac{\partial P_{S_2}}{\partial x_i}\) is independent of this ambiguity. From Eq. (21b), \(\frac{\partial u_{i,S_1}}{\partial t}\) is determined, irrespective of \(f_2(t)\), in such a way that \(\frac{\partial (\omega_{S_1} u_{i,S_1})}{\partial x_i}/\partial t = 0\) for the above choice of \(P_{S_2}\). Thus, the solution \(u_{i,S_1}\) of Eqs. (21a) and (21b) is determined by Eq. (21b) with the supplementary condition (25) instead of Eq. (21a). From Eq. (21c), \((5/2)\frac{\partial \tau_{S_1}}{\partial t} - \frac{\partial P_{S_1}}{\partial t} \) or \((5/2)\frac{\partial \tau_{S_1}}{\partial t} - f_1(t) \) is determined, i.e.,
\[
(5/2)\frac{\partial \tau_{S_1}}{\partial t} - f_1(t) \frac{\partial t}{\partial \tilde{t}} = G(x_i, \tilde{t}), \quad (26)
\]

where
\[
G(x_i, \tilde{t}) = \frac{5}{2} \frac{u_{i,S_1}}{\partial x_j} + \frac{5 \gamma_2}{4} \frac{\partial^2 \tau_{S_1}}{\partial x_j^2}. \quad (27)
\]

\(^5\) The time-derivative term vanishes owing to the boundary condition mentioned in the first paragraph of Section 3.1.
Thus, $\tau_{S1}$ is determined in the unbounded-domain problem, but $\tau_{S1}$ has ambiguity owing to $f_1(\tilde{t})$ in the bounded-domain problem. The undetermined function $f_1(\tilde{t})$ is determined in the following way.

In the bounded-domain problem whose boundary is a simple boundary, the mass of the gas in the domain is invariant with respect to $\tilde{t}$. The condition at the leading order is

$$\frac{d}{d\tilde{t}} \int_V \omega_{S1} d\mathbf{x} = 0,$$

where $V$ indicates the domain (or its volume in the later). With the aid of Eq. (23), we have

$$\frac{d f_1(\tilde{t})}{d\tilde{t}} V - \frac{d}{d\tilde{t}} \int_V \tau_{S1} d\mathbf{x} = 0.$$  

(29)

On the other hand, from Eq. (26),

$$-\frac{d f_1(\tilde{t})}{d\tilde{t}} V + \frac{5}{2} \frac{d}{d\tilde{t}} \int_V \tau_{S1} d\mathbf{x} = \int_V G(x_i, \tilde{t}) d\mathbf{x}. $$

(30)

From Eqs. (29) and (30), we obtain $d f_1(\tilde{t})/d\tilde{t}$ and $d \int_V \tau_{S1} d\mathbf{x}/d\tilde{t}$ as

$$\frac{d f_1(\tilde{t})}{d\tilde{t}} = \left\{ \begin{array}{l} \frac{2}{3V} \int_V G(x_i, \tilde{t}) d\mathbf{x}, \\
\frac{d}{d\tilde{t}} \int_V \tau_{S1} d\mathbf{x} = \frac{2}{3} \int_V G(x_i, \tilde{t}) d\mathbf{x}. \end{array} \right. $$

(31)

That is, $f_1(\tilde{t})$ in the bounded-domain problem [and thus the solution $\tau_{S1}$ of Eq. (21c)] is determined.

The analysis of the higher-order equations is similar; for example, from Eqs. (22a)–(22c), $u_{iS2}$, $\tau_{S2}$, and $P_{S3}$ are determined in the unbounded-domain problem, but $f_2(\tilde{t})$, $u_{iS2}$, $\tau_{S2}$, and $P_{S3}$, except for an additive function of $\tilde{t}$ in $P_{S3}$, are determined in the bounded-domain problem. Let $u_{iS2}$, $\tau_{S2}$, and $f_2(\tilde{t})$ be given at $\tilde{t}$ in such a way that Eq. (22a) is satisfied. Taking the divergence of Eq. (22b) and using Eq. (22a) and the results obtained above, we find that $P_{S3}$ is governed by the Poisson equation

$$\frac{\partial^2 P_{S3}}{\partial x_i^2} = \text{Inhomogeneous term},$$

(32)

where the inhomogeneous term consists of $u_{iS2}$, $P_{S2}$, and the functions determined in the preceding analysis. On a simple boundary, the derivative of $P_{S3}$ normal to it being known, $P_{S3}$ is determined by this equation, except for an additive function of $\tilde{t}$ [say, $f_3(\tilde{t})$] in the bounded-domain problem. Then, from

---

Note that, with the aid of Eq. (23), the time-derivative term $\frac{d}{d\tilde{t}} P_{S2}/\partial \tilde{t} - \frac{5}{2} \frac{d\omega_{S2}}{d\tilde{t}}$ in Eq. (22c) is transformed into $\frac{d}{d\tilde{t}} \tau_{S2}/\partial \tilde{t} - \frac{5}{2} \frac{d\omega_{S2}}{d\tilde{t}}$ and $\frac{d}{d\tilde{t}} \tau_{S1}/\partial \tilde{t}$.

The time derivative $\frac{d\omega_{S1}}{d\tilde{t}}$ is known from $\tau_{S1}/\partial \tilde{t}$, $d f_1(\tilde{t})/d\tilde{t}$, and Eq. (23).

Shift the discussion of the boundary condition for $P_{S2}$ to the next order.
Eq. (22b), $\partial u_{iS2}/\partial t$ at $\tilde{t}$ is determined irrespective of $f_3(\tilde{t})$. From Eq. (22c), $\partial(3P_{S2} - 5\omega_{S2})/\partial t$ or $\partial(5\tau_{S2} - 2P_{S2})/\partial t$ at $\tilde{t}$ is determined. Thus, $u_{iS2}$ and $\tau_{S2}$ (except for the additive function $2f_2/5$ in the bounded-domain problem) [thus, $\omega_{S2}$ (except for the additive function $3f_2/5$)] are determined. In the bounded-domain problem, where the boundary is a simple boundary, the condition of invariance of the mass of the gas in the domain at the corresponding order is

$$\frac{d}{dt} \int_V \omega_{S2} d\mathbf{x} = 0. \quad (33)$$

With the aid of Eq. (23), $df_2(\tilde{t})/d\tilde{t}$ at $\tilde{t}$ is determined as $df_1(\tilde{t})/d\tilde{t}$ is done.

To summarize, the solution $(u_{iS1}, P_{S1}, \tau_{S1}, P_{S2})$ of the initial and boundary-value problem of Eqs. (20)–(21c) is determined, with an additive arbitrary function $f_2(\tilde{t})$ in $P_{S2}$ in a bounded-domain problem of a simple boundary, when the initial data of $u_{iS1}, P_{S1}, \tau_{S1},$ and $P_{S2}$ satisfy Eqs. (21a) and (25). The additive function $f_2(\tilde{t})$ does not affect the other variables. The function $f_2(\tilde{t})$ is determined in the next-order analysis. In other words, the solution $(u_{iS1}, P_{S1}, \tau_{S1})$ of Eqs. (20)–(21c) is determined consistently by Eqs. (20), (21b), and (21c) with the supplementary condition (25), instead of Eq. (21a), when the initial data of $u_{iS1}, P_{S1},$ and $\tau_{S1}$ satisfy Eq. (21a). Naturally, the initial $P_{S2}$ is required to satisfy Eq. (25). This process is natural for numerical computation.

### 3.1.2 Ghost-effect equations (M-3.275)–(M-3.278b):

Consider the initial and boundary-value problem of Eqs. (M-3.275)–(M-3.278b), i.e.,

$$\dot{\rho}_{SB0} = \rho_0(\tilde{t}), \quad (34)$$

$$\dot{\rho}_{SB1} = \rho_1(\tilde{t}), \quad (35)$$

$$\frac{\partial \dot{\rho}_{SB0}}{\partial t} + \frac{\partial \rho_{SB0} \hat{v}_{iSB1}}{\partial x_i} = 0, \quad (36a)$$

$$\frac{\partial \rho_{SB0} \hat{v}_{iSB1}}{\partial t} + \frac{\partial \rho_{SB0} \hat{v}_{jSB1} \hat{v}_{iSB1}}{\partial x_j}$$

$$= -\frac{1}{2} \frac{\partial \rho_{SB2}}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_j} \left[ \Gamma_1(\hat{T}_{SB0}) \left( \frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3} \frac{\partial \hat{v}_{kSB1}}{\partial x_k} \delta_{ij} \right) \right]$$

$$+ \frac{1}{2\rho_0} \frac{\partial}{\partial x_j} \left\{ \Gamma_7(\hat{T}_{SB0}) \left[ \frac{\partial \hat{T}_{SB0}}{\partial x_i} \frac{\partial \hat{T}_{SB0}}{\partial x_j} - \frac{1}{3} \left( \frac{\partial \hat{T}_{SB0}}{\partial x_k} \right)^2 \delta_{ij} \right]\right\}, \quad (36b)$$

$$\frac{3}{2} \frac{\partial \rho_{SB0} \hat{T}_{SB0}}{\partial t} + \frac{5}{2} \frac{\partial \rho_{SB0} \hat{v}_{iSB1} \hat{T}_{SB0}}{\partial x_i} = \frac{5}{4} \frac{\partial}{\partial x_i} \left( \Gamma_2(\hat{T}_{SB0}) \frac{\partial \hat{T}_{SB0}}{\partial x_i} \right), \quad (36c)$$

The contribution of the Knudsen-layer correction to the mass in the domain is of a higher order, though it is required to $\omega_{S2}$.
where \( \hat{p}_0 \) and \( \hat{p}_1 \) depend only on \( \hat{t} \), and

\[
\begin{align*}
\hat{p}_{SB0} &= \hat{p}_{SB0} \hat{T}_{SB0}, \\
\hat{p}_{SB1} &= \hat{p}_{SB1} \hat{T}_{SB1} + \hat{p}_{SB0} \hat{T}_{SB1}, \\
\hat{p}_{SB2} &= \hat{p}_{SB2} \hat{T}_{SB2} + \hat{p}_{SB1} \hat{T}_{SB1} + \hat{p}_{SB0} \hat{T}_{SB2},
\end{align*}
\]

(37)

\[
\hat{p}_{SB2} = \hat{p}_{SB2} + \frac{2}{3 \hat{p}_0} \frac{\partial}{\partial x_k} \left( \Gamma_3 \partial \hat{p}_{SB0} \frac{\partial \hat{T}_{SB0}}{\partial x_k} \right).
\]

(38)

Let \( \hat{p}, \hat{v}_i \), and \( \hat{T} \) (thus, \( \hat{p} = \hat{p} \hat{T} \)) at time \( \hat{t} \) be given; thus, \( \hat{p}_{SB0}, \hat{v}_{iSB1}, \hat{T}_{SB0} \) (\( \hat{p}_{SB0} \)), etc., including \( \hat{p}_{SB2} \), are given. Then \( \partial \hat{p}_{SB0}/\partial \hat{t}, \partial \hat{p}_{SB1}/\partial \hat{t}, \) and \( \partial \hat{T}_{SB0}/\partial \hat{t} \) at \( \hat{t} \) are given by Eqs. (36a)-(36c); thus, the future \( \hat{p}_{SB0}, \hat{v}_{iSB1}, \) and \( \hat{T}_{SB0} \) (also \( \hat{p}_{SB0} \)) at \( \hat{t} \), is required to be independent of \( x_i \) owing to Eq. (34). Taking this point into account, we discuss how the solution is determined. For convenience of the discussion, transform Eq. (36c) in the form

\[
\frac{\partial \hat{p}_{SB0}}{\partial \hat{t}} = \mathcal{P},
\]

(39)

where

\[
\mathcal{P} = -\frac{5}{3} \hat{p}_{SB0} \frac{\partial \hat{v}_{iSB1}}{\partial x_i} + \frac{5}{6} \frac{\partial}{\partial x_i} \left( \Gamma_2 \hat{T}_{SB0} \frac{\partial \hat{T}_{SB0}}{\partial x_i} \right).
\]

First, consider the case where \( \hat{p} \) (thus, \( \hat{p}_{SB0}, \hat{p}_{SB1}, \) etc.) is specified at some point, e.g., at infinity. Then, from Eq. (34), \( \hat{p}_0(\hat{t}) \) is a given function of \( \hat{t} \), and \( \hat{p}_{SB0} \) is determined. The initial value of \( \hat{p}_{SB0} \) is uniform, i.e., \( \hat{p}_{SB0} = \hat{p}_0(0) \). On the other hand, from Eq. (39), the variation of \( \partial \hat{p}_{SB0}/\partial \hat{t} \) is also determined by the data of \( \hat{p}_{SB0}, \hat{T}_{SB0}, \hat{v}_{iSB1}, \) and their space derivatives at \( \hat{t} \). This must coincide with the corresponding data given by Eq. (34), i.e., \( \partial \hat{p}_{SB0}/\partial \hat{t} = \partial \hat{p}_0/\partial \hat{t} \).

Substituting this relation into Eq. (39), we have

\[
\frac{\partial}{\partial x_i} \left( \hat{p}_{SB0} \hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2} \frac{\partial \hat{T}_{SB0}}{\partial x_i} \right) = -\frac{3}{5} \hat{p}_0 \frac{\partial \hat{p}_0}{\partial \hat{t}},
\]

(40)

which requires a relation among \( \hat{p}_{SB0}, \hat{T}_{SB0}, \) and \( \hat{v}_{iSB1} \) for all \( \hat{t} \), since \( \partial \hat{p}_0/\partial \hat{t} \) is given. This condition is equivalently replaced by the following two conditions: The initial data of \( \hat{p}_{SB0}, \hat{T}_{SB0}, \) and \( \hat{v}_{iSB1} \) are required to satisfy Eq. (40), and the time derivative of Eq. (40) has to be satisfied for all \( \hat{t} \), i.e.,

\[
\frac{\partial^2}{\partial \hat{t} \partial x_i} \left( \hat{p}_{SB0} \hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2} \frac{\partial \hat{T}_{SB0}}{\partial x_i} \right) = -\frac{3}{5} \hat{p}_0 \frac{\partial^2 \hat{p}_0}{\partial \hat{t}^2}.
\]

(41)

With the aid of Eqs. (36a)-(36c) and (39), the left-hand side of Eq. (41) is expressed in the form without the time-derivative terms, i.e., \( \partial \hat{p}_{SB0}/\partial \hat{t}, \partial \hat{T}_{SB0}/\partial \hat{t}, \) and \( \partial \hat{v}_{iSB1}/\partial \hat{t} \), as follows:

\[
\frac{\partial^2}{\partial \hat{t} \partial x_i} \left( \hat{p}_{SB0} \hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2} \frac{\partial \hat{T}_{SB0}}{\partial x_i} \right) = -\frac{1}{2} \hat{p}_{SB0} \frac{\partial}{\partial x_i} \left( \frac{1}{\hat{p}_{SB0}} \frac{\partial \hat{p}_{SB2}}{\partial x_i} \right) + \text{fn}_1,
\]

13
where \( f_{n1} \) is a given function of \( \rho_{SB0}, \dot{v}_{iSB1}, \dot{T}_{SB0} \), and their space derivatives. Thus, the condition (41) is reduced to an equation for \( \rho_{SB2} \), i.e.,

\[
\frac{\partial}{\partial x_i} \left( \frac{1}{\rho_{SB0}} \frac{\partial \rho_{SB2}}{\partial x_i} \right) = F_n,
\]

where

\[
F_n = \frac{2}{\rho_0} \left( f_{n1} + \frac{3}{5} d P_0 \right).
\]

The boundary condition for \( \rho_{SB2} \) in Eq. (42) on a simple boundary is derived by multiplying Eq. (36b) by the normal \( n_i \) to the boundary. In this process, the contribution of its time-derivative terms vanishes.\(^{10}\) Thus, \( \rho_{SB2} \) (or \( \rho_{SB0} \)) is determined in the present case, where \( \rho \) (thus, \( \rho_{SB0} \)) is specified at some point. The solution \( \rho_{SB2} \) of Eq. (42) being substituted into Eq. (36b), Eqs. (36a)–(36c) with the first relation in Eq. (37) are reduced to the equations for \( \rho_{SB0}, \dot{T}_{SB0}, \) and \( \dot{v}_{iSB1} \) which naturally determine \( \partial \rho_{SB0}/\partial t, \partial \dot{T}_{SB0}/\partial t, \) and \( \partial \dot{v}_{iSB1}/\partial t \). Further, if the initial data of \( \rho_{SB0}, \dot{T}_{SB0}, \) and \( \dot{v}_{iSB1} \) are chosen in such a way that \( \rho_{SB0} T_{SB0}(= \rho_{SB0}) = \rho_0 \) and that Eq. (40) is satisfied, the variation \( \partial \rho_{SB2}/\partial t \) of \( \rho_{SB2}(= \rho_{SB0} T_{SB0}) \) given by these equations is consistent with Eq. (34), since Eq. (42) or (41) with the condition (40) at the initial state guarantees Eq. (40), i.e., \( \partial \rho_{SB2}/\partial t = d \rho_0/\partial t \), for all \( t \).

Equations (34) and (36a)–(36c) with Eqs. (37) and (38) determine \( \rho_{SB0}, \dot{T}_{SB0}, \rho_{SB0}, \dot{v}_{iSB1}, \) and \( \rho_{SB2} \) consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. In the above system, \( \rho_{SB2} \) is determined. On the other hand, the variation \( \partial \rho_{SB2}/\partial t \) is determined independently by the counterpart of Eq. (39) at the order after next. The situation is similar to that at the leading order, where Eqs. (34), with a given \( \rho_0 \), and (39) determine \( \rho_{SB0} \) independently. The analysis can be carried out in a similar way. Let \( \rho_{SB2} \) determined by Eq. (42) be indicated by \( (\rho_{SB2})_0 \) and the equation for \( \partial \rho_{SB2}/\partial t \), or the counterpart of Eq. (39) at the order after next, be put in the form

\[
\frac{\partial \rho_{SB2}}{\partial t} = P_2,
\]

where \( P_2 \) is a given function of \( \rho_{SBm}, \dot{v}_{iSBm+1}, \dot{T}_{SBm} \) \((m \leq 2)\), and their space derivatives. For the consistency, \( \partial (\rho_{SB2})_0/\partial t \) is substituted for \( \partial \rho_{SB2}/\partial t \) in Eq. (43), i.e.,

\[
P_2 = \frac{\partial (\rho_{SB2})_0}{\partial t},
\]

where \( \partial (\rho_{SB2})_0/\partial t \) is known. This requires a relation among \( \rho_{SBm}, \dot{v}_{iSBm+1}, \dot{T}_{SBm} \) \((m \leq 2)\), and their space derivatives. This condition is equivalently replaced by the following two conditions: Equation (44) is applied only for the initial state, and the time derivative of Eq. (44), i.e.,

\[
\frac{\partial P_2}{\partial t} = \frac{\partial^2 (\rho_{SB2})_0}{\partial t^2},
\]

\(^{10}\)The discussion is similar to that in Footnote 5.
has to be satisfied for all \( \tilde{t} \). The \( \partial \tilde{\rho}_{SBm}/\partial \tilde{t}, \partial \tilde{v}_{i_{SBm+1}}/\partial \tilde{t}, \partial \tilde{T}_{SBm}/\partial \tilde{t} \) \( (m \leq 2) \) in \( \partial P_{2}/\partial \tilde{t} \) being replaced by the counterparts of Eqs. (36a)–(36c) and (39) at the corresponding order, an equation for \( \tilde{\rho}_{SB4} \) for all \( \tilde{t} \) is derived.\(^{11}\) The conclusion is that an additional initial condition and the condition for \( \tilde{\rho}_{SB4} \) are introduced and, instead, that the condition (42) for \( \tilde{\rho}_{SB2} \) is required only for the initial data. The higher-order consideration does not affect the determination of the solution \( \tilde{\rho}_{SB0}, \tilde{T}_{SB0}, \) and \( \tilde{v}_{i_{SB1}} \) (thus also \( \tilde{\rho}_{SB0} \)).

In this way, the solution of Eqs. (34), (36a)–(38) is determined consistently by Eqs. (36a)–(38) with the aid of the supplementary condition (42), instead of Eq. (34), when the initial data of \( \tilde{\rho}_{SB0}, \tilde{T}_{SB0}, \) and \( \tilde{v}_{i_{SB1}} \) satisfy Eqs. (34) and (40), where \( \tilde{\rho}_{0}(\tilde{t}) \) is a known function of \( \tilde{t} \) from the boundary condition.

Secondly, consider a bounded-domain problem of a simple boundary. In contrast to the first case, \( \partial \tilde{\rho}_{0}/\partial \tilde{t} \) is unknown because no condition is imposed on \( \tilde{\rho}_{SB0} \) on a simple boundary. However, in a bounded-domain problem of a simple boundary, the mass of the gas in the domain is invariant with respect to \( \tilde{t} \), i.e., at the leading order,

\[
\frac{d}{d\tilde{t}} \int_V \tilde{\rho}_{SB0} \, dx = 0,
\]

(45)

where \( V \) indicates the domain under consideration. Using the first relation of Eq. (37), i.e., \( \tilde{\rho}_{SB0} = \tilde{\rho}_{0}/\tilde{T}_{SB0} \), in Eq. (45), we have

\[
\frac{d\tilde{\rho}_{0}}{d\tilde{t}} \int_V \frac{1}{\tilde{T}_{SB0}} \, dx = \tilde{\rho}_{0} \int_V \frac{1}{\tilde{T}_{SB0}^2} \frac{\partial \tilde{T}_{SB0}}{\partial \tilde{t}} \, dx.
\]

(46)

Using Eq. (36c) for \( \partial \tilde{T}_{SB0}/\partial \tilde{t} \) in Eq. (46), we find that the variation \( d\tilde{\rho}_{0}/d\tilde{t} \) is expressed with \( \tilde{\rho}_{0}, \tilde{T}_{SB0}, \) and \( \tilde{v}_{i_{SB1}} \) as follows:

\[
\frac{d\tilde{\rho}_{0}}{d\tilde{t}} = P(\tilde{t}),
\]

(47)

where

\[
P(\tilde{t}) = \tilde{\rho}_{0} \int_V \frac{1}{\tilde{T}_{SB0}^2} \left[ \frac{5}{6\tilde{\rho}_{SB0}} \frac{\partial}{\partial x_i} \left( \Gamma_2(\tilde{T}_{SB0}) \frac{\partial \tilde{T}_{SB0}}{\partial x_i} \right) - \frac{5}{3} \tilde{v}_{i_{SB1}} \frac{\partial \tilde{T}_{SB0}}{\partial x_i} \right] \, dx \times \left( \int_V \frac{1}{\tilde{T}_{SB0}} \, dx \right)^{-1}.
\]

(48)

With this expression of \( d\tilde{\rho}_{0}/d\tilde{t} \), we can carry out the analysis in a similar way to that in the first case.

The variation \( d\tilde{\rho}_{0}/d\tilde{t} \) or \( \partial \tilde{\rho}_{SB0}/\partial \tilde{t} \) is also determined by Eq. (39). The two \( \partial \tilde{\rho}_{SB0}/\partial \tilde{t} \)’s given by Eq. (47) with Eq. (48) and Eq. (39) have to be consistent.

\(^{11}\) The conditions on the odd-order \( \tilde{\rho}^{2n+1}_{SB} \)’s are derived by the analysis starting from the condition (35) that \( \tilde{\rho}_{SB1} \) is independent of \( x_i \).
Thus, substituting Eq. (47) with Eq. (48) into \( \partial \hat{\rho}_{SB0} / \partial \hat{t} \) in Eq. (39), we have

\[
\frac{\partial}{\partial x_i} \left( \hat{\rho}_{SB0} \hat{v}_{SB1} - \frac{\Gamma_2(T_{SB0})}{2} \frac{\partial T_{SB0}}{\partial x_i} \right) = -\frac{3}{5} P(\hat{t}),
\]

(49)

where \( P(\hat{t}) \) is given by Eq. (48). This must hold for all \( \hat{t} \) for consistency. This condition is equivalently replaced by the following two conditions: The initial data of \( \hat{\rho}_{SB0}, \hat{T}_{SB0}, \hat{v}_{SB1} \) are required to satisfy Eq. (49), and the time derivative of Eq. (49) has to be satisfied for all \( \hat{t} \), i.e.,

\[
\frac{\partial^2}{\partial \hat{t}^2} \left( \hat{\rho}_{SB0} \hat{v}_{SB1} - \frac{\Gamma_2(T_{SB0})}{2} \frac{\partial T_{SB0}}{\partial x_i} \right) = -\frac{3}{5} dP(\hat{t}).
\]

(50)

Using Eqs. (36a), (36b), and (39) for the time derivatives \( \partial \hat{\rho}_{SB0} / \partial \hat{t}, \partial \hat{v}_{SB1} / \partial \hat{t}, \) and \( \partial \hat{p}_{SB0} / \partial \hat{t} \) in Eq. (50), we find that \( \hat{p}_{SB2} \) at \( \hat{t} \) is determined by the equation

\[
\frac{\partial}{\partial x_i} \left( \frac{1}{\hat{\rho}_{SB0}} \frac{\partial \hat{p}_{SB2}}{\partial x_i} \right) + \mathcal{L} \left( \frac{\partial \hat{p}_{SB2}}{\partial x_i} \right) = \text{Fn},
\]

(51)

where \( \text{Fn} \) is a given functional of \( \hat{\rho}_{SB0}, \hat{v}_{SB1}, \hat{T}_{SB0}, \) and their space derivatives, and \( \mathcal{L}(\partial \hat{p}_{SB2} / \partial x_i) \) is a given linear functional of \( \partial \hat{p}_{SB2} / \partial x_i \), i.e.,

\[
\mathcal{L} \left( \frac{\partial \hat{p}_{SB2}}{\partial x_i} \right) = -\frac{1}{\hat{\rho}_0} \int_V \frac{1}{T_{SB0}} \frac{\partial \hat{T}_{SB0}}{\partial x_i} \frac{\partial \hat{p}_{SB2}}{\partial x_i} d\mathbf{x} \left( \int_V \frac{1}{T_{SB0}} d\mathbf{x} \right)^{-1}.
\]

On a simple boundary, the derivative of \( \hat{p}_{SB2} \) normal to the boundary is specified. Thus, \( \hat{p}_{SB2}^* \) is determined except for an additive function of \( \hat{t} \). The solution \( \hat{p}_{SB2}^* \) of Eq. (51) being substituted into Eq. (36b), the result is independent of the additive function. Thus, Eqs. (36a)–(36c) with the first relation in Eq. (37) and the above \( \hat{p}_{SB2} \) substituted are reduced to those for \( \hat{\rho}_{SB0}, \hat{T}_{SB0}, \) and \( \hat{v}_{SB1}, \) which naturally determine \( \partial \hat{\rho}_{SB0} / \partial \hat{t}, \partial \hat{T}_{SB0} / \partial \hat{t}, \) and \( \partial \hat{v}_{SB1} / \partial \hat{t} \). Further, if the initial data of \( \hat{\rho}_{SB0}, \hat{T}_{SB0}, \) and \( \hat{v}_{SB1} \) being chosen in such a way that \( \hat{\rho}_{SB0} \hat{T}_{SB0}(= \hat{\rho}_{SB0}) = \hat{\rho}_0 \) and that Eq. (49) is satisfied, the variation \( \partial \hat{\rho}_{SB0} / \partial \hat{t} \) of \( \hat{p}_{SB0}(= \hat{\rho}_{SB0}) \) given by these equations is consistent with Eq. (34), since Eq. (51) or (50) with the condition (49) at the initial state guarantees Eq. (49), i.e., \( \partial \hat{p}_{SB0} / \partial \hat{t} = d\hat{p}_0 / d\hat{t}, \) for all \( \hat{t} \).

Equations (34) and (36a)–(36c) with Eqs. (37) and (51) determine \( \hat{\rho}_{SB0}, \hat{T}_{SB0}, \hat{p}_{SB0}, \hat{v}_{SB1}, \) and \( \hat{p}_{SB2}, \) except for an additive function of \( \hat{t} \) in \( \hat{p}_{SB2}, \) consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. First, the undetermined additive function in \( \hat{p}_{SB2} \) is determined by the condition of invariance of the mass of the gas in the domain at the order after next as \( d\hat{p}_0 / d\hat{t} \) is determined.\(^{12}\) The \( \partial \hat{p}_{SB2} / \partial \hat{t} \) or \( \hat{p}_{SB2} \) determined in this way is indicated by \( \partial (\hat{p}_{SB2}^*) / \partial \hat{t} \) or \( (\hat{p}_{SB2})^* \). On the other hand, the

\(^{12}\)The Knudsen-layer correction to \( \hat{\rho}_{SB1}, \) already determined [see Footnote 11], contributes to the mass at this order.
variation $\partial \hat{p}_{SB2}/\partial \hat{t}$ is determined independently by Eq. (43) or the counterpart of Eq. (39) at the order after next. The two results must coincide. The discussion from here is the same as that given from the sentence starting from Eq. (43) to the end of the paragraph. The results are that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced, and that the condition (51) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{p}_{SB0}, \hat{T}_{SB0}$, and $\hat{v}_{i,SB1}$ (thus also $\hat{p}_{SB0}$).

In this way, the solution of Eqs. (34), (36a)-(36c) is determined consistently by Eqs. (36a)-(36c) with the aid of the supplementary condition (51), instead of Eq. (34), when the initial data of $\hat{\rho}_{SB0}, \hat{T}_{SB0}$, and $\hat{v}_{i,SB1}$ satisfy Eqs. (34) and (49).

3.2 Notes on basic equations in classical fluid dynamics

3.2.1 Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

The mass, momentum, and energy-conservation equations of fluid flow are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i} (\rho v_i) = 0,$$

(52)

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial X_j} (\rho v_i v_j + p_{ij}) = 0,$$

(53)

$$\frac{\partial}{\partial t} \left[ \rho \left( e + \frac{1}{2} v_i^2 \right) \right] + \frac{\partial}{\partial X_j} \left[ \rho v_j \left( e + \frac{1}{2} v_i^2 \right) + v_i p_{ij} + q_j \right] = 0,$$

(54)

where $\rho$ is the density, $v_i$ is the flow velocity, $e$ is the internal energy per unit mass, $p_{ij}$, which is symmetric with respect to $i$ and $j$, is the stress tensor, and $q_i$ is the heat-flow vector. The pressure $p$ and the internal energy $e$ are given by the equations of state as functions of $T$ and $\rho$, i.e.,

$$p = p(T, \rho), \quad e = e(T, \rho).$$

(55)

Especially, for a perfect gas,

$$p = R \rho T, \quad e = e(T).$$

(56)

Equations (53) and (54) are rewritten with the aid of Eq. (52) in the form

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} + \frac{\partial p_{ij}}{\partial X_j} = 0,$$

(57)

$$\rho \frac{\partial}{\partial t} \left( e + \frac{1}{2} v_i^2 \right) + \rho v_j \frac{\partial}{\partial X_j} \left( e + \frac{1}{2} v_i^2 \right) + \frac{\partial}{\partial X_j} (v_i p_{ij} + q_j) = 0.$$  

(58)
The operator $\partial/\partial t + v_j \partial/\partial x_j$, which expresses the time variation along the fluid particle, is denoted by $D/Dt$, i.e.,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}.$$  

Multiplying Eq. (57) by $v_i$ we obtain the equation for the variation of kinetic energy as

$$\rho \frac{D}{Dt} \left( \frac{1}{2} v_i^2 \right) = -v_i \frac{\partial p_{ij}}{\partial X_j}. \quad (59)$$

Another form of Eq. (54), where Eq. (59) is subtracted from Eq. (58), is given as

$$\rho \frac{De}{Dt} = -p_{ij} \frac{\partial v_i}{\partial X_j} - \frac{\partial q_j}{\partial X_j}. \quad (60)$$

Noting the thermodynamic relation

$$\frac{De}{Dt} = T \frac{Ds}{Dt} + \rho \frac{D\rho}{Dt}, \quad (61)$$

where $s$ is the entropy per unit mass, and Eq. (52), Eq. (60) is rewritten as

$$\rho \frac{Ds}{Dt} = -\frac{1}{T} \left[ (p_{ij} - p\delta_{ij}) \frac{\partial v_i}{\partial X_j} + \frac{\partial q_j}{\partial X_j} \right]. \quad (62)$$

Equation (62) expresses the variation of the entropy of a fluid particle.

Equations (52)-(55) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor $p_{ij}$ and the heat-flow vector $q_i$ are assumed in some ways. The Navier–Stokes set of equations (or the Navier–Stokes equations) is Eqs. (52)-(55) where $p_{ij}$ and $q_i$ are given by

$$p_{ij} = p\delta_{ij} - \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) - \mu_B \frac{\partial v_k}{\partial X_k} \delta_{ij}, \quad (63)$$

$$q_i = -\lambda \frac{\partial T}{\partial X_i}, \quad (64)$$

where $\mu$, $\mu_B$, and $\lambda$ are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of $T$ and $\rho$. The Euler set of equations (or the Euler equations) is Eqs. (52)-(55) where $p_{ij}$ and $q_i$ are given by

$$p_{ij} = p\delta_{ij}, \quad q_i = 0, \quad (65)$$

or the Navier–Stokes equations with $\mu = \mu_B = \lambda = 0$.

For the Navier–Stokes equations, in view of the relations (63) and (64), the
entropy variation is expressed in the form\(^{13}\)

\[
\frac{D s}{D t} = \frac{1}{T} \left[ \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \mu B \left( \frac{\partial v_k}{\partial X_k} \right)^2 + \frac{\partial}{\partial X_i} \left( \lambda \frac{\partial T}{\partial X_i} \right) \right].
\]

For the Euler equations, for which \(p_{ij}\) and \(q_i\) are given by Eq. (65), the entropy of a fluid particle is invariant, i.e.,

\[
\rho \frac{D s}{D t} = 0.
\]

For an incompressible fluid, the first relation of Eq. (55) is replaced by\(^{14}\)

\[
\frac{D \rho}{D t} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0.
\]

Thus, from Eqs. (52) and (68),

\[
\frac{\partial v_i}{\partial X_i} = 0.
\]

Equation (63) for the Navier–Stokes-stress tensor reduces to

\[
p_{ij} = p \delta_{ij} - \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right).
\]

The first term on the right-hand side of Eq. (60) reduces to

\[
-p_{ij} \frac{\partial v_i}{\partial X_j} = - \left[ p \delta_{ij} - \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right] \frac{\partial v_i}{\partial X_j}
\]

\[
= \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2.
\]

Thus, Eq. (60) reduces to

\[
\rho \frac{D e}{D t} = \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left( \lambda \frac{\partial T}{\partial X_j} \right).
\]

\(^{13}\)Note the following transformation:

\[
\frac{\partial v_i}{\partial X_j} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \frac{1}{2} \frac{\partial v_i}{\partial X_j} \frac{\partial v_j}{\partial X_i} \delta_{ij} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right).
\]

The second term in the last expression is easily seen to vanish.

\(^{14}\)The density is invariant along fluid-particle paths. If \(\rho\) is of uniform value \(\rho_0\) initially, it is a constant, i.e.,

\[
\rho = \rho_0.
\]

In a time-independent (or steady) problem, the density is constant along streamlines.

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To summarize, the Navier–Stokes equations for incompressible fluid are

\[
\frac{\partial v_i}{\partial X_i} = 0, \quad (72a)
\]

\[
\frac{\rho}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j} \left[ \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right], \quad (72b)
\]

\[
\frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left( \frac{\lambda}{\rho} \frac{\partial T}{\partial X_j} \right), \quad (72c)
\]

with the incompressible condition (68) being supplemented, i.e.,

\[
\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \quad (73)
\]

### 3.2.2 Boundary condition for the Euler set

In Section M-3.5, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentum-conservation equations of the Euler set are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (74)
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (75)
\]

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y}, \quad (76)
\]

where \(\rho\) is the density, which is uniform, \((u, v)\) is the flow velocity, and \(p\) is the pressure. Owing to Eq. (74), the stream function \(\Psi\) can be introduced as

\[
u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \quad (77)
\]
Eliminating \( p \) from Eqs. (75) and (76), we have\(^{15}\)

\[
u \frac{\partial \Omega}{\partial x} + v \frac{\partial \Omega}{\partial y} = 0, \tag{78}\]

where \( \Omega \) is the vorticity, i.e.,

\[
\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}. \tag{79}\]

From Eqs. (77) and (78),

\[
\frac{\partial \Psi}{\partial y} \frac{\partial \Omega}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial \Omega}{\partial y} = 0. \tag{80}\]

This equation shows that \( \Omega \) is a function of \( \Psi \), i.e.,

\[
\Omega = F(\Psi). \tag{81}\]

\(^{15}\)The following equation is formed from them:

\[
\partial \text{Eq. (75)} / \partial y - \partial \text{Eq. (76)} / \partial x = 0.
\]

\(^{16}\)This can be seen with the aid of theorems on implicit functions (see References M-[47, 48, 267]). The proof is outlined here. The \( \Omega \) and \( \Psi \) are functions of \( x \) and \( y \):

\[
\Omega = \Omega(x, y), \quad \Psi = \Psi(x, y). \quad (\ast)
\]

Solving the second equation with respect to \( x \), we have

\[
x = \dot{x}(\Psi, y). \quad (***)
\]

With this relation into Eq. (\( \ast \)),

\[
\begin{align*}
\Omega &= \Omega(\dot{x}(\Psi, y), y) = \hat{\Omega}(\Psi, y), \\
\Psi &= \Psi(\dot{x}(\Psi, y), y) = \hat{\Psi}(\Psi, y).
\end{align*} \quad (\ast\ast\ast)
\]

That is, \( \Omega \) is expressed as a function of \( \Psi \) and \( y \). From Eqs. (\( \ast\ast\ast \)) and (\( \ast\ast \)),

\[
\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = \frac{\partial \hat{\Omega}(\dot{x}(\Psi, y), y)}{\partial y} = \frac{\partial \hat{\Omega}(x, y)}{\partial x} \frac{\partial \dot{x}(\Psi, y)}{\partial y} + \frac{\partial \hat{\Omega}(x, y)}{\partial y}, \tag{\( \ast\ast\ast\ast \))}
\]

\[
\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = 0. \tag{\( \ast\ast\ast\ast \))}
\]

On the other hand,

\[
\frac{\partial \Psi(\Psi, y)}{\partial y} = \frac{\partial \Psi(\dot{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \dot{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y}.
\]

Thus,

\[
\frac{\partial \Psi(x, y)}{\partial y} \frac{\partial \dot{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y} = 0. \tag{4}
\]

From Eqs. (80), (\( \ast\ast\ast\)) and (4), we have

\[
\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = 0, \quad \text{or} \quad \Omega = \hat{\Omega}(\Psi).
\]

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This functional relation between $\Omega$ and $\Psi$ is a local relation, and therefore $F$ may be a multivalued function of $\Psi$. From Eqs. (79) and (81),

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = F(\Psi).$$

(82)

Consider a boundary-value problem in a simply-connected bounded domain, where $\Psi$ is given on the boundary ($\Psi = \Psi_B$). Introduce a coordinate $s$ ($0 \leq s < S$) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where $\partial \Psi_B / \partial s < 0$, and the fluid flows out from the domain on the boundary where $\partial \Psi_B / \partial s > 0$. When $F$ is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose $F$ on the part where $\partial \Psi_B / \partial s < 0$ or $\partial \Psi_B / \partial s > 0$. For example, take the case where $\partial \Psi_B / \partial s < 0$ for $0 < s < S_m$ and $\partial \Psi_B / \partial s > 0$ for $S_m < s < S$, and choose the distribution $\Omega_B(s)$ of $\Omega$ along the boundary for the part $0 < s < S_m$. By the choice of $\Omega_B$, the function $F(\Psi)$ is determined in the following way. Inverting the relation $\Psi = \Psi_B(s)$ between $\Psi$ and $s$ on the part $0 < s < S_m$, i.e., $s(\Psi)$, and noting the relation (81), we find that $F$ is given by

$$F(\Psi) = \Omega_B(s(\Psi)).$$

(83)

Then, the boundary-value problem is fixed. That is, Eq. (82) is fixed as

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \Omega_B(s(\Psi)),$$

(84)

and the boundary condition is given as $\Psi = \Psi_B(s)$. This system is a standard from the point of counting of the number of boundary conditions. Obviously, from Eq. (79), the solution of the above system automatically satisfies condition $\Omega = \Omega_B(s)$ along the boundary for $0 < s < S_m$. We cannot choose the distribution of $\Omega$ on the boundary for $S_m < s < S$.

The energy-conservation equation of the incompressible Euler set is given by Eq. (71) with $\mu = \lambda = 0$, i.e.,

$$u \frac{\partial e}{\partial x} + v \frac{\partial e}{\partial y} = 0, \quad \text{or} \quad \frac{\partial \Psi}{\partial y} \frac{\partial e}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial e}{\partial y} = 0,$$

(85)

where $e$ is the internal energy. Thus, $e$ is a function of $\Psi$, i.e.,

$$e = F_1(\Psi).$$

(86)

In the above boundary-value problem, therefore, $e$ can be specified on the the part $(0 < s < S_m)$ of the boundary, but no condition can be specified on other part $(S_m < s < S)$ and vice versa.  

---

\footnote{There is still some ambiguity. The case where there is a region with closed stream lines $\Psi(x, y) = \text{const}$ inside the domain is not excluded.}

\footnote{From the second relation on $e$ of Eq. (55) and the uniform-density condition, the condition on $e$ can be replaced by the condition on the temperature $T$.}
To summarize, we can specify three conditions for $\Psi$, $\Omega$, and $e$ on the part $\partial \Psi_B / \partial s < 0$ ($\partial \Psi_B / \partial s > 0$) of boundary but one condition for $\Psi$ on the other part $\partial \Psi_B / \partial s > 0$ ($\partial \Psi_B / \partial s < 0$). The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

### 3.2.3 Ambiguity of pressure in the incompressible Navier–Stokes system

It may be better to note ambiguity of the solution of the initial and boundary-value problem of the incompressible Navier–Stokes equations in a bounded domain of simple boundaries.

Consider the Navier–Stokes equations for an incompressible fluid, i.e.,

\begin{align}
\frac{\partial v_i}{\partial x_i} &= 0, \quad \text{(87a)} \\
\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad \text{(87b)} \\
\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} &= \mu \frac{\partial}{\partial X_j} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) + \frac{\partial}{\partial X_j} \left( \lambda \frac{\partial T}{\partial X_j} \right), \quad \text{(87c)} \\
\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} &= 0, \quad \text{(87d)}
\end{align}

where $e$, $\mu$, and $\lambda$ are functions of $T$ and $\rho$.

Consider the initial and boundary-value problem of Eqs. (87a)–(87d) in a bounded domain $D$ on the boundary $\partial D$ of which $v_i$ and $T$ are specified as $v_i = v_{wi}$ and $T = T_w$ ($v_{wi}$ and $T_w$ are, respectively, the surface velocity and temperature of the boundary satisfying $\int_{\partial D} v_{wi} n_i dS = 0$, $n_i$ : the unit normal vector to the boundary) and no condition is imposed on $\rho$ and $p$. Let $(v_i(s), \rho(s), T(s), p(s))$ be a solution of the initial and boundary-value problem. Let $P(s)$ be an arbitrary function of $t$, independent of $x_i$, that vanishes at initial time $t = 0$, i.e., $P(0) = f(t)$ with $f(0) = 0$. Put

$$(v_i, \rho, T, p) = (v_i(s), \rho(s), T(s), p(s) + P(s)).$$

Then, $e$, $\mu$, and $\lambda$ corresponding to the new $(v_i, \rho, T, p)$ are equal to $e(s)$, $\mu(s)$, and $\lambda(s)$ respectively, because they are determined by $\rho$ and $T$. The new $(v_i, \rho, T, p)$ satisfy the equations (87a)–(87d) and the initial and boundary conditions.

### 3.2.4 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set. This statement should be made precise. The difference is briefly explained in the book “Molecular Gas Dynamics” in
connection with the equations derived by the S expansion from the Boltzmann equation in Sections M-3.2.4 and M-3.7.2. Here, we explicitly show the process of analysis from the compressible Navier–Stokes set. The resulting set of equations no longer has ambiguity of pressure in contrast to the incompressible Navier–Stokes set. Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the nondimensional variables introduced by Eq. (M-1.74) in Section M-1.10 as follows:

$$Sh \frac{\partial \omega}{\partial t} + \frac{\partial (1 + \omega)u_i}{\partial x_i} = 0,$$

(88)

$$Sh \frac{\partial (1 + \omega)u_i}{\partial t} + \frac{\partial}{\partial x_j} \left( (1 + \omega)u_iu_j + \frac{1}{2} P_{ij} \right) = 0,$$

(89)

$$Sh \frac{\partial}{\partial t} \left[ (1 + \omega) \left( \frac{3}{2} (1 + \tau) + u_i^2 \right) \right] + \frac{\partial}{\partial x_j} \left[ (1 + \omega)u_j \left( \frac{3}{2} (1 + \tau) + u_i^2 \right) + u_i (\delta_{ij} + P_{ij}) + Q_j \right] = 0.$$

(90)

The nondimensional stress tensor $P_{ij}$, and heat-flow vector $Q_i$, are expressed as

$$P_{ij} = P \delta_{ij} - \frac{\mu_0 (2RT_0)^{1/2}}{p_0 L} (1 + \bar{\mu}) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right),$$

(91a)

$$Q_i = - \frac{\lambda_0 T_0}{Lp_0 (2RT_0)^{1/2}} (1 + \bar{\lambda}) \frac{\partial \tau}{\partial x_i}.$$  

(91b)

Here, $\bar{\mu}$ and $\bar{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

$$\mu = \mu_0 (1 + \bar{\mu}), \quad \lambda = \lambda_0 (1 + \bar{\lambda}),$$

where $\mu_0$ and $\lambda_0$ are, respectively, the values of the viscosity $\mu$ and the thermal conductivity $\lambda$ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of $\tau$ and $\omega$.

The first relation of the equation of state [Eq. (56)] is expressed as

$$P = \omega + \tau + \omega \tau.$$  

(92)

Take a small parameter $\varepsilon$, and consider the case where

$$u_i = O(\varepsilon), \quad \omega = O(\varepsilon), \quad \tau = O(\varepsilon), \quad Sh = O(\varepsilon), \quad \mu_0 (2RT_0)^{1/2} \frac{p_0}{p_0 L} = \gamma_1 \varepsilon, \quad \lambda_0 T_0 \frac{Lp_0 (2RT_0)^{1/2}}{4} = \frac{5}{4} \gamma_2 \varepsilon,$$

(93a)

thus,

$$P = O(\varepsilon), \quad \bar{\mu} = O(\varepsilon), \quad \bar{\lambda} = O(\varepsilon).$$

For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B = 0$. 

19For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B = 0$. 

24
According to the definition of $u_i$ in Eq. (M-1.74), $\varepsilon$ is of the order of the Mach number. In view of this and the definition of the Prandtl number $Pr = 5R\mu/2\lambda$ (see Section M-3.1.9), $\gamma_1$ and $\gamma_2$ are, respectively, of the orders of $1/Re$ and $1/PrRe$ ($Re$: the Reynolds number). According to Eq. (M-1.48a), the condition $Sh = O(\varepsilon)$ in Eq. (93a) means that the time scale $t_0$ of the variation of variables is of the order of $L/(2RT_0)^{1/2}\varepsilon$, which is of the order of time scale of viscous diffusion. Thus, we are considering the case where the Mach number is small, the Reynolds and Prandtl numbers are of the order of unity, and the time scale of variation of the system is of the order of the time scale of viscous diffusion. We can take $t_0 = L/(2RT_0)^{1/2}\varepsilon$ without loss of generality. Then,

$$Sh = \varepsilon.$$

Corresponding to the above situation, $u_i, \omega, P,$ and $\tau$ are expanded in power series of $\varepsilon$, i.e.,

$$u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots,$$  \hspace{1cm} (95a)

$$\omega = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots,$$  \hspace{1cm} (95b)

$$P = P_1\varepsilon + P_2\varepsilon^2 + \cdots,$$  \hspace{1cm} (95c)

$$\tau = \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots,$$  \hspace{1cm} (95d)

$$\bar{\mu} = \bar{\mu}_1\varepsilon + \bar{\mu}_2\varepsilon^2 + \cdots,$$  \hspace{1cm} (95e)

$$\bar{\lambda} = \bar{\lambda}_1\varepsilon + \bar{\lambda}_2\varepsilon^2 + \cdots,$$  \hspace{1cm} (95f)

$$P_{ij} = P_{1\delta_{ij}}\varepsilon + P_{ij2}\varepsilon^2 + \cdots,$$  \hspace{1cm} (95g)

$$Q_i = Q_{i2}\varepsilon^2 + \cdots.$$  \hspace{1cm} (95h)

Substituting Eqs. (95a)-(95h) with Eqs. (93b) and (94) into Eqs. (88)-(90) with Eqs. (91a) and (91b), and arranging the same-order terms of $\varepsilon$, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \quad \frac{\partial P_1}{\partial x_i} = 0, \quad \frac{\partial u_{i1}}{\partial x_i} = 0,$$

$$\frac{\partial \omega_1}{\partial t} + \frac{\partial u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} = 0,$$

$$\frac{\partial u_{i1}}{\partial t} + \frac{\partial u_{i1}u_{j1}}{\partial x_j} + \frac{1}{2} \frac{\partial P_2}{\partial x_i} - \gamma_1 \frac{\partial}{\partial x_j} \left( \frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{k1}}{\partial x_k} \delta_{ij} \right) = 0,$$

$$\frac{3}{2} \frac{\partial P_1}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{5}{2} u_{j2} + \frac{5}{2} P_1 u_{j1} - \frac{5}{4} \gamma_2 \frac{\partial \tau_1}{\partial x_j} \right) = 0,$$

and so on. At the leading order, the equations derived from Eqs. (88) and (90) degenerate into the same equation $\partial u_{i1}/\partial x_i = 0$. Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

$$\frac{\partial P_1}{\partial x_i} = 0.$$  \hspace{1cm} (96)
\[
\frac{\partial u_{i1}}{\partial x_i} = 0,
\]

\[
\frac{\partial u_{i1}}{\partial t} + u_{j1} \frac{\partial u_{i1}}{\partial x_j} = - \frac{1}{2} \frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2} \frac{\partial^2 u_{i1}}{\partial x_j^2},
\]

\[
\frac{5}{2} \frac{\partial \tau_1}{\partial t} - \frac{\partial P_1}{\partial t} + \frac{5}{2} u_{i1} \frac{\partial \tau_1}{\partial x_i} = \frac{5}{4} \gamma_2^2 \frac{\partial^2 \tau_1}{\partial x_j^2},
\]

\[
\frac{\partial u_{i2}}{\partial x_i} = - \frac{\partial \omega_1}{\partial t} - \frac{\partial \omega_1 u_{i1}}{\partial x_i},
\]

\[
\frac{\partial u_{i2}}{\partial t} + u_{j2} \frac{\partial u_{i2}}{\partial x_j} + u_{j1} \frac{\partial u_{i1}}{\partial x_j} = - \frac{1}{2} \left( \frac{\partial P_3}{\partial x_i} - \omega_1 \frac{\partial P_2}{\partial x_j} \right) + \frac{\gamma_1}{2} \frac{\partial}{\partial x_j} \left( \frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{i2}}{\partial x_k} \delta_{ij} \right)
\]

\[\frac{\gamma_1}{2} \omega_1^2 \frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2} \frac{\partial}{\partial x_j} \left( \frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} \right) \right],
\]

\[
3 \frac{\partial P_2}{\partial t} + \frac{3}{2} u_{j1} \frac{\partial P_2}{\partial x_j} + \frac{5}{2} \left( \frac{\partial u_{i2}}{\partial x_j} - \frac{\partial \omega_2}{\partial t} - \frac{\partial(\omega_1 u_{i2} + \omega_2 u_{j1})}{\partial x_j} \right)
\]

\[
= \frac{5 \gamma_2}{4} \frac{\partial}{\partial x_i} \left( \frac{\partial \tau_2}{\partial x_i} + \frac{\lambda_1}{2} \frac{\partial \tau_1}{\partial x_i} \right) + \frac{\gamma_1}{2} \left( \frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} \right)^2,
\]

where

\[
P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1 \tau_1.
\]

These equations are very similar to Eqs. (M-3.265)–(M-3.268) [or Eqs. (20)–(23)] obtained by the S expansion of the Boltzmann equation in Section M-3.7.2 (or Section 3.1.1).\(^{20}\) The solution is determined in the same way as the solution of the S-expansion system is done in Section 3.1.1. What should be noted is the determination of \(P_1, P_2, \cdots\) in a bounded-domain problem. They are determined by the condition of invariance of the mass of the gas in the domain with the aid of higher-order equations in the same way as \(P_{S1}, P_{S2}, \cdots\) in the S-expansion system (see Section 3.1.1).

In order to compare Eqs. (97a)–(97c) and (99) with the incompressible Navier–Stokes equations (87a)–(87d), we will rewrite the latter equations for the situation where the former equations are derived. The starting equations are Eqs. (88)–(91b)\(^{21}\) and the nondimensional form of Eq. (68), i.e.,

\[
\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = 0,
\]

\(^{20}\)Equations (96)–(98a) and (99) are of the same form as Eqs. (M-3.265)–(M-3.267a) and (M-3.268). Thus the discussion from the next paragraph to the end of Section 3.2.4 applies to the two systems.

\(^{21}\)As the internal energy \(e, 3RT/2 = 3RT_0(1 + \tau)/2\) is chosen for consistency.
instead of Eq. (92).\textsuperscript{22} The analysis is carried out in a similar way and the equations corresponding to Eqs. (97a)–(97c) are\textsuperscript{23}

\[
\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{101a}
\]

\[
\frac{\partial u_{i1}}{\partial t} + u_{j1} \frac{\partial u_{i1}}{\partial x_j} = - \frac{1}{2} \frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2} \frac{\partial^2 u_{i1}}{\partial x_i^2}, \tag{101b}
\]

\[
\frac{3}{2} \frac{\partial \tau_1}{\partial t} + \frac{3}{2} u_{i1} \frac{\partial \tau_1}{\partial x_i} = - \frac{5}{4} \frac{\partial \omega_1}{\partial \hat{t}} + \frac{5}{4} \frac{\partial \omega_1}{\partial x_i} \frac{\partial \omega_1}{\partial x_i} + \frac{\partial \omega_1}{\partial x_i} \frac{\partial u_{i1}}{\partial x_i}, \tag{101c}
\]

Equations (101a) and (101b) are, respectively, of the same form as Eqs. (97a) and (97b). Equation (97c) is rewritten with the aid of Eqs. (96) and (99) as

\[
\frac{3}{2} \frac{\partial \tau_1}{\partial t} + \frac{3}{2} u_{i1} \frac{\partial \tau_1}{\partial x_i} - \left( \frac{\partial \omega_1}{\partial \hat{t}} + u_{i1} \frac{\partial \omega_1}{\partial x_i} \right) = \frac{5}{4} \frac{\partial \tau_1}{\partial x_i}. \tag{102}
\]

The difference of Eq. (97c) or (102) from Eq. (101c) is

\[
\frac{\partial \omega_1}{\partial \hat{t}} + u_{i1} \frac{\partial \omega_1}{\partial x_i},
\]

which vanishes for an incompressible fluid. The work done per unit time on unit volume of fluid by pressure, given by \(-p_0(2RT_0)^{1/2}L^{-1}(1 + P)u_i/\partial x_i\), is transformed with the aid of Eqs. (96), (97a), and (98a) in the following way:

\[
\frac{W}{p_0(2RT_0)^{1/2}L^{-1}} = - \frac{\partial (1 + P)u_i}{\partial x_i}
\]

\[
= - \frac{\partial u_{i1}}{\partial x_i} \varepsilon - \left( P_1 \frac{\partial u_{i1}}{\partial x_i} + u_{i1} \frac{\partial P_1}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} \right) \varepsilon^2 + \cdots
\]

\[
= - \frac{\partial u_{i2}}{\partial x_i} \varepsilon^2 + \cdots
\]

\[
= \left( \frac{\partial \omega_1}{\partial \hat{t}} + u_{i1} \frac{\partial \omega_1}{\partial x_i} \right) \varepsilon^2 + \cdots.
\]

The work vanishes up to the order considered here for an incompressible fluid, because \(\partial u_i/\partial x_i = 0\) and \(\partial P_1/\partial x_i = 0\) (see Footnotes 22 and 23). That is, Eq. (97c) differs from Eq. (101c) by the amount of the work done by pressure. Thus, naturally, the temperature \(\tau_1\) fields in the two cases are different owing to this difference.

To summarize, the mass and momentum-conservation equations (97a) and (97b) of the set derived from the compressible Navier–Stokes set [Eqs. (88)–(91b) and (92)] under the situation given by Eqs. (93a) and (93b) with small \(\varepsilon\) are of the same form as those equations (101a) and (101b) of the corresponding set derived from the incompressible Navier–Stokes set [Eqs. (88)–(91b) and (100)].

\textsuperscript{22} From Eqs. (88) and (100), we have \(\partial u_i/\partial x_i = 0\).

\textsuperscript{23} We also obtain \(\partial P_1/\partial x_i = 0\).
but the energy-conservation equations (97c) and (101c) of the two sets differ by the work done by pressure. The density $\omega_1$ obtained from Eqs. (96)-(97c) with the first relation of Eq (99) does not generally satisfy the incompressible condition (100) with $\omega = \omega_1$ and $u_i = u_{i1}$. Both the density and temperature fields ($\omega_1$, $\tau_1$) are different in the two sets. The variation of the density $\omega_1$ along a particle path is due to the first relation of Eq. (99). Even if the temperature $\tau_1$ varies according to Eq. (101c), the density $\omega_1$ determined by the first relation of Eq. (99) does not generally satisfy the incompressible condition. Incidentally, in a bounded domain problem with simple boundaries, the pressure has ambiguity of an additive function of time for the incompressible set in contrast to the pressure for a compressible set [see Section 3.2.3 and the paragraph just after Eq. (99)].

Finally, it may be noted that under the situation (93a), the solenoidal condition for $u_{i1}$, i.e., Eq. (97a) or (101a), is derived only from the mass conservation equation (88) without the help of the incompressible condition (100).

3.2.5 Equations derived from the compressible Euler set when the Mach number and the temperature variation are small

Take the Euler set of equations, Eqs. (M-3.250a)-(M-3.250c) and the equation of state, in the nondimensional form derived from the Boltzmann equation in the limit $k \to 0$:

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{\partial \hat{\rho} \hat{v}_j}{\partial x_j} = 0, \tag{103a}$$

$$\frac{\partial \hat{\rho} \hat{v}_i}{\partial t} + \frac{\partial \hat{\rho} \hat{v}_j \hat{v}_i}{\partial x_j} + \frac{1}{2} \frac{\partial \hat{p}}{\partial x_i} = 0, \tag{103b}$$

$$\frac{\partial}{\partial t} \left[ \hat{\rho} \left( \hat{v}_i^2 + \frac{3}{2} \hat{T} \right) \right] + \frac{\partial}{\partial x_j} \left[ \hat{\rho} \hat{v}_j \left( \hat{v}_i^2 + \frac{5}{2} \hat{T} \right) \right] = 0, \tag{103c}$$

$$\hat{p} = \hat{\rho} \hat{T}, \tag{103d}$$

where the subscript $H0$ is eliminated for simpleness of notation. We consider the situation where the state of the gas deviates slightly from a uniform equilibrium state at rest. That is,

$$\hat{\rho} = 1 + \hat{\omega}, \quad \hat{p} = 1 + \hat{P}, \quad T = 1 + \hat{\tau}, \quad \hat{v}_i = \hat{u}_i, \tag{104}$$

24When the density $\rho$ is uniform initially, for which $\rho$ is a constant for an incompressible fluid, the viscosity and thermal conductivity are constants, and heat production by viscosity is neglected, Eqs. (101a)-(101c) can be compared directly with Eqs. (97a)-(97c) and (99), without carrying expansion, and the same results are obtained.

25It is easily seen that the velocity $u_{i1}$ vanishes, the pressure $P_1$ is a constant, and the temperature $\tau_1$ (thus, the density $\omega_1$) varies with time in initial-value problems where the velocity is zero and the temperature is nonuniform [strictly, non-harmonic] initially, and the pressure is time-independent at infinity. Thus, the incompressible condition is not satisfied. See also the example given in Section K-4.10.3, where the velocity vanishes and the density varies with time, and further, the temperature field is quite different from the incompressible case owing to the time-dependent boundary condition on $P_{S1}$, corresponding to $P_1$ here [note Footnote 20].
where the perturbed quantities $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ are small, say of the order of $\varepsilon$. They are expanded as

$$
\hat{h} = \hat{h}_1 \varepsilon + \hat{h}_2 \varepsilon^2 + \cdots,
$$

(105)

where $\hat{h} = \hat{\omega}$, $\hat{P}$, $\hat{\tau}$, or $\hat{u}_i$.

We discuss the two cases with different time scale. The first case is

$$
\frac{\partial \hat{h}}{\partial \hat{t}} = O(\hat{h}).
$$

(106)

Substituting the expansions (105) of the variables $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ into the Euler equations (103a)-(103d) and arranging the same-order terms with Eq. (106) in mind, we find that the leading-order variables are governed by the following set of equations:

$$
\begin{align}
\frac{\partial \hat{\omega}_1}{\partial \hat{t}} + \frac{\partial \hat{u}_{1j}}{\partial x_j} &= 0, \\
\frac{\partial \hat{u}_{1j}}{\partial \hat{t}} + \frac{1}{2} \frac{\partial \hat{P}_1}{\partial x_i} &= 0, \\
\frac{\partial \hat{P}_1}{\partial \hat{t}} + \frac{5}{3} \frac{\partial \hat{u}_j}{\partial x_j} &= 0,
\end{align}
$$

(107a-c)

$$
\hat{P}_1 = \hat{\omega}_1 + \hat{\tau}_1.
$$

(107d)

This set is the well-known acoustic equations (see Section M-3.7.1), which are explained in a standard textbook of gas dynamics, e.g., M-Liepman & Roshko [1957].

The second case is the case where the variables are slowly varying or the time scale of variation of the variables is long and of the order $1/\varepsilon$:

$$
\frac{\partial \hat{h}}{\partial \hat{t}} = \varepsilon O(\hat{h}).
$$

(108)

Here, we introduce the shrunk time $\hat{t}_\varepsilon$:

$$
\hat{t}_\varepsilon = \varepsilon \hat{t}.
$$

(109)

Then,

$$
\frac{\partial \hat{h}}{\partial \hat{t}_{\varepsilon}} = O(\hat{h}).
$$

(110)

Substituting the expansion (105) of the variables $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ into the Euler equations (103a)-(103d) and arranging the same-order terms with Eq. (110) in
mind, we obtain the equations that determine the leading-order variables as

\[
\frac{\partial \hat{P}_1}{\partial x_i} = 0, \quad (111a)
\]

\[
\frac{\partial \hat{u}_{i_1}}{\partial x_j} = 0, \quad (111b)
\]

\[
\frac{\partial \hat{u}_{i_1}}{\partial \hat{t}_\varepsilon} + \hat{u}_{j_1} \frac{\partial \hat{u}_{i_1}}{\partial x_j} + \frac{1}{2} \frac{\partial \hat{P}_2}{\partial x_i} = 0, \quad (111c)
\]

\[
\frac{5}{2} \frac{\partial \hat{\tau}_1}{\partial \hat{t}_\varepsilon} - \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \frac{5}{2} \hat{u}_{j_1} \frac{\partial \hat{\tau}_1}{\partial x_j} = 0, \quad (111d)
\]

\[
\hat{P}_1 = \hat{\omega}_1 + \hat{\tau}_1. \quad (111e)
\]

From Eq. (111a), \(\hat{P}_1\) is a function of \(\hat{t}_\varepsilon\) only, and thus is determined by the boundary condition. The relation

\[
\frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} = \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j_1} \frac{\partial \hat{P}_1}{\partial x_j}, \quad (112)
\]

obvious from Eq. (111a), is conveniently used in the following discussion. The energy equation (111d) is transformed as

\[
3 \left( \frac{\partial \hat{\tau}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j_1} \frac{\partial \hat{\tau}_1}{\partial x_j} \right) - \frac{\partial \hat{\omega}_1}{\partial \hat{t}_\varepsilon} - \hat{u}_{j_1} \frac{\partial \hat{\omega}_1}{\partial x_j} = 0, \quad (113)
\]

by using Eqs. (111c) and (112) for \(\partial \hat{P}_1/\partial \hat{t}_\varepsilon\). From Eqs. (113) and (111e), the variation of \(\hat{\omega}_1\) along the fluid-particle path is expressed as follows:

\[
\frac{\partial \hat{\omega}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j_1} \frac{\partial \hat{\omega}_1}{\partial x_j} = \frac{3}{5} \left( \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j_1} \frac{\partial \hat{P}_1}{\partial x_j} \right) = \frac{3}{5} \frac{d \hat{P}_1}{d \hat{t}_\varepsilon}. \quad (114)
\]

Eqs. (113) and (114) are the linearized forms of the isentropic variations of \(\hat{\omega}_1\) versus \(\hat{\tau}_1\) and \(\hat{P}_1\) along the fluid-particle path. The energy equation (113) is conveniently compared with the energy equation of incompressible fluid. For the latter, the last two terms are absent and the temperature is invariant along the fluid-particle path. The difference is the work done by pressure, which can be shown as is done in Section 3.2.4.

The behavior of the gas governed by Eqs. (111a)-(111e) is summarized as follows:

(1): Equations (111b) and (111c) for the velocity field are of the same form as those of incompressible fluid.

(2): Depending on the condition of the boundary, \(\hat{P}_1\) can be time dependent or independent. (i) If \(\hat{P}_1\) is time dependent, the density \(\hat{\omega}_1\) varies along the

\[\text{[Footnote]}^{26}\text{Under the assumptions (104) and (108) or (110), the solenoidal condition (111b) for } \hat{u}_{i_1}\text{ is derived solely from the mass conservation equation (103a). It should not be confused with the incompressible condition.}\]

\[\text{[Footnote]}^{27}\text{For example, the pressure is specified at infinity in an unbounded problem.}\]
fluid-particle path owing to Eq. (114). (ii) If $\hat{P}_1$ is time independent, the temperature $\hat{\tau}_1$ and the density $\hat{\omega}_1$ are invariant along the fluid-particle path owing to Eqs. (111d) and (114).

(Section 3.2.5: Version 8-00)

4 Chapter M-4

4.1 Notes on application of the solution in Section M-4.3

In the application of the quasi-unidirectional solution in Section M-4.3, some cares are required. For some ranges from the entrance and from the exit of the pipe, the assumptions made in the first paragraph of Section M-4.3 and the assumption\(^{28}\) just after Eq. (M-4.64) are not generally satisfied. For a long pipe,\(^{29}\) it is expected that the three regions, entrance, central, and exit regions, can be analyzed separately and that the results can be smoothly connected. In the central region, we try to use the solution of Section M-4.3. This process being successfully done, and the contributions of the entrance and exit regions to the pressure and temperature variations being estimated to be much smaller than the contributions of the central region, the solution in Section M-4.3 gives the global behavior of flow through a long pipe. It is often applied without confirmation that the end effects are so small as to be neglected. The solution of the central region, for which the solution of Section M-4.3 is used, has to be confirmed that it satisfies the above assumptions made in the analysis. For example, if the vacuum condition (or the vanishing density condition) is directly applied to the exit when exit is connected to a large vacuum chamber, one finds the average flow velocity on the cross section of the exit is infinite owing to the mass flow conservation through the pipe. This obviously violates the latter assumption. The solution cannot be applied up to such low density (or pressure) region. The contribution of the region where the assumption is violated has to be investigated in more complete formulation.

In a pipe problem, the temperature is controlled locally by the temperature of the pipe, but the pressure is controlled only at the entrance and the exit. The local pressure in between is determined by the mass conservation condition as shown in Section M-4.3. Thus, the nondimensional local pressure gradient cannot be specified at our disposal. For example, let us examine how the solution on the basis of the local linear theory breaks down in a straight pipe with a uniform cross section and a uniform temperature. According to Eq. (M-4.77) required by the mass flow conservation through the pipe, the quantities at the cross section A and the cross section B are related as follows:

\[
p_A \left( \frac{L}{p} \frac{d\rho}{dX} \right)_A \hat{M}_P(k_A) = p_B \left( \frac{L}{p} \frac{d\rho}{dX} \right)_B \hat{M}_P(k_B),
\]

\(^{28}\)This condition is consistent with the preceding assumptions.
\(^{29}\)This means that the length of the pipe is much larger than the linear dimension of its cross section.
where the subscripts A and B indicate the values at the cross sections A and B respectively, $X_1$ is the coordinate along the pipe, and the unnecessary common factors $T_w$ and $L$ are eliminated from the formula (M-4.77). The nondimensional pressure gradient at the cross section B is expressed with that at A as

$$\left( \frac{L}{p} \frac{dp}{dX_1} \right)_B = \frac{p_A \hat{M}_p(k_A)}{p_B \hat{M}_p(k_B)} \left( \frac{L}{p} \frac{dp}{dX_1} \right)_A,$$

where the ratio $\hat{M}_p(k_A)/\hat{M}_p(k_B)$ is bounded from below by a positive constant when $k_B > k_A$ and becomes infinite as $k_A \to 0$ (see, for example, Table M-5.3, M-Sone & Yamamoto [1968]).

Thus, $\left| (L/p_B)(dp/dX_1)_B \right|$ becomes very large or infinite even when $\left| (L/p_A)(dp/dX_1)_A \right|$ is small if $p_A/p_B \gg 1$ or $k_A \ll 1$, and the slowly varying assumption $\left| (L/p)(dp/dX_1) \right| \ll 1$ is violated at the cross section B. Thus, the solution in Section M-4.3 is generally no longer valid there, and more complete analysis is required. In practical applications, the quantities that are assumed to be small may be small but are not very small. Thus, they may easily reach nonsmall values in another cross section.

(Section 4.1: Version 8-00)

### 4.2 Gas over a Plane Interface: Supplement to M-4.4

Here, the discussion of the half-space problem under the boundary condition (M-1.26) for a simple boundary in Section M-4.4 is extended to that under the boundary condition (M-1.30) or (180) for an interface of a gas and its condensed phase. That is, a plane simple boundary is replaced by a plane condensed phase of the gas, and the possible solution including the possible state at infinity is discussed in the situation when no evaporation or condensation is taking place on the condensed phase. This is the problem first discussed by Golse under the complete condensation condition (M-Bardos, Golse & Sone [2006]), which is a special case of the boundary condition (M-1.30). The analysis goes parallel to that in Section M-4.4. The full explanation is given with the difference being shown in Footnotes, though it may be redundant.

Consider a semi-infinite expanse of a gas ($X_1 > 0$) bounded by its stationary plane condensed phase with a uniform temperature $T_w$ at $X_1 = 0$. There is no external force acting on the gas. The state of the gas is time-independent and uniform with respect to $X_2$ and $X_3$, i.e., $f = f(X_1, \xi)$, and it approaches an equilibrium state as $X_1 \to \infty$, i.e.,

$$f \to \frac{\rho_\infty}{(2\pi RT_\infty)^{3/2}} \exp \left( -\frac{(\xi_i - v_i\infty)^2}{2RT_\infty} \right) \text{ as } X_1 \to \infty,$$

where $\rho_\infty$, $v_i\infty$, and $T_\infty$ are bounded. The boundary condition on the interface

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30 Generally, $\hat{M}_p(k)$ first decreases from infinity as $k$ increases from zero, reaching the minimum value at some $k$ (around $k = \sqrt{\pi}$ or $Ku = 2$ in Table M-5.3) [Knudsen minimum] and increases to the finite value at $k = \infty$. The probability distribution

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is given by Eq. (180) with the conditions (181a)-(181c) and (184), i.e.,
\[ f(0, \xi) = g_\xi + \int_{\xi_1 < 0} K_f(\xi, \xi_*) f(0, \xi_*) d\xi_* \quad (\xi_1 > 0). \] (116)

Here, we are interested in the case where no evaporation or condensation is taking place on the condensed phase,\(^{31}\) i.e.,
\[ \rho v_1 = \int \xi_1 f d\xi = 0 \quad \text{at} \quad X_1 = 0. \] (117)

We will show that the solution of the Boltzmann equation (M-1.5), i.e.,
\[ \xi_1 \frac{\partial f}{\partial X_1} = J(f, f), \] (118)

describing the above situation exists only when
\[ v_{i\infty} = 0, \quad \rho_{i\infty} = \rho_w, \quad T_{i\infty} = T_w, \]
where \(\rho_w\) is the saturation gas density at temperature \(T_w\), and that the solution is uniquely given by the Maxwellian
\[ f = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp \left( -\frac{\xi^2_i}{2RT_w} \right). \] (119)

From the integral of the Boltzmann equation (118) over the whole space of \(\xi\) [or the conservation equation (M-1.12)], i.e.,
\[ \frac{d}{dX_1} \left( \int \xi_1 f d\xi \right) = 0, \]
and Eq. (117), we find that the mass flux vanishes for \(X_1 \geq 0\), i.e.,
\[ \int \xi_1 f d\xi = 0 \quad (0 \leq X_1 < \infty). \] (120)

With this result in the condition (115) at infinity, we have
\[ \int \xi_1 \xi_1^2 f d\xi = 0 \quad \text{at infinity.} \] (121)

The integral of the Boltzmann equation (118) multiplied by \(\xi_1^2\) over the whole space of \(\xi\) [or the conservation equation (M-1.14)] gives
\[ \frac{d}{dX_1} \left( \int \xi_1 \xi_1^2 f d\xi \right) = 0. \] (122)

\(^{31}\)No mass flux across the boundary irrespective of a situation is the definition of a simple boundary.
Thus, from Eqs. (121) and (122), we have
\[ \int \xi_1 \xi_2^2 f d\xi = 0 \quad (0 \leq X_1 < \infty). \] (123)

For the boundary condition (180) with the conditions (181a)-(181c) and (184), the following inequality holds at \( X_1 = 0 \) [Eq. (203) with \( \rho v_1 = 0, v_{wi} = 0, n_i = (1,0,0) \)]:\footnote{The same equality holds for a simple boundary except that \( \rho_w \) in \( f_w \) is a free parameter for this case (see Section M-4.4).}
\[ \int \xi_1 f \ln(f/f_w) d\xi \leq 0, \] (124)
where \( f_w \) is the Maxwellian with the temperature \( T_w \) and velocity \( v_{wi} (= 0) \) of the condensed phase and the saturation gas density \( \rho_w \) at temperature \( T_w \), i.e.,
\[ f_w = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp \left( -\frac{\xi_i^2}{2RT_w} \right). \] (125)

With the aid of Eqs. (120) and (123),
\[ \int \xi_1 f \ln(f/c_0) d\xi \leq \int \xi_1 f \ln(f_w/c_0) d\xi \]
\[ = -\frac{1}{2RT_w} \int \xi_1 \xi_2^2 f d\xi = 0 \text{ at } X_1 = 0, \] (126)
where \( c_0 \) is a constant to make the argument of the logarithmic function dimensionless, whose choice does not influence the result.

On the other hand, from the H theorem, i.e., Eq. (M-1.36), in a time-independent one-dimensional case,
\[ -\int \xi_1 f \ln(f/c_0) d\xi \bigg|_{X_1=0} + \int \xi_1 f \ln(f/c_0) d\xi \bigg|_{X_1=\infty} = \int_0^\infty GdX_1 \leq 0, \] (127)
where
\[ G = -\frac{1}{4m} \int (f'f'_s - ff_s) \ln \left( \frac{f'f'_s}{ff_s} \right) Bd\Omega d\xi_s d\xi \leq 0. \]

From Eqs. (115), (120), and (121), the second term on the left-hand side of Eq. (127) vanishes, that is,
\[ -\int \xi_1 f \ln(f/c_0) d\xi \bigg|_{X_1=0} = \int_0^\infty GdX_1 \leq 0. \] (128)

Combining the two inequalities (126) and (128), we have
\[ 0 \leq -\int \xi_1 f \ln(f/c_0) d\xi \bigg|_{X_1=0} = \int_0^\infty GdX_1 \leq 0. \]
Therefore, we have
\[ \int_{0}^{\infty} G dX_1 = 0, \quad \text{thus}, \quad G = 0, \quad (129) \]
and
\[ \int_{X_1=0}^{\infty} \xi_1 f \ln(f/c_0) d\xi = 0. \quad \]

From Eq. (129), \( f \) is Maxwellian in \( 0 < X_1 < \infty \), and Eq. (118) is reduced to \( \xi_1 \partial f/\partial X_1 = 0 \). That is, \( f \) is a uniform Maxwellian. From the condition (115) at infinity and Eq. (120), the solution is to be in the form
\[ f = \frac{\rho_{\infty}}{(2\pi RT_{\infty})^{3/2}} \exp \left( -\frac{\xi_1^2 + (\xi_2 - v_{2\infty})^2 + (\xi_3 - v_{3\infty})^2}{2RT_{\infty}} \right) \quad (0 < X_1 < \infty). \quad (130) \]

From the uniqueness condition of Eq. (118c), the Maxwellian that satisfies the boundary condition (181c) is given by Eq. (125). Thus, the parameters in Eq. (130) have to be\[ v_{2\infty} = v_{3\infty} = 0, \quad \rho_{\infty} = \rho_w, \quad T_{\infty} = T_w, \quad \]
and the solution is given by Eq. (119).

The same statement holds for the linearized Boltzmann equation with the corresponding general boundary condition (M-1.112) on an interface of the gas and its condensed phase. The temperature \( T_w \) of the condensed phase and the saturation gas density \( \rho_w \) at temperature \( T_w \) are, respectively, taken here as the reference temperature \( T_0 \) or \( \tau_w = 0 \) and the reference density \( \rho_0 \) or \( \omega_w = 0 \).

The linearized Boltzmann equation is given in the form
\[ \zeta_1 \frac{\partial \phi}{\partial \eta} = L(\phi) \quad (0 < \eta < \infty). \quad (131) \]
The boundary condition on the interface is given by Eq. (M-1.112) with the supplementary conditions (i), (ii-a), and (ii-b) as
\[ E(\zeta) \phi(\eta, \zeta) = \int_{\zeta_1 < 0} K_1(\zeta, \zeta_*) \phi(\eta, \zeta_*) E(\zeta) d\zeta \quad (\zeta_1 > 0) \quad \text{at} \ \eta = 0. \quad (132) \]
The condition at infinity is
\[ \phi(\eta, \zeta) \to \omega_\infty + 2\zeta_i u_{i\infty} + \left( \zeta_i^2 - \frac{3}{2} \right) \tau_\infty \quad \text{as} \ \eta \to \infty, \quad (133) \]
where \( \omega_\infty, u_{i\infty}, \) and \( \tau_\infty \) are some constants and \( \eta = x_1/k = 2X_1/\sqrt{\pi} \ell_0 \). Then, the solution of the boundary-value problem (131)-(133) exists when and only when
\[ \omega_\infty = 0, \quad u_{i\infty} = 0, \quad \tau_\infty = 0, \quad (134) \]

\[ 33 \text{For a simple boundary, we can choose } \rho_\infty \text{ at our disposal, because } \rho \text{ in Eq. (M-1.27c) is arbitrary.} \]
\[ 34 \text{We take the reference density } \rho_w \text{ in contrast with the case of a simple boundary. This is only for convenience of explanation. For this choice, } \omega_w \text{ term disappears in Eq. (132) but } \omega_\infty \text{ term appears in Eq. (133).} \]
and the unique solution is given by
\[ \phi = 0. \]  
\[ (135) \]

The proof can be given in the same way as the preceding proof for the nonlinear case. From the conservation equation (M-1.90), i.e., \( \partial u_1/\partial \eta = 0 \), and the condition of absence of evaporation or condensation on the condensed phase (\( u_1 = \int \zeta_1 \phi E d\zeta = 0 \) at \( \eta = 0 \)), we have
\[ u_1 = \int \zeta_1 \phi E d\zeta = 0 \quad (0 \leq \eta < \infty). \]  
\[ (136) \]
Thus,
\[ u_{i\infty} = 0. \]  
\[ (137) \]
From Eqs. (133) and (137),
\[ \int \zeta_1 \phi^2 E d\zeta = 0 \quad \text{at infinity.} \]  
\[ (138) \]
According to the second part of Section M-A.10,
\[ \int \zeta_1 \phi^2 E d\zeta \leq 0 \quad \text{at } \eta = 0. \]  
\[ (139) \]
The linearized-Boltzmann-equation version of the equation for the H function given by Eq. (M-1.115) is expressed as
\[ \frac{\partial}{\partial \eta} \int \zeta_1 \phi^2 E d\zeta = LG, \]  
\[ (140) \]
where
\[ LG = -\frac{1}{2} \int EE_\omega (\phi' + \phi' - \phi - \phi')^2 \hat{B} d\omega d\zeta d\zeta \leq 0. \]  
\[ (141) \]
From Eqs. (138), (139), and (140) with Eq. (141), we find that \( LG \) is to be zero and that \( \phi \) is a summational invariant or the linearized form of a Maxwellian, i.e.,
\[ \phi = \omega + 2(\zeta_2 u_2 + \zeta_3 u_3) + \left( \zeta_4^2 - \frac{3}{2} \right) \tau, \]
where Eq. (136) is used. Then, Eq. (131) reduces to \( \zeta_1 \partial \phi / \partial \eta = 0 \), and therefore, \( \omega, u_2, u_3, \) and \( \tau \) are constant. In view of Eq. (133), the constants \( \omega, u_2, u_3, \) and \( \tau \) are given as
\[ \omega = \omega_\infty, \quad u_2 = u_{2\infty}, \quad u_3 = u_{3\infty}, \quad \tau = \tau_\infty, \]
\[ \phi = \omega_\infty + 2(\zeta_2 u_{2\infty} + \zeta_3 u_{3\infty}) + \left( \zeta_4^2 - \frac{3}{2} \right) \tau_\infty. \]

The boundary where this equality holds irrespective of a situation is the definition of a simple boundary.

This is the linearized-Boltzmann-equation version of the inequality (203) and valid for both types of boundaries, a simple boundary and an interface. For the case of an interface, an additional condition (M-A.271), which corresponds to Eq. (184) in the nonlinear case, is imposed on the kernel \( \hat{K}_{ij} \) (see also Footnote 52 in Section 6.4.2).

35The boundary where this equality holds irrespective of a situation is the definition of a simple boundary.

36This is the linearized-Boltzmann-equation version of the inequality (203) and valid for both types of boundaries, a simple boundary and an interface. For the case of an interface, an additional condition (M-A.271), which corresponds to Eq. (184) in the nonlinear case, is imposed on the kernel \( \hat{K}_{ij} \) (see also Footnote 52 in Section 6.4.2).
Owing to the supplementary condition (ii-b) to the boundary condition (M-1.112) together with Eq. (137), we have\(^{37}\)

\[
\begin{align*}
\omega_\infty &= 0, \quad u_1\infty = 0, \quad u_2\infty = 0, \quad u_3\infty = 0, \quad \tau_\infty = 0, \\
\phi &= 0.
\end{align*}
\]

(Section 4.2: Version 5-00)

5 Chapter M-9

5.1 Processes of solution of the equations with the ghost effect of infinitesimal curvature (July 2007)

The way in which Eqs. (M-9.33)-(M-9.39b) or Eqs. (M-9.49a)-(M-9.50c), including the time-dependent case with the additional time-derivative terms given by Eq. (M-9.42) or the mathematical expressions next to Eq. (M-9.59), contain the pressure terms, \((\bar{P}_{S_0}, \bar{P}_{S_2})\) or \((P_{S_1}, P_{S_2})\), is different from the way in which the Navier–Stokes equations (M-3.265)-(M-3.266) do the pressure terms, \((\bar{P}_{S_1}, P_{S_2})\). In Section M-9.4, we consider the time-independent solution of Eqs. (M-9.49a)-(M-9.50c) [Eqs. (M-9.56)-(M-9.57d)] that is uniform with respect to \(\tilde{\chi}\). Here, it may be better to explain how a solution of Eqs. (M-9.33)-(M-9.39b) or Eqs. (M-9.49a)-(M-9.50c) in a general case or a time-dependent solution that depends on \(\chi\) or \(\tilde{\chi}\) is obtained. Incidentally, the boundary conditions for the time-dependent case are derived in the same way as in Section M-3.7.3. Naturally from the derivation of the equations, the domain of a gas is in a straight pipe or channel of infinite length whose axis is in the \(x\) or \(\tilde{\chi}\) direction.

5.1.1 Equations (M-9.33)-(M-9.39b):

Take Eqs. (M-9.33)-(M-9.39b) with the additional time-derivative terms given by Eq. (M-9.42), i.e.,\(^{38}\)

\[
\frac{\partial \bar{p}_{S_0}}{\partial t} = \frac{\partial \bar{p}_{S_0}}{\partial z} = 0,
\]

\[
\frac{\partial \bar{p}_{S_0}}{\partial t} + \frac{\partial \bar{p}_{S_0} \bar{v}_{x S_0}}{\partial \chi} + \frac{\partial \bar{p}_{S_0} \bar{v}_{y S_1}}{\partial y} + \frac{\partial \bar{p}_{S_0} \bar{v}_{z S_1}}{\partial z} = 0,
\]

\[
\frac{\partial \bar{v}_{x S_0}}{\partial t} + \bar{\rho}_{S_0} \left( \frac{\partial \bar{v}_{x S_0}}{\partial \chi} + \frac{\partial \bar{v}_{y S_1}}{\partial y} + \frac{\partial \bar{v}_{z S_1}}{\partial z} \right)
\]

\[
= -\frac{1}{2} \frac{\partial \bar{p}_{S_0}}{\partial \chi} + \frac{1}{2} \frac{\partial}{\partial y} \left( \Gamma_1 \frac{\partial \bar{v}_{x S_0}}{\partial y} \right) + \frac{1}{2} \frac{\partial}{\partial z} \left( \Gamma_1 \frac{\partial \bar{v}_{x S_0}}{\partial z} \right),
\]

\(^{37}\)Owing to the difference of the supplementary condition (ii-b) of Eq. (M-1.112) [or Eq. (132)] for an interface from the condition (iii) of Eq. (M-1.107) for a simple boundary, \(\omega\) is determined for an interface. For a simple boundary, \(\omega_\infty\) can be chosen at our disposal.

\(^{38}\)Equation (M-9.33) is replaced by its equivalent form (142).
\[
\dot{\rho}_{\text{eo}} \frac{\partial \hat{v}_{y\text{eo}}}{\partial t} + \dot{\rho}_{\text{eo}} \left( \hat{v}_{z\text{eo}} \frac{\partial \hat{v}_{y\text{eo}}}{\partial \chi} + \hat{v}_{y\text{eo}} \frac{\partial \hat{v}_{y\text{eo}}}{\partial y} + \hat{v}_{z\text{eo}} \frac{\partial \hat{v}_{y\text{eo}}}{\partial z} - \frac{1}{c^2} \hat{v}_{x\text{eo}}^2 \right) \\
= -\frac{1}{2 \rho_{\text{eo}}} \sum_{i=1}^{8} \left( \frac{\partial}{\partial y} \left( \frac{\partial T_{y\text{eo}}}{\partial y} \right)^2 \right) + \frac{1}{2 \rho_{\text{eo}}} \left( \frac{\partial}{\partial z} \left( \frac{\partial T_{z\text{eo}}}{\partial z} \right)^2 \right)
\]

\[
\rho_{\text{eo}} \frac{\partial \hat{v}_{z\text{eo}}}{\partial t} + \dot{\rho}_{\text{eo}} \left( \hat{v}_{x\text{eo}} \frac{\partial \hat{v}_{z\text{eo}}}{\partial \chi} + \hat{v}_{y\text{eo}} \frac{\partial \hat{v}_{z\text{eo}}}{\partial y} + \hat{v}_{z\text{eo}}^2 \right) \\
= -\frac{1}{2 \rho_{\text{eo}}} \sum_{i=1}^{8} \left( \frac{\partial}{\partial y} \left( \frac{\partial T_{z\text{eo}}}{\partial y} \right)^2 \right) + \frac{1}{2 \rho_{\text{eo}}} \left( \frac{\partial}{\partial z} \left( \frac{\partial T_{z\text{eo}}}{\partial z} \right)^2 \right)
\]

\[
\frac{5 \dot{\rho}_{\text{eo}}}{2} \frac{\partial T_{z\text{eo}}}{\partial t} - \frac{5 \dot{\rho}_{\text{eo}}}{2} \left( \hat{v}_{x\text{eo}} \frac{\partial T_{z\text{eo}}}{\partial \chi} + \hat{v}_{y\text{eo}} \frac{\partial T_{z\text{eo}}}{\partial y} + \hat{v}_{z\text{eo}} \frac{\partial T_{z\text{eo}}}{\partial z} \right)
\]

and the subsidiary relations

\[
\dot{\rho}_{\text{eo}}(\chi, t) = \rho_{\text{eo}} \dot{T}_{z\text{eo}},
\]
\[
\dot{\rho}_2 = \rho_2 + \frac{2\Gamma_1}{3} \left( \frac{\partial \dot{v}_{x0}}{\partial \chi} + \frac{\partial \dot{v}_{y1}}{\partial y} + \frac{\partial \dot{v}_{z1}}{\partial z} \right) + \frac{\Gamma_7}{3\rho_0} \left[ (\frac{\partial \dot{T}_{\rho0}}{\partial y})^2 + (\frac{\partial \dot{T}_{\rho0}}{\partial z})^2 \right] \\
+ \frac{2}{3\rho_0} \left[ \frac{\partial}{\partial y} \left( \Gamma_3 \frac{\partial \dot{T}_{\rho0}}{\partial y} \right) + \frac{\partial}{\partial z} \left( \Gamma_3 \frac{\partial \dot{T}_{\rho0}}{\partial z} \right) \right] \\
- \frac{2\Gamma_9}{3\rho_0} \left[ (\frac{\partial \dot{v}_{x0}}{\partial y})^2 + (\frac{\partial \dot{v}_{x0}}{\partial z})^2 \right],
\]

(148b)

where \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_7, \Gamma_8,\) and \(\Gamma_9\) are short forms of the functions \(\Gamma_1(\dot{T}_{\rho0}),\)
\(\Gamma_2(\dot{T}_{\rho0}), \ldots, \Gamma_9(\dot{T}_{\rho0})\) of \(\dot{T}_{\rho0}\) defined in Section M-A.2.9.

Consider the solution of the initial and boundary-value problem of Eqs. (142)–(148b).

Let \(\dot{\rho}, \dot{v}_i,\) and \(\dot{T}\) (thus, \(\dot{p} = \rho\dot{T}\)) at time \(\dot{t}\) be given; thus, \(\dot{\rho}_{e0}, \dot{v}_{x0}, \dot{v}_{y1}, \dot{v}_{z1}, \dot{T}_{\rho0}\) (\(\dot{\rho}_{e0}\), \(\dot{\rho}_{e0}\), etc.), including \(\dot{p}_{e2}\), are given. Then \(\frac{\partial \dot{\rho}_{e0}}{\partial \dot{t}}, \frac{\partial \dot{v}_{x0}}{\partial \dot{t}}, \frac{\partial \dot{v}_{y1}}{\partial \dot{t}}, \frac{\partial \dot{v}_{z1}}{\partial \dot{t}}, \) and \(\frac{\partial \dot{T}_{\rho0}}{\partial \dot{t}}\) at \(\dot{t}\) are given by Eqs. (143)–(148b); thus, the future \(\dot{\rho}_{e0}, \dot{v}_{x0}, \dot{v}_{y1}, \dot{v}_{z1},\) and \(\dot{T}_{\rho0}\) (also \(\dot{p}_{e0}\)) are determined. However, the future \(\dot{p}_{e0}\) is required to be independent of \(y\) and \(z\), as well as \(\dot{p}_{e0}\) at \(\dot{t}\), owing to Eq. (142). Taking this into account, we will discuss how the solution is obtained by this system consistently.

First, transform Eq. (147) with the aid of Eqs. (143) and (148a) in the following form:

\[
\frac{\partial \dot{p}_{e0}}{\partial \dot{t}} = \mathcal{P},
\]

(149)

where

\[
\mathcal{P} = \frac{5}{3} \dot{p}_{e0} \left( \frac{\partial \dot{v}_{x0}}{\partial \chi} + \frac{\partial \dot{v}_{y1}}{\partial y} + \frac{\partial \dot{v}_{z1}}{\partial z} \right) - \dot{v}_{x0} \frac{\partial \dot{p}_{e0}}{\partial \chi} \\
+ 5 \left[ \frac{\partial}{\partial y} \left( \Gamma_2 \frac{\partial \dot{T}_{\rho0}}{\partial y} \right) + \frac{\partial}{\partial z} \left( \Gamma_2 \frac{\partial \dot{T}_{\rho0}}{\partial z} \right) \right] \\
+ \frac{2}{3} \Gamma_1 \left[ (\frac{\partial \dot{v}_{x0}}{\partial y})^2 + (\frac{\partial \dot{v}_{x0}}{\partial z})^2 \right].
\]

(150)

For \(\dot{p}_{e0}\) to be independent of \(y\) and \(z\) [see Eq. (142)], \(\mathcal{P}\) as well as the initial data of \(\dot{p}_{e0}\) is required to be independent of \(y\) and \(z\). Noting that \(\dot{p}_{e0}\) is independent of \(y\) and \(z\), and taking the average of Eq. (150) over the cross section \(S\) of the pipe or channel,

\[^{30}\text{[i]}\text{In a channel, where the gas extends from } z = -\infty \text{ to } z = \infty, \text{the integral } \int_S \dot{A} dy dz \text{ per unit length in } z, \text{ per a period in } z, \text{ etc. should be considered. Otherwise, it can be infinite.}\]

\[^{30}\text{[ii]}\text{Note that } \dot{v}_{y1} n_y + \dot{v}_{z1} n_z = 0 \text{ on a simple boundary where } n_i = (0, n_y, n_z) \text{ is the normal to the boundary.}\]

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respect to $y$ and $z$, i.e.,

$$
\mathcal{P} = -\frac{5}{3} \frac{\partial \dot{v}_s \rho}{\partial \chi} \frac{\partial \dot{\rho}}{\partial \rho} - \dot{v}_{x \rho} \frac{\partial \dot{\rho}}{\partial \rho} + \frac{5}{6} \frac{\partial \rho}{\partial y} \left( \Gamma_2 \frac{\partial \dot{T}_s}{\partial y} \right) + \frac{\partial \rho}{\partial z} \left( \Gamma_2 \frac{\partial \dot{T}_s}{\partial z} \right) \\
+ \frac{2}{3} \Gamma_1 \left[ \left( \frac{\partial \dot{v}_s \rho}{\partial y} \right)^2 + \left( \frac{\partial \dot{v}_s \rho}{\partial z} \right)^2 \right],
$$

(151)

where

$$
\mathcal{X} = \int_S \rho \, d\rho \, dy \, dz / \int_S \rho \, d\rho \, dy \, dz.
$$

The expression (151) is noted to be independent of $\dot{v}_{y \rho}$ and $\dot{v}_{z \rho}$. The two expressions (150) and (151) must give the same result, i.e.,

$$
\mathcal{P} = \mathcal{P}^*,
$$

or

$$
-\frac{5}{3} \rho \frac{\partial \dot{v}_s \rho}{\partial \chi} \left( \frac{\partial \dot{v}_{y \rho}}{\partial \rho} + \frac{\partial \dot{v}_{z \rho}}{\partial \rho} \right) - \dot{v}_{x \rho} \frac{\partial \dot{\rho}}{\partial \rho} + \frac{5}{6} \frac{\partial \rho}{\partial y} \left( \Gamma_2 \frac{\partial \dot{T}_s}{\partial y} \right) + \frac{\partial \rho}{\partial z} \left( \Gamma_2 \frac{\partial \dot{T}_s}{\partial z} \right) \\
+ \frac{2}{3} \Gamma_1 \left[ \left( \frac{\partial \dot{v}_s \rho}{\partial y} \right)^2 + \left( \frac{\partial \dot{v}_s \rho}{\partial z} \right)^2 \right] = \mathcal{P},
$$

(152)

when Eq. (142) holds, and vice versa. The condition (152) for all $i$ is equivalently replaced by the two conditions that the initial data of $\dot{\rho}_{\rho \rho}, \dot{T}_{\rho \rho}, \dot{v}_{x \rho}, \dot{v}_{y \rho}, \dot{v}_{z \rho}$, and $\dot{v}_{z \rho}$ satisfy Eqs. (142) and (152) and that the time derivative of Eq. (152) holds for all $i$, i.e.,

$$
\frac{\partial \mathcal{P}}{\partial t} = \frac{\partial \mathcal{P}}{\partial t}.
$$

(153)

Using Eqs. (143)–(146) and (149) for $\partial \dot{\rho}_{\rho \rho} / \partial \rho$, $\partial \dot{\dot{v}}_{x \rho} / \partial \rho$, $\partial \dot{\dot{v}}_{y \rho} / \partial \rho$, $\partial \dot{\dot{v}}_{z \rho} / \partial \rho$, and $\partial \dot{\dot{v}}_{z \rho} / \partial \rho$ ($\dot{\rho}_{\rho \rho} \partial \dot{T}_{\rho \rho} / \partial \rho = \partial \dot{\rho}_{\rho \rho} / \partial \rho - \dot{T}_{\rho \rho} \partial \dot{\rho}_{\rho \rho} / \partial \rho$) in $\partial \mathcal{P} / \partial t$ derived from Eq. (150), we find that $\partial \mathcal{P} / \partial t$ is expressed with $\dot{\rho}_{\rho \rho}, \dot{v}_{x \rho}, \dot{v}_{y \rho}, \dot{v}_{z \rho}, \dot{\rho}_{\rho \rho}$, and $\dot{\rho}_{\rho \rho}$ in the form

$$
\frac{\partial \mathcal{P}}{\partial t} = \frac{5}{6} \dot{\rho}_{\rho \rho} \left[ \frac{\partial}{\partial y} \left( \frac{1}{\dot{\rho}_{\rho \rho}} \frac{\partial \dot{\rho}_{\rho \rho}}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{1}{\dot{\rho}_{\rho \rho}} \frac{\partial \dot{\rho}_{\rho \rho}}{\partial z} \right) \right] + \text{F}_{\rho 1},
$$

(154)

where $\text{F}_{\rho 1}$ is a given function of $\dot{\rho}_{\rho \rho}, \dot{v}_{x \rho}, \dot{v}_{y \rho}, \dot{v}_{z \rho}, \dot{\rho}_{\rho \rho}$, and their space derivatives. The expression (151) of $\mathcal{P}$ being independent of $\dot{v}_{y \rho}$ and $\dot{v}_{z \rho}$, its time derivative $\partial \mathcal{P} / \partial t$ does not contain $\dot{\dot{v}}_{y \rho} / \partial \rho$ and $\dot{\dot{v}}_{z \rho} / \partial \rho$. Therefore, with the aid of Eqs. (143), (144), and (147), $\partial \mathcal{P} / \partial t$ is expressed with $\dot{\rho}_{\rho \rho}, \dot{v}_{x \rho}, \dot{v}_{y \rho}, \dot{v}_{z \rho}, \dot{\rho}_{\rho \rho}$, and their space derivatives, i.e.,

$$
\frac{\partial \mathcal{P}}{\partial t} = \text{F}_{\rho 2}(\dot{\rho}_{\rho \rho}, \dot{v}_{x \rho}, \dot{v}_{y \rho}, \dot{v}_{z \rho}, \dot{\rho}_{\rho \rho}, \dot{\rho}_{\rho \rho}, \text{and their space derivatives}),
$$

(155)
where $F_{n2}$ is a given functional of its arguments. From Eqs. (153), (154), and (155), we have

$$\frac{\partial}{\partial y} \left( \frac{1}{\rho_{\delta 0}} \frac{\partial \phi_{\delta 2}}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{1}{\rho_{\delta 0}} \frac{\partial \phi_{\delta 2}}{\partial z} \right) = F_n, \quad (156)$$

where $F_n = 6(F_{n2} - F_{n1})/5\rho_{\delta 0}$, and therefore, $F_n$ is a given functional of $\rho_{\delta 0}$, $\dot{v}_{\delta 0}$, $\dot{v}_{y1}$, $\dot{v}_{z1}$, $\ddot{\rho}_{\delta 0}$, and their space derivatives. This is the equation for $\phi_{\delta 2}$ over a cross section of the pipe or channel.

The boundary condition for $\phi_{\delta 2}$ on a simple boundary is obtained by multiplying Eqs. (144)–(146) by the normal $n_i = (0, n_y, n_z)$ to the boundary; In this process, the contribution of their time-derivative terms vanishes because $\dot{v}_{y1} n_y + \dot{v}_{z1} n_z = 0$; Then, the $n_y \partial \phi_{\delta 2}/\partial y + n_z \partial \phi_{\delta 2}/\partial z$ is imposed as the boundary condition. Thus, $\phi_{\delta 2}$ is determined by Eq. (156) except for an additive function of $t$ and $\chi$. With this $\phi_{\delta 2}$ substituted into Eqs. (145) and (146), $\partial \rho_{\delta 0}/\partial \bar{t}$, $\partial v_{\delta 0}/\partial \bar{t}$, $\partial v_{y1}/\partial \bar{t}$, $\partial v_{z1}/\partial \bar{t}$, and $\partial \phi_{\delta 0}/\partial \bar{t}$ are determined by Eqs. (143)–(148b) independently of the additive function in $\phi_{\delta 2}$ in such a way that $\partial(\partial \rho_{\delta 0}/\partial y)/\partial \bar{t} = \partial(\partial \phi_{\delta 2}/\partial z)/\partial \bar{t} = 0$ and $\partial(\partial P/\partial y)/\partial \bar{t} = \partial(\partial P/\partial z)/\partial \bar{t} = 0$. That is, the solution $(\rho_{\delta 0}, v_{\delta 0}, v_{y1}, v_{z1}, \dot{\bar{T}}_{\delta 0})$ of Eqs. (142)–(148b) is determined by Eqs. (143)–(148b) with the aid of the supplementary condition (156), instead of Eq. (142), when the initial condition for $\rho_{\delta 0}$, $v_{\delta 0}$, $v_{y1}$, $v_{z1}$, and $\bar{T}_{\delta 0}$ is given in such a way that $\dot{\rho}_{\delta 0} = \dot{\rho}_{\delta 0} \dot{\bar{T}}_{\delta 0}$ and $P$ are independent of $y$ and $z$.

Equations (142)–(148b) are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. The equation for $\partial \phi_{\delta 2}/\partial \bar{t}$, corresponding to Eq. (149), is derived at the order after next. However, owing to the consistency of $\rho_{\delta 0}$, $\dot{\rho}_{\delta 2}$ is already determined by Eq. (156) except for an additive function of $\chi$ and $t$. The situation is similar to that at the leading order. That is, $\dot{\rho}_{\delta 0}$ and $\dot{\rho}_{\delta 2}$ are, respectively, determined by Eqs. (142) and (156), each with an additive function of $\chi$ and $t$ and also by Eqs. (149) and the counterpart of Eq. (149) at the order after next. Thus, the higher-order analysis can be carried out in a similar way. The results are that an additional initial condition and an equation for $\dot{\rho}_{\delta 2}$, the counterpart of Eq. (156), are introduced and that the condition (156) is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\rho_{\delta 0}, \dot{\bar{T}}_{\delta 0}, \dot{v}_{\delta 0}, \dot{v}_{y1}$, and $\dot{v}_{z1}$ (thus also $\dot{\rho}_{\delta 0}$).

To summarize, the solution $(\rho_{\delta 0}, v_{\delta 0}, v_{y1}, v_{z1}, \dot{\bar{T}}_{\delta 0})$ of Eqs. (142)–(148b) is determined by Eqs. (143)–(148b) with the aid of the supplementary condition (156), instead of Eq. (142), when the initial data of $\rho_{\delta 0}$, $\dot{v}_{\delta 0}$, $\dot{v}_{y1}$, $\dot{v}_{z1}$, and $\bar{T}_{\delta 0}$ are given in such a way that $\dot{\rho}_{\delta 0} = \dot{\rho}_{\delta 0} \dot{\bar{T}}_{\delta 0}$ and $P$ are independent of $y$ and $z$. The results are not affected by the higher-order analysis.\footnote{If $P$ is independent of $y$ and $z$, $P = \Psi$ by definition.}
5.1.2 Equations (M-9.49a)–(M-9.50e):

Take Eqs. (M-9.49a)–(M-9.50e) with the additional time-derivative terms given in the first mathematical expressions after Eq. (M-9.59), i.e.,
\[
\begin{align*}
\frac{\partial P_{01}}{\partial \chi} &= \frac{\partial P_{01}}{\partial y} = \frac{\partial P_{01}}{\partial z} = 0, \quad P_{01} = \omega + \tau, \\
\frac{\partial P_{02}}{\partial y} &= \frac{\partial P_{02}}{\partial z} = 0, \\
\frac{\partial u_x}{\partial \chi} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} &= 0, \\
\frac{\partial u_y}{\partial t} + \frac{u_x}{\partial \chi} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} &= -\frac{1}{2} \frac{\partial P_{02}}{\partial \chi} + \gamma_1 \left( \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right), \\
\frac{\partial u_y}{\partial t} + \frac{u_x}{\partial \chi} + \frac{u_y}{\partial y} + \frac{u_y}{\partial y} + \frac{u_z}{\partial z} &= \frac{u^2}{C^2} = -\frac{1}{2} \frac{\partial P_{20}}{\partial y} + \gamma_1 \left( \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right), \\
\frac{\partial u_z}{\partial t} + \frac{u_x}{\partial \chi} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} &= -\frac{1}{2} \frac{\partial P_{20}}{\partial \chi} + \gamma_1 \left( \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right), \\
\frac{\partial \tau}{\partial t} &= \frac{2}{5} \frac{\partial P_{01}}{\partial \chi} + \frac{u_x}{\partial \chi} + \frac{u_y}{\partial y} + \frac{u_z}{\partial z} = \gamma_2 \left( \frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2} \right).
\end{align*}
\] (157a–158e)

The qualitative difference of this set of equations from the set (142)–(148b) is the absence of the time-derivative term in Eq. (158a) that corresponds to Eq. (143).

Consider the solution of the initial and boundary-value problem of Eqs. (157a)–(158e). Let \( u_x, u_y, u_z, \) and \( \tau \) at \( t \) be given in such a way that Eq. (158a) is satisfied. Integrating Eq. (158a) over the cross section of the channel or pipe \( \int_S \text{Eq. (158a)} \, dy \, dz \), we find that \( \int_S u_x \, dy \, dz \) depends only on \( t \), i.e.,
\[
\int_S (u_x/\partial \chi) \, dy \, dz = 0, 
\] (159)
where \( S \) indicates the cross section. Applying Eqs. (157b), (158a), and (159) to the equation \( \partial \int_{\chi} \text{Eq. (158b)} \, dy \, dz / \partial \chi \), we have \( \partial^2 P_{02} / \partial \chi^2 \) as
\[
\frac{\partial^2 P_{02}}{\partial \chi^2} = \frac{\partial}{\partial \chi} \left[ -2 \frac{\partial u_x^2}{\partial \chi} + \gamma_1 \left( \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) \right].
\] (160)
where
\[
\bar{A} = \int_S A \, dy \, dz / \int_S dy \, dz.
\]
Thus, \( \partial P_{02} / \partial \chi \) and \( P_{02} \) are determined if they are specified at a point in the gas. Here, we consider this case.\(^{42}\) Using Eq. (158a) in the sum of \( \partial \text{Eq. (158b)} / \partial \chi \),

\(^{41}\)See Footnote 39, with \( \hat{v}_{51} \) and \( \hat{v}_{52} \) being replaced by \( u_y \) and \( u_z \).\(^{42}\) (i) Imagine the case of the Poiseuille flow.

(ii) Here, \( P \) (thus, \( P_{01} \)) is specified at some point. Then, \( P_{01} \) is a given function of \( t \).
\[ \partial \text{[Eq. (158c)]}/\partial y, \text{ and } \partial \text{[Eq. (158d)]}/\partial z, \text{ we obtain the equation for } P_{20} \text{ in the form} \]
\[
\frac{\partial^2 P_{20}}{\partial y^2} + \frac{\partial^2 P_{20}}{\partial z^2} = F_n(u_x, u_y, u_z, \text{ and their space derivatives}), \tag{161}
\]
where \( F_n \) is a given functional of the variables in the parentheses, and the time derivatives are absent owing to Eq. (158a). Thus, the right-hand side of Eq. (161) is known. This equation is the Poisson equation for \( P_{20} \) over the cross section \( S \). Its boundary condition is obtained in a way similar to how the condition for \( \hat{\rho}_{S2} \) in Eq. (156) is derived. Thus, \( P_{20} \) over each cross section is determined except for an additive function of \( \hat{\tau} \) and \( \bar{\chi} \). This ambiguity does not influence \( \partial P_{20}/\partial y \) and \( \partial P_{20}/\partial z \).

With \( P_{02} \) and \( P_{20} \) prepared above into Eqs. (158b)-(158e), the time derivatives \( \partial u_x/\partial \hat{\tau}, \partial u_y/\partial \hat{\tau}, \partial u_z/\partial \hat{\tau}, \) and \( \partial \tau/\partial \hat{\tau} \) are determined in such a way that \( \partial (\partial u_x/\partial \hat{\chi} + \partial u_y/\partial y + \partial u_z/\partial z)/\partial \hat{\tau} = 0 \) owing to the above choice of \( P_{20} \).

Thus, the solution \((u_x, u_y, u_z, \tau)\) of Eqs. (157b), (158a)-(158e) is determined by Eqs. (158b)-(158e) with the aid of the supplementary conditions (160) and (161) for \( P_{02} \) and \( P_{20} \), instead of Eqs. (157b) and (158a). This process is natural for numerical computation. The undetermined additive function of \( \bar{\chi} \) and \( \hat{\tau} \) in \( P_{20} \), which does not affect the solution \((u_x, u_y, u_z, \tau)\), is determined by the higher-order equation derived from that for \( \partial \hat{\nu}_{S2}/\partial \hat{t} \) (see Section 5.1.1), in a way similar to that in which \( P_{02} \) is determined by Eq. (158b). In the higher-order equation, \( P_{20} \) plays the same role as \( P_{02} \) in Eq. (158b); Equation (161) corresponds to Eq. (157b), and \( P_{20} \) and \( P_{02} \) are determined by these equations, each with an additive function of \( \bar{\chi} \) and \( \hat{\tau} \).

6 Appendix M-A

6.1 Note on the loss term of the collision integral [From Eq. (M-A.18) to Eq. (M-A.21)]

Consider the following collision term of the Boltzmann equation (M-A.18):\(^{44}\)

\[
\frac{d^2}{2m} \int_{\text{all } e, \text{ all } \xi} |(\xi_\ast - \xi) \cdot e| [f(\xi')f(\xi_\ast') - f(\xi)f(\xi_\ast)]d\Omega(e)d\xi_\ast, \tag{162}
\]
where

\[
\xi' = \xi + [\alpha \cdot (\xi_\ast - \xi)]\alpha, \quad \xi_\ast' = \xi_\ast - [\alpha \cdot (\xi_\ast - \xi)]\alpha. \tag{163}
\]

The change (M-A.20) of the variable of integration from \( e \) to \( \alpha \), i.e.,

\[
|(\xi_\ast - \xi) \cdot e|d\Omega(e) = \frac{2}{d^2} B \text{d} \Omega(\alpha), \tag{164}
\]

\(^{43}\)Note that \( P_{01} \) is known (Footnote 42).

\(^{44}\)The factor \( d^2/2m \) can be rewritten as \( nd_m^2/2\rho \), where \( n \) is the number of molecules in unit volume. The numerator \( nd_m^2 \) is of the order of the inverse of the mean free path (Section M-1.5). Note Footnote M-4 in Section M-A.1.
is introduced instead of expressing $\alpha$ in Eq. (163) in terms of $e$. The part of the integral of Eq. (162)
\[ \frac{d_m^2}{2m} \int_{\text{all } e, \text{ all } \xi_*} |(\xi_* - \xi) \cdot e| f(\xi) f(\xi_*) \text{d}\Omega(e) \text{d}\xi_* , \]
which comes from $I_{-\infty}$ in Eq. (M-A.8) and corresponds to the loss term (see Section M-1.2) of the collision integral of the Boltzmann equation (M-1.5) or (M-A.21), does not contain $\alpha$, and the change (164) of the variable of integration is not required.\textsuperscript{46} Thus, the result is determined uniquely irrespective of the relation between $\alpha$ and $e$, that is, the loss term of the collision integral is independent of the intermolecular potential when $d_m$ is of a finite value. That is, the loss term of the collision integral is determined only by $d_m^2/2m$ and $f(\xi)$, and is the same as that for the hard-sphere molecule with the same $d_m$.

(Section 6.1: Version 6-00)

6.2 Note on the loss term of the kernel representation of the linearized collision integral [Section M-A.2.10]

In Section M-A.2.10, we discussed the kernel representation of the linearized collision integral $L(\phi)$ introduced in Section M-1.10, and gave its explicit form for a hard-sphere molecule. From the discussion in Section 6.1, the kernel representation of the loss term of the linearized collision integral for a hard-sphere molecule applies to any intermolecular potential with a finite $d_m$.

In Section M-A.2.10, the linearized collision integral $L(\phi)$ is expressed by Eqs. (M-137a)-(M-A.139c) as
\[
L(\phi) = \int E_*(\phi' + \phi'_* - \phi - \phi_*) \hat{B} \text{d}\Omega(\alpha) \text{d}\zeta_* ,
\]
\[ = L^G(\phi) - L^{L2}(\phi) - \nu_L(\zeta)\phi ,
\]
where
\[
L^G(\phi) = \int E_*(\phi' + \phi'_*) \hat{B} \text{d}\Omega(\alpha) \text{d}\zeta_* ,
\]
\[ = \int K_2(\zeta, \zeta_* )\phi(\zeta_*) \text{d}\zeta_* ,
\]
\[
\nu_L(\zeta) = \int E_* \hat{B} \text{d}\Omega(\alpha) \text{d}\zeta_* .
\]

The loss term is the sum of Eqs. (166b) and (166c) multiplied by $\phi$, i.e., $L^{L2}(\phi) + \nu_L(\zeta)\phi$.

\textsuperscript{46}Transformation [M-A.20] or (164) is carried out to make the variable of integration to be the same. Thus, it is simply one of the changes of variable $e$ of integration to some variable.
\( \nu_L(\zeta) \phi \). The kernel \( K_2(\zeta, \zeta_*) \) and the function \( \nu_L(\zeta) \) for a hard-sphere molecule are given by Eqs. (M-A.149b) and (M-A.149c) as

\[
K_2(\zeta, \zeta_*) = \frac{|\zeta_* - \zeta|}{2\sqrt{2\pi}} \exp\left(-\zeta_*^2\right), \tag{167a}
\]
\[
\nu_L(\zeta) = \frac{1}{2\sqrt{2}} \left[ \exp(-\zeta^2) + \left(2\zeta + \frac{1}{\zeta}\right) \int_{0}^{\zeta} \exp(-\zeta_*^2) d\zeta_* \right], \tag{167b}
\]

where \( \zeta = |\zeta| \).

These formulas apply to any potential with a finite \( d_m \) as well as to a hard-sphere molecule.

(Section 6.2: Version 6-00)

### 6.3 Parity of the collision integral: Supplement to Section M-A.2.7

In Section M-A.2.7, we discussed the parity of the linearized collision integral. It may be better to explain a similar property of the collision integral defined by Eq. (M-1.9), i.e.,

\[
\hat{J}(\hat{f}, \hat{g}) = \frac{1}{2} \int (\hat{f}' \hat{g}' + \hat{f}_* \hat{g}' - \hat{f} \hat{g}_* - \hat{f}_* \hat{g}) \hat{B} d\Omega(\alpha) d\zeta_*, \tag{168}
\]

\[
\hat{B} = \hat{B}(|\alpha \cdot V|/|V|, |V|), \quad \hat{f} = \hat{f}(\zeta), \quad \hat{f}_* = \hat{f}(\zeta_*), \quad \hat{f}' = \hat{f}(\zeta'), \quad \hat{f}'_* = \hat{f}(\zeta'_*),
\]

and a similar notation for \( \hat{g}, \hat{g}_*, \hat{g}', \) and \( \hat{g}'_* \),

\[
\zeta = \zeta_i + \alpha_j V_j \alpha_i, \quad \zeta_* = \zeta_i - \alpha_j V_j \alpha_i, \quad \zeta_i = V_i + \zeta_i.
\]

Here, we discuss the relation of the parity of \( \hat{J}(\hat{f}, \hat{g}) \) with respect to a component (\( \zeta_1, \zeta_2, \) or \( \zeta_3 \)) of the variable \( \zeta \) to that of \( \hat{f} \) and \( \hat{g} \). Put the integral (168) in the sum

\[
\hat{J}(\hat{f}, \hat{g}) = \frac{1}{2} (IV + III - II - I), \tag{169}
\]

\[\text{Only the term } \nu_L(\zeta) \phi \text{ is often called the loss term, and the rest, i.e., } \mathcal{L}_2(\phi) - \mathcal{L}_2(\phi), \text{ is called the gain term by misunderstanding. This is probably because the loss term of the original collision integral (162) is often written in the form } \nu_c f, \text{ where } \nu_c \text{ is the collision frequency defined by Eq. (M-1.18) as}
\]

\[
\nu_c = m^{-1} \int_{\alpha, \zeta} f(\xi_\ast) B d\Omega(\alpha) d\xi_\ast = (d_m^2/2m) \int_{\alpha, \zeta} (|\zeta_\ast - \xi| \cdot e|f(\xi_\ast) d\Omega(e) d\xi_\ast.
\]

Not to mention, \( \mathcal{L}_2(\phi) \) is derived from \( \nu_c f \).
where

\[ I = \int \hat{f}_s \hat{g} \hat{B} d\Omega(\alpha) dV, \quad (170a) \]
\[ II = \int \hat{f}_s \hat{g} \hat{B} d\Omega(\alpha) dV, \quad (170b) \]
\[ III = \int \hat{f}_s' \hat{g}' \hat{B} d\Omega(\alpha) dV, \quad (170c) \]
\[ IV = \int \hat{f}_s'' \hat{g}'' \hat{B} d\Omega(\alpha) dV, \quad (170d) \]

and discuss each term separately. In Eqs. (170a)-(170d), the variable of integration is changed from \( \zeta \) to \( V (= \zeta - \zeta') \). The following change of the variables

\[ \tilde{V}_1 = -V_1, \quad \tilde{V}_s = V_s, \quad \tilde{\alpha}_1 = -\alpha_1, \quad \tilde{\alpha}_s = \alpha_s \quad (s = 2, 3) \]

is performed in the integrals \( I, II, III, \) and \( IV \). Noting that

\[ \zeta_s = V_i + \zeta, \quad |\tilde{V}_1| = |V_i|, \quad \tilde{\alpha}_i \tilde{V}_i = \alpha_i V_i, \]

we can transform the integrals \( I, II, III, \) and \( IV \) in the following way, where the subscript \( s \) indicates \( s = 2 \) and 3:

\[ I(\zeta_1, \zeta_s) = \int \hat{f}(V_1 + \zeta_1, V_s + \zeta_s) \hat{g}(\zeta_1, \zeta_s) \hat{B} (|\alpha_i V_i|/|V_i|) d\Omega(\alpha) dV \]
\[ = \int \hat{f}(-\tilde{V}_1 + \zeta_1, \tilde{V}_s + \zeta_s) \hat{g}(\zeta_1, \zeta_s) \hat{B} (|\tilde{\alpha}_i \tilde{V}_i|/|\tilde{V}_i|) d\Omega(\tilde{\alpha}) d\tilde{V}; \quad (173a) \]

Interchanging the arguments of \( \hat{f} \) and \( \hat{g} \) in \( I \), we have

\[ II(\zeta_1, \zeta_s) = \int \hat{f}(\zeta_1, \zeta_s) \hat{g}(-\tilde{V}_1 + \zeta_1, \tilde{V}_s + \zeta_s) \hat{B} (|\tilde{\alpha}_i \tilde{V}_i|/|\tilde{V}_i|) d\Omega(\tilde{\alpha}) d\tilde{V}; \quad (173b) \]

\[ III(\zeta_1, \zeta_s) = \int \hat{f}(V_1 + \zeta_1 - \alpha_2 V_2 \alpha_1) \hat{g}(\zeta_1 + \alpha_2 V_2 \alpha_1) \hat{B} (|\alpha_i V_i|/|V_i|) d\Omega(\alpha) dV \]
\[ = \int \hat{f}(-\tilde{V}_1 + \zeta_1 + \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_1, \tilde{V}_s + \zeta_s - \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_s) \]
\[ \times \hat{g}(\zeta_1 - \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_1, \zeta_s + \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_s) \hat{B} (|\tilde{\alpha}_i \tilde{V}_i|/|\tilde{V}_i|) d\Omega(\tilde{\alpha}) d\tilde{V}; \quad (173c) \]

Interchanging the arguments of \( \hat{f} \) and \( \hat{g} \) in \( III \), we have

\[ IV(\zeta_1, \zeta_s) = \int \hat{f}(\zeta_1 - \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_1, \zeta_s + \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_s) \]
\[ \times \hat{g}(-\tilde{V}_1 + \zeta_1 + \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_1, \tilde{V}_s + \zeta_s - \tilde{\alpha}_2 \tilde{V}_2 \tilde{\alpha}_s) \]
\[ \times \hat{B} (|\tilde{\alpha}_i \tilde{V}_i|/|\tilde{V}_i|) d\Omega(\tilde{\alpha}) d\tilde{V}. \quad (173d) \]

\[ ^{47} \text{The separation is made only for convenience of explanation.} \]
Now we examine the parity of the integrals I, II, III, and IV with respect to \( \zeta_1 \) on the basis of Eqs. (173a)–(173d). Here, we introduce the notation: (i) the parity of \( \hat{f} \) (or \( \hat{g} \)) is indicated by the subscript attached to it, i.e., the subscript \( E \) is attached when it is even and the subscript \( O \) when it is odd; (ii) the first subscript of I, II, III, and IV indicates the parity of \( \hat{f} \) in them and the second indicates the parity of \( \hat{g} \). First, when \( \hat{f} \) and \( \hat{g} \) are even functions of \( \zeta_1 \).

\[
I_{EE}(\zeta_1, \zeta_s) = \int \hat{f}_E(-\bar{V}_1 + \zeta_1, \bar{V}_s + \zeta_s) \hat{g}_E(\zeta_1, \zeta_s) \\
\times \hat{B}(|\bar{\alpha}_i \bar{V}_i|, |\bar{V}_1|, |\bar{V}_s|) d\Omega(\bar{\alpha}) d\bar{V} \\
= \int \hat{f}_E(\bar{V}_1 - \zeta_1, \bar{V}_s + \zeta_s) \hat{g}_E(-\zeta_1, \zeta_s) \\
\times \hat{B}(|\bar{\alpha}_i \bar{V}_i|, |\bar{V}_1|, |\bar{V}_s|) d\Omega(\bar{\alpha}) d\bar{V} \\
= I_{EE}(\zeta_1, \zeta_s),
\]

(174a)

where the last relation holds owing to the first relation of Eq. (173a); Interchanging the arguments of \( \hat{f}_E \) and \( \hat{g}_E \) in \( I_{EE} \), we have

\[
II_{EE}(\zeta_1, \zeta_s) = II_{EE}(\zeta_1, \zeta_s);
\]

(174b)

\[
III_{EE}(\zeta_1, \zeta_s) = \int \hat{f}_E(-\bar{V}_1 + \zeta_1 + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \bar{V}_s + \zeta_s - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \\
\times \hat{g}_E(\zeta_1 - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \zeta_s + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \hat{B}(|\bar{\alpha}_i \bar{V}_i|, |\bar{V}_1|, |\bar{V}_s|) d\Omega(\bar{\alpha}) d\bar{V} \\
= \int \hat{f}_E(\bar{V}_1 - \zeta_1 - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \bar{V}_s + \zeta_s - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \\
\times \hat{g}_E(-\zeta_1 + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \zeta_s + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \hat{B}(|\bar{\alpha}_i \bar{V}_i|, |\bar{V}_1|, |\bar{V}_s|) d\Omega(\bar{\alpha}) d\bar{V} \\
= III_{EE}(\zeta_1, \zeta_s);
\]

(174c)

Interchanging the arguments of \( \hat{f}_E \) and \( \hat{g}_E \) in \( III_{EE} \), we have

\[
IV_{EE}(\zeta_1, \zeta_s) = IV_{EE}(\zeta_1, \zeta_s).
\]

(174d)

When both \( \hat{f} \) and \( \hat{g} \) are odd with respect to \( \zeta_1 \),

\[
I_{OO}(\zeta_1, \zeta_s) = \int \hat{f}_O(-\bar{V}_1 + \zeta_1, \bar{V}_s + \zeta_s) \hat{g}_O(\zeta_1, \zeta_s) \hat{B}(|\bar{\alpha}_i \bar{V}_i|, |\bar{V}_1|, |\bar{V}_s|) d\Omega(\bar{\alpha}) d\bar{V} \\
= \int \hat{f}_O(\bar{V}_1 - \zeta_1, \bar{V}_s + \zeta_s) \hat{g}_O(-\zeta_1, \zeta_s) \hat{B}(|\bar{\alpha}_i \bar{V}_i|, |\bar{V}_1|, |\bar{V}_s|) d\Omega(\bar{\alpha}) d\bar{V} \\
= I_{OO}(\zeta_1, \zeta_s);
\]

(175a)

Interchanging the arguments of \( \hat{f}_O \) and \( \hat{g}_O \) in \( II_{OO} \), we have

\[
II_{OO}(\zeta_1, \zeta_s) = II_{OO}(\zeta_1, \zeta_s);
\]

(175b)
\[ III_{OO}(\zeta_1, \zeta_s) = \int \hat{f}_O(-\bar{V}_1 + \zeta_1 + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \bar{V}_s + \zeta_s - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \]
\[ \times \hat{g}_O(\zeta_1 - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \zeta_s + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \]
\[ \times \hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = \int \hat{f}_O(\bar{V}_1 - \zeta_1 - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \bar{V}_s + \zeta_s - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \]
\[ \times \hat{g}_O(-\zeta_1 + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \zeta_s + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = III_{OO}(-\zeta_1, \zeta_s); \quad (175c) \]

Interchanging the arguments of \( \hat{f} \) and \( \hat{g} \) in \( III_{OO} \), we have
\[ IV_{OO}(\zeta_1, \zeta_s) = IV_{OO}(-\zeta_1, \zeta_s). \quad (175d) \]

When \( \hat{f} \) is even and \( \hat{g} \) is odd with respect to \( \zeta_1 \),
\[ I_{EO}(\zeta_1, \zeta_s) = \int \hat{f}_E(-\bar{V}_1 + \zeta_1, \bar{V}_s + \zeta_s)\hat{g}_O(\zeta_1, \zeta_s)\hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = -\int \hat{f}_E(\bar{V}_1 - \zeta_1, \bar{V}_s + \zeta_s)\hat{g}_O(-\zeta_1, \zeta_s)\hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = -I_{EO}(-\zeta_1, \zeta_s); \quad (176a) \]

\[ II_{EO}(\zeta_1, \zeta_s) = \int \hat{f}_E(\zeta_1, \zeta_s)\hat{g}_O(-\bar{V}_1 + \zeta_1, \bar{V}_s + \zeta_s)\hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = -\int \hat{f}_E(-\zeta_1, \zeta_s)\hat{g}_O(\bar{V}_1 - \zeta_1, \bar{V}_s + \zeta_s)\hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = -II_{EO}(-\zeta_1, \zeta_s); \quad (176b) \]

\[ III_{EO}(\zeta_1, \zeta_s) = \int \hat{f}_E(-\bar{V}_1 + \zeta_1 + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \bar{V}_s + \zeta_s - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \]
\[ \times \hat{g}_O(\zeta_1 - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \zeta_s + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = -\int \hat{f}_E(\bar{V}_1 - \zeta_1 - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \bar{V}_s + \zeta_s - \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \]
\[ \times \hat{g}_O(-\zeta_1 + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_1, \zeta_s + \bar{\alpha}_j \bar{V}_j \bar{\alpha}_s) \hat{B}(|\bar{\alpha}_1 \bar{V}_1|/|\bar{V}_i|, |\bar{V}_i|)d\Omega(\bar{\alpha})d\bar{V} \]
\[ = -III_{EO}(-\zeta_1, \zeta_s); \quad (176c) \]
\[ N_{OE}(\zeta_1, \zeta_s) = \int \hat{f}_E(\zeta_1 - \tilde{\alpha}_j \tilde{V}_j \tilde{a}_1, \zeta_s + \tilde{\alpha}_j \tilde{V}_j \tilde{a}_s) \]
\[ \times \hat{g}_O(-\tilde{V}_1 + \zeta_1 + \tilde{\alpha}_j \tilde{V}_j \tilde{a}_1, \tilde{V}_s + \zeta_s - \tilde{\alpha}_j \tilde{V}_j \tilde{a}_s) \]
\[ \times \hat{B}(|\tilde{a}_j \tilde{V}_1|/|\tilde{V}_1|, |\tilde{V}_s|)|\Omega(\tilde{\alpha})|d\tilde{V} \]
\[ = - \int \hat{f}_E(-\zeta_1 + \tilde{\alpha}_j \tilde{V}_j \tilde{a}_1, \zeta_s + \tilde{\alpha}_j \tilde{V}_j \tilde{a}_s) \]
\[ \times \hat{g}_O(\tilde{V}_1 - \zeta_1 - \tilde{\alpha}_j \tilde{V}_j \tilde{a}_1, \tilde{V}_s + \zeta_s - \tilde{\alpha}_j \tilde{V}_j \tilde{a}_s) \]
\[ \times \hat{B}(|\tilde{a}_j \tilde{V}_1|/|\tilde{V}_1|, |\tilde{V}_s|)|\Omega(\tilde{\alpha})|d\tilde{V} \]
\[ = -N_{OE}(-\zeta_1, \zeta_s). \]  

For \( IOE, II_{OE}, III_{OE}, \) and \( IV_{OE}, \) interchanging the role of \( \hat{f} \) and \( \hat{g}, \) respectively, in \( II_{EO}, I_{EO}, IV_{EO}, \) and \( III_{EO}, \) we have
\[ IOE(\zeta_1, \zeta_s) = -IOE(\zeta_1, \zeta_s), \quad (177a) \]
\[ II_{OE}(\zeta_1, \zeta_s) = -II_{OE}(\zeta_1, \zeta_s), \quad (177b) \]
\[ III_{OE}(\zeta_1, \zeta_s) = -III_{OE}(\zeta_1, \zeta_s), \quad (177c) \]
\[ IV_{OE}(\zeta_1, \zeta_s) = -IV_{OE}(\zeta_1, \zeta_s). \quad (177d) \]

The parity is common to \( I, II, III, \) and \( IV. \) Therefore, the parity of \( \hat{J}(\hat{f}, \hat{g}) \)
is the same as \( I, \) i.e.,
\[ \hat{J}(\hat{f}_E, \hat{g}_E)(\zeta_1, \zeta_s) = \hat{J}(\hat{f}_E, \hat{g}_E)(-\zeta_1, \zeta_s), \quad (178a) \]
\[ \hat{J}(\hat{f}_O, \hat{g}_O)(\zeta_1, \zeta_s) = \hat{J}(\hat{f}_O, \hat{g}_O)(-\zeta_1, \zeta_s), \quad (178b) \]
\[ \hat{J}(\hat{f}_E, \hat{g}_O)(\zeta_1, \zeta_s) = -\hat{J}(\hat{f}_E, \hat{g}_O)(-\zeta_1, \zeta_s), \quad (178c) \]
\[ \hat{J}(\hat{f}_O, \hat{g}_E)(\zeta_1, \zeta_s) = -\hat{J}(\hat{f}_O, \hat{g}_E)(-\zeta_1, \zeta_s). \quad (178d) \]

Obviously, the same parity holds for the other components, i.e., \( \zeta_2, \zeta_3, \) of \( \xi. \)

(Section 6.3: Version 4-00)

6.4 Supplement to Section M-A.10

6.4.1 On the equality condition of Eq. (M-A.266)

Here we will discuss the equality condition in the Darrozes–Guiraud inequality in Section M-A.10 in more detail. The equality in the Jensen inequality (M-A.265) is proved to hold when and only when \( \phi \) is independent of \( \xi \) (see, e.g., Reference M-129). It should be noted that the uniqueness condition of the equality applies only to the region of \( \xi \) where \( \psi > 0 \) and that no condition is required of \( \phi \) where \( \psi = 0. \) Choose a \( \xi \) in \( (\xi - v_m)n_i > 0, \) and consider the condition for equality in Eq. (M-A.266). According to the above note, the equality holds only when \( f(\xi_\ast)/f_0(\xi_\ast) \) is a constant (say, \( c_1 \)) in the region \( D_1 \) of \( \xi_\ast, \) joint or disjoint, where \( K_B(\xi, \xi_\ast) > 0. \) If we choose another \( \xi, K_B(\xi, \xi_\ast) > 0 \) in a different range \( D_2 \) of \( \xi_\ast, \) and \( f(\xi_\ast)/f_0(\xi_\ast) = c_2 \) (\( c_2 : \text{const} \)) is required in \( D_2. \) The constants
\[ f(\xi) = c_0 f_0(\xi) \quad \text{for all } \xi. \] (179)

Incidentally, the kernel \( K_B \) that is positive almost everywhere (Footnote M-5 in Section M-1.2) is classified as positive, and Eq. (179) holds almost everywhere of \( \xi \). When the overlap-covering condition is not satisfied, the above Maxwellian is not necessarily required for the equality.\(^{50}\)

The equality condition of Eq. (M-A.267) is seen to be the same as that of Eq. (M-A.266) in the following way. Obviously, \( B = A \leftrightarrow \int V a(\xi)[B(\xi) - A(\xi)]d\xi = 0 \) if \( A(\xi) \leq B(\xi) \) and \( a(\xi) > 0 \). Taking

\[ A(\xi) = \int \frac{K_B(\xi, \xi_*) f_0(\xi_*)}{f_0(\xi)} f(\xi), \quad B(\xi) = \int_{(\xi_*, v_{wi}) n_i < 0} \frac{K_B(\xi, \xi_*) f_0(\xi_*)}{f_0(\xi)} f(\xi), \]

and \((\xi_i - v_{wi}) n_i > 0\) as the domain \( V \) of integration, and comparing Eq. (M-A.266) and its next equation without number, we find the equivalence of the equality conditions of Eqs. (M-A.266) and (M-A.267). The above discussion being common for a strictly convex function \( F \), the equality condition applies to the Darrozes–Guiraud inequality (M-A.262) and Eq. (M-A.268).

(Section 6.4.1: Version 5-00)

### 6.4.2 Extension of the Darrozes–Guiraud inequality to an interface

Darrozes–Guiraud inequality (M-A.262) or (M-A.267) is proved for a function \( f \) satisfying the boundary condition (M-1.26) on a simple boundary (M-Darrozes & Guiraud [1966]). Here, we discuss its extension to \( f \) that satisfies the boundary condition (M-1.30) on an interface of a gas and its condensed phase.

---

\(^{48}\)(i) In the common region, \( f(\xi_*)/f_0(\xi_*) \) cannot take two values. On a set with measure zero, whether \( f(\xi_*)/f_0(\xi_*) \) is determined or not can be ignored. (See Footnote M-5 in Section M-1.2 for the set with measure zero.)

(ii) If the intersection is empty or measure zero, the integrations with respect to \( \xi \) at different \( \xi_\ast \)’s, are not influenced by the \( f(\xi_*)/f_0(\xi_*) \) determined by the other \( \xi \).

(iii) The equality only on a set of \( \xi \) with measure zero is ignored. Thus, the above set of \( \xi \), where \( f(\xi_*)/f_0(\xi_*) \) is constant is required to have some extent with measure nonzero with respect to \( \xi \) including the intersections.

\(^{49}\)The collection has to have some extent mentioned in Footnote 48 (iii).

\(^{50}\)In fact, Takata (private communication) constructed a kernel \( K_B \) which is zero in \((\xi - v_{wi}) n_i - C_{(\xi_i - v_{wi}) n_i + C_{*}} > 0 \) \((C \text{ and } C_{*} \text{: some positive constants}) \) and satisfies the conditions [M-1.27a]–[M-1.27c], for which the equality holds for another function.
Figure 1: Kernel $K_B(\xi, \xi_*)$ that requires $f(\xi) = c_0 f_0(\xi)$ for all $\xi$. The quarter in the figure is the range $(\xi_* - v_{wi})n_i < 0$ and $(\xi_i - v_{wi})n_i > 0$ in the space $(\xi_*, \xi)$. Let $K_B > 0$ in the regions A, B, C, and D at least, and their ranges of $\xi_*$ cover $(\xi_* - v_{wi})n_i < 0$. Then, $f(\xi_*)/f_0(\xi_*)$ is constant in each of A, B, C, and D (say, $a$ in A, $b$ in B, $c$ in C, and $d$ in D). Some ranges in A and B being on a common $\xi$ having some extent, $a = b$. In view of the intersection of the ranges of $\xi_*$ of B and C and that of B and D, $c = b (= a)$, and $d = b (= a)$. Thus, $f(\xi_*)/f_0(\xi_*) = a$ in $(\xi_* - v_{wi})n_i < 0$. It may be noted that the regions of $\xi_*$ of A and C are required to be only in contact with each other because the intersection of the ranges of $\xi_*$ of C and B is not measure zero.
The boundary condition on the interface is given as\(^{51}\)

\[
f(\xi) = g_I(\xi) + \int_{(\xi_i - v_{wi})n_i > 0} K_I(\xi, \xi_*) f(\xi_*) d\xi_*, \quad [(\xi_i - v_{wi})n_i > 0], \tag{180}
\]

where \(g_I\) and \(K_I\) are independent of \(f\). Further, \(g_I\) and \(K_I\) satisfy the following conditions [see Eqs. (M-1.31a)-(M-1.31c)]:

(i) Nonnegativity of \(g_I\)

\[
g_I(\xi) \geq 0 \quad [(\xi_i - v_{wi})n_i > 0]. \tag{181a}
\]

(ii) Nonnegativity of \(K_I\)

\[
K_I(\xi, \xi_*) \geq 0 \quad [(\xi_i - v_{wi})n_i > 0, (\xi_{i*} - v_{wi})n_i < 0]. \tag{181b}
\]

(iii) Condition of establishment of the equilibrium state

\[
f_w(\xi) = g_I(\xi) + \int_{(\xi_{i*} - v_{wi})n_i < 0} K_I(\xi, \xi_*) f_w(\xi_*) d\xi_*, \quad [(\xi_{i*} - v_{wi})n_i > 0], \tag{181c}
\]

where \(f_w\) is the Maxwellian determined by the temperature \(T_w\) and velocity \(v_{wi}\) of the interface and the saturation gas density \(\rho_w\) at temperature \(T_w\), i.e.,

\[
f_w(\xi) = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_{wi})^2}{2RT_w}\right). \tag{182}
\]

It is also required here that if \(f(\xi)\) for \((\xi_{i*} - v_{wi})n_i < 0\) is the corresponding part of another Maxwellian [say, \(f_r(\xi)\)], it does not give \(f_r(\xi)\) for \((\xi_i - v_{wi})n_i > 0\), which will be called the uniqueness condition of Eq. (181c) for shortness.

In the following discussion, we impose another condition in addition to Eqs. (181a)-(181c), i.e., putting

\[
\alpha(\xi_*) = -\int_{(\xi_{i*} - v_{wi})n_i > 0} \frac{(\xi_i - v_{wi})n_i}{(\xi_{i*} - v_{wi})n_{i*}} K_I(\xi, \xi_*) d\xi_*, \quad [(\xi_{i*} - v_{wi})n_j < 0], \tag{183}
\]

we assume\(^{52}\) that

\[
0 \leq \alpha(\xi_*) \leq 1 \quad [(\xi_{i*} - v_{wi})n_j < 0]. \tag{184}
\]

\(^{51}\) The variables \(X\) and \(t\) are not shown here because they are not important in the present discussion [see Footnote M-10 (ii) in Section M-1.5].

\(^{52}\) This condition corresponds to Eq. (M-1.27b) for a simple boundary. The simple boundary consists of molecules of different kinds from the gas molecules, and they stay there forever. The gas molecules impinging on the boundary are reflected without time delay (in the time scale of our interest), and there is no net mass flux to the boundary in this process. The condition (M-1.27b) is derived from this situation, as explained in Footnote M-13 in Section M-1.6.1. In the case of an interface, the condition (184) is derived similarly, if we consider that some of the molecules impinging on the interface do not reflect and stay there. However, the interface is the condensed phase of the gas and consists of the same kind of molecules as the gas. On the interface, molecules leave it depending on the condition of the interface even if there is no impinging molecules; this is the \(g_I\) part in Eq. (180). When a molecule impinges on the interface, it interacts with molecules of the interface, and some molecules leave the interface. Whether the impinging molecule is reflected or kicks out another molecule has no
Incidentally, from Eqs. (181a)-(181c),

\[ f_w(\xi) - g_I(\xi) \geq 0. \]  

(185)

We will show that the inequality (M-A.267) with \( f_0 \) being replaced by \( f_w \), i.e.,

\[ \int \xi (\xi - v_{wi}) n_i f_w(\xi) F[f(\xi)/f_w(\xi)] d\xi \leq 0, \]  

(186)

holds when \( F(x) \) is such a strictly convex function (see Footnote M-52 in Section M-A.10) that

\[ F(x) \geq 0 \text{ and } F(1) = 0. \]

The equality of the relation (186) holds when \( f(\xi) = f_w(\xi) \), and this relation is required except for some boundary conditions shown later. The inequality is proved with the aid of the Jensen inequality (see Eq. (M-A.265) or M-Jensen [1906], M-Lieb & Moss [2001], M-Parzen [1960], or M-Redin [1976])

\[ F \left( \int \phi \psi d\xi / \int \psi d\xi \right) \leq \int \psi F(\phi) d\xi / \int \psi d\xi \quad (\psi \geq 0), \]  

(187)

where \( F(x) \) is a strictly convex function, and \( \phi \) and \( \psi \) (\( \psi \geq 0 \)) are arbitrary functions of \( \xi \). The equality sign holds when \( \phi \) is independent of \( \xi \); it is also required where \( \psi > 0 \) for the equality.

Let \( F(x) \) be a nonnegative strictly convex function that takes value zero at \( x = 1 \), i.e.,

\[ F(x) \geq 0, \quad F(1) = 0. \]  

(188)

Consider the function \( F(f(\xi)/f_w(\xi)) \), where \( f_w(\xi) \) is given by Eq. (182). The function \( F(f(\xi)/f_w(\xi)) \) for \( (\xi - v_{wi}) n_i > 0 \) is bounded by an integral of \( f(\xi) \) difference. Further, depending on the condition (e.g., speed or direction) of the impinging molecule and that of the interface, more than one molecule may be kicked out or no molecule may be kicked out or reflected. Thus, it is not clear that the condition (184) holds or not. However, it is sure that the size of the kernel \( K_I \) is limited owing to the conditions (181a)-(181c), e.g., \( K_I = 0 \) if \( g_I = f_w \) (the complete condensation). See also Footnote 55 in Section 6.4.2.

(iia) The case \( \alpha(\xi_i) = 1 \) for \( (\xi_i - v_{wi}) n_i < 0 \) is excluded by the uniqueness condition of Eq. (184). In fact, multiplying Eq. (180) by \( (\xi_j - v_{wj}) n_j \) and integrating with respect to \( \xi_i \) over \( (\xi_j - v_{wj}) n_j > 0 \), we obtain \( g_I(\xi) = 0 \). Thus, \( C f_w \) (\( C \) : a constant) also satisfies Eq. (180).

(ii-b) When \( \alpha(\xi_i) = 0 \) for \( (\xi_j - v_{wj}) n_j < 0 \), the kernel \( K_I(\xi_i, \xi_j) \) degenerates, i.e., \( K_I(\xi_i, \xi_j) = 0 \) for \( (\xi_j - v_{wj}) n_j > 0 \). This is the case of the complete condensation.

Note that \( x = 1 \) is the unique zero point of \( F(x) \).
for \((\xi_i - v_{wi})n_i < 0\) with the aid of Eq. (180) in the following way:

\[
F\left(\frac{f(\xi)}{f_w(\xi)}\right) = F\left(\frac{g_I(\xi)}{f_w(\xi)} + \int_{(\xi_i - v_{wi})n_i < 0} \frac{K_I(\xi, \xi_*)}{f_w(\xi)} f(\xi_*) d\xi_*\right)
\]

\[
= F\left[\frac{g_I(\xi)}{f_w(\xi)} + \left(1 - \frac{g_I(\xi)}{f_w(\xi)}\right) \int_{\xi_i < 0} \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{[1 - g_I(\xi)/f_w(\xi)] f_w(\xi)} f(\xi_*) d\xi_*\right]
\]

\[
\leq \frac{g_I}{f_w} F(1) + \left(1 - \frac{g_I}{f_w}\right) F\left(\int_{(\xi_i - v_{wi})n_i < 0} \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{[1 - g_I(\xi)/f_w(\xi)] f_w(\xi)} f(\xi_*) d\xi_*\right)
\]

\[
= \left(1 - \frac{g_I(\xi)}{f_w(\xi)}\right) F\left(\int_{(\xi_i - v_{wi})n_i < 0} \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{[1 - g_I(\xi)/f_w(\xi)] f_w(\xi)} f(\xi_*) d\xi_*\right)
\]

\[
\vert \{(\xi_i - v_{wi})n_i > 0\}.
\]

(189)

Here, we, for a moment, consider the point of \(\xi \vert \{(\xi_i - v_{wi})n_i > 0\}\) where

\[
f_w(\xi) - g_I(\xi) > 0,
\]

for which

\[
\int_{(\xi_i - v_{wi})n_i < 0} \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{[1 - g_I(\xi)/f_w(\xi)] f_w(\xi)} d\xi_* = 1 \vert \{(\xi_i - v_{wi})n_i > 0\},
\]

because of Eq. (181c); in the second and third lines, the simple \(\leq\) sign of the subscript of the integral sign \(\int\) indicates \((\xi_i - v_{wi})n_i < 0\); the convex property of \(F(x)\) is used from the second line to the third, and \(F(1) = 0\) is used from the third to the fourth.

Now, we apply the Jensen inequality (187) to the function \(F\) on the fourth line in Eq. (189). Here, we choose \(\phi(\xi_*)\) and \(\psi(\xi_*)\) as

\[
\phi(\xi_*) = \frac{f(\xi_*)}{f_w(\xi_*)},
\]

\[
\psi(\xi_*) = \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{[1 - g_I(\xi)/f_w(\xi)] f_w(\xi)} \geq 0 \vert \{(\xi_i - v_{wi})n_i > 0, \ (\xi_i - v_{wi})n_i < 0\}.
\]

It should be noted that \(\phi(\xi_*)\) is defined for the whole range of \(\xi_*\) and that \(\psi(\xi_*)\) depends also on \(\xi\) and satisfies the relation, irrespective of \(\xi_\):

\[
\int_{(\xi_i - v_{wi})n_i < 0} \psi(\xi_*) d\xi_* = 1 \vert \{(\xi_i - v_{wi})n_i > 0\}.
\]

Then, \(F(f(\xi)/f_w(\xi))\) for \((\xi_i - v_{wi})n_i > 0\) is bounded as

\[
F\left(\frac{f(\xi)}{f_w(\xi)}\right) \leq \left(1 - \frac{g_I}{f_w}\right) F\left(\int_{(\xi_i - v_{wi})n_i < 0} \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{[1 - g_I(\xi)/f_w(\xi)] f_w(\xi)} F(\zeta) d\xi_*\right)
\]

\[
\leq \int_{(\xi_i - v_{wi})n_i < 0} \frac{K_I(\xi, \xi_*) f_w(\xi_*)}{f_w(\xi)} F\left(\frac{f(\xi_*)}{f_w(\xi_*)}\right) d\xi_* \vert \{(\xi_i - v_{wi})n_i > 0\}.
\]

(190)
Up to this point, we limited our discussion to the point of $\xi_0 [\xi_i - v_{wi} n_i > 0]$ where

$$f_w(\xi) - g_I(\xi) > 0.$$ If it vanishes at some $\xi_A [\xi_i - v_{wi} n_i > 0]$, i.e.,

$$f_w(\xi_A) - g_I(\xi_A) = 0,$$

the integral $\int_{(\xi_i - v_{wi}) n_i < 0} K_f(\xi, \xi_s) f_w(\xi_s) d\xi_s$ vanishes there, i.e.,

$$\int_{(\xi_i - v_{wi}) n_i < 0} K_f(\xi_A, \xi_s) f_w(\xi_s) d\xi_s = 0,$$

because of the condition (181c). The function $f_w(\xi_s)$ being positive for all $\xi_s$, the kernel $K_f(\xi_A, \xi_s)$ must vanish for $(\xi_i - v_{wi}) n_i < 0$, i.e.,

$$K_f(\xi_A, \xi_s) = 0 \quad [(\xi_i - v_{wi}) n_i < 0].$$

Thus, from the boundary condition (180),

$$f(\xi_A) = g_I(\xi_A) = f_w(\xi_A).$$

Therefore, the function $F(f(\xi_A)/f_w(\xi_A))$ vanishes, i.e.,

$$F(f(\xi_A)/f_w(\xi_A)) = F(1) = 0.$$ (193)

From Eqs. (192) and (193), the equality holds between the left-most side and the right-most of Eq. (190) at $\xi = \xi_A$. In conclusion, the inequality

$$F \left( \frac{f(\xi)}{f_w(\xi)} \right) \leq \int_{(\xi_i - v_{wi}) n_i < 0} \frac{K_f(\xi, \xi_s) f_w(\xi_s)}{f_w(\xi)} F \left( \frac{f(\xi)}{f_w(\xi_s)} \right) d\xi_s \quad [(\xi_i - v_{wi}) n_i > 0],$$

(194)

holds without the assumption of $f_w(\xi) - g_I(\xi) > 0$.

When $f(\xi)/f_w(\xi) = 1$ for all $\xi$, $F(f(\xi)/f_w(\xi))$ vanishes in Eq. (194), and the equality holds there. We look for the other possibilities of the equality. The first inequality in Eq. (190) comes from that of Eq. (189), for which the equality holds at $\xi = \xi_A$ when (i) $g_I(\xi_A)/f_w(\xi_A) = 0$ or (ii) $g_I(\xi_A)/f_w(\xi_A) = 1$, or (iii) the arguments of two $F$’s on the third line of Eq. (189) are equal, i.e.,

$$\int_{(\xi_i - v_{wi}) n_i < 0} \frac{K_f(\xi_A, \xi_s) f_w(\xi_s)}{[1 - g_I(\xi_A)/f_w(\xi_A)] f_w(\xi_A)} f(\xi_s) d\xi_s = 1,$$ (195)

for some $f(\xi_s)$. In the third case, the equality relation being imposed between the first and the second line on the right-hand side of Eq. (190) under the condition (195), we find that

$$f(\xi_s) = f_w(\xi_s) \text{ in } B_A(\xi_s),$$

where $B_A(\xi_s)$ is the region of $\xi_s$ in which $K_f(\xi_A, \xi_s) > 0$. 

55
If \( g_I(\xi) / f_w(\xi) = 0 \) for \( (\xi_i - \nu_{w_i}) n_i > 0 \), the boundary condition (180) reduces to
\[
 f(\xi) = \int_{(\xi_i - \nu_{w_i}) n_i < 0} K_I(\xi, \xi_*) f(\xi_*) d\xi_*. \tag{196}
\]

Then, the Maxwellian \( a_0 f_w(\xi) \) (\( a_0 \) : a constant) also satisfies the boundary condition (180), which is not allowed by the uniqueness condition of Eq. (181c). Thus, this case is excluded. If \( g_I(\xi) / f_w(\xi) = 1 \) for \( (\xi_i - \nu_{w_i}) n_i > 0 \), the kernel
\( K_I(\xi, \xi_*) \) vanishes for \( (\xi_i - \nu_{w_i}) n_i > 0 \) and \( (\xi_3 - \nu_{w_3}) n_3 < 0 \) from the discussion in the preceding paragraph. That is, \( f(\xi) = f_w(\xi) \) in \( (\xi_i - \nu_{w_i}) n_i > 0 \) irrespective of \( f(\xi) \) in \( (\xi_i - \nu_{w_i}) n_i < 0 \) (this is the case of the complete condensation condition). For this case the equality holds in Eq. (194). If the third condition holds for \( (\xi_i - \nu_{w_i}) n_i > 0 \), we have
\[
 f_w(\xi) = g_I(\xi) + \int_{(\xi_i - \nu_{w_i}) n_i < 0} K_I(\xi, \xi_*) f(\xi_*) d\xi_* \quad [(\xi_i - \nu_{w_i}) n_i > 0]. \tag{197}
\]

From the discussion of the preceding paragraph,
\[
 f(\xi_*) = f_w(\xi_*) \text{ in } B(\xi_*), \tag{198}
\]
where \( B(\xi_*) \) is the region of \( \xi_* \) in which \( K_I(\xi, \xi_*) > 0 \) for some \( \xi \). This condition is paraphrased as
\[
 f(\xi_*) = f_w(\xi_*) \text{ except in the region } \alpha(\xi_*) = 0. \tag{199}
\]

Whether \( f(\xi_*) = f_w(\xi_*) \) or \( \alpha(\xi_*) = 0 \) in \( (\xi_3 - \nu_{w_3}) n_3 < 0 \),
\[
 f(\xi) = f_w(\xi) \quad [(\xi_i - \nu_{w_i}) n_i > 0].
\]

Let us consider the case where the three situations (i), (ii), and (iii) listed just before Eq. (195) take place for different \( \xi \), say, (i) for \( \xi \) in \( A_1 \), (ii) for \( \xi \) in \( A_2 \), and (iii) for \( \xi \) in \( A_3 \). The \( A_2 \) part does not contribute to the restriction on \( f(\xi_*) \). When \( A_1 \) is empty, the condition is the same as for the case of Eq. (197), i.e., Eq. (198) or (199). When \( A_1 \) is not empty, from the discussion for \( \xi \) in \( A_3 \), \( f(\xi_*) = f_w(\xi_*) \) in the region of \( \xi_* \) where \( K_I(\xi, \xi_*) > 0 \) for some \( \xi \) in \( A_3 \) [say, \( B_3(\xi_*) \)], and the condition for the remaining \( \xi_* \) is determined only by the behavior of \( K_I(\xi) \) for \( \xi \) in \( A_1 \), that is, the region \( f(\xi_*) / f_w(\xi_*) = \text{const} \) [say, \( B_1(\xi_*) \)] is looked for in the range \( (\xi_3 - \nu_{w_3}) n_3 < 0 \) in the same way as in Section 6.4.1 and if \( B_1 \) has a common region with \( B_3 \), \( f(\xi_*) = f_w(\xi_*) \) in \( B_1 \). In the region of the remaining \( \xi_* \) [say, \( R(\xi_*) \)], \( f(\xi_*) \) other than \( f_w(\xi_*) \) can exist. The region \( a(\xi_*) = 1 \) in \( R(\xi_*) \) is denoted by \( R_{a=1} \) for the convenience in the later citation.

When \( A_3 \) is empty, the boundary condition (180) is expressed as
\[
 f(\xi) = \begin{pmatrix}
 0 \\
 f_w(\xi)
\end{pmatrix} + \int_{(\xi_3 - \nu_{w_3}) n_3 < 0} \begin{pmatrix}
 K_I(\xi, \xi_*) \\
 0
\end{pmatrix} f(\xi_*) d\xi_* \quad [(\xi \in A_1) \quad (\xi \in A_2)], \tag{200}
\]
of the Darrozes–Guiraud inequality, we have to examine the equality of the second inequality
\( f(B) \) the same as that of Eq. (194). Thus, the equality condition of the inequality of Eq. (201) is

\[
\int_{(\xi_i - v_{wi}) n_i < 0} f_w(\xi) \frac{K_f(\xi, \xi_*) f_w(\xi_*)}{f_w(\xi)} \, d\xi = 1 \quad \text{for} \ (\xi_i - v_{wi}) n_i > 0 \text{ and } \xi \text{ in } A_1.
\]

The boundary condition (200) obviously satisfies the conditions (181a)–(181c). In this case, the restriction on \( f(\xi_*) \) is determined by \( K_f \) in \( A_1 \). Substituting \( f(\xi_*) = C_D f_w(\xi_*) \) \( (\xi_i - v_{wi}) n_i < 0, C_D \) independent of \( \xi_*, \) which is the strongest restriction on \( f(\xi_*) \), into Eq. (200), we have \( f(\xi) = C_D f_w(\xi) \) [in \( A_1 \)] and \( f(\xi) = f_w(\xi) \) [in \( A_2 \)] for \( (\xi_i - v_{wi}) n_i > 0 \). For this \( f(\xi) \), the equality holds in Eq. (194). Thus, for the boundary condition (200) as well as the complete condensation condition, the equality in Eq. (194) holds for \( f(\xi) \) other than \( f(\xi) = f_w(\xi) \) \( f(\xi_*) = C_D f_w(\xi_*) \) for \( (\xi_i - v_{wi}) n_i < 0 \) for Eq. (200), and \( f(\xi_*) \) is arbitrary for \( (\xi_i - v_{wi}) n_i < 0 \) for the complete condensation. This is an example of \( f(\xi_*) \) that satisfies the equality in Eq. (194).

With the aid of the inequality (194) and Eq. (183), we have

\[
\int_{(\xi_i - v_{wi}) n_i > 0} (\xi_i - v_{wi}) n_i f_w(\xi) F \left( \frac{f(\xi)}{f_w(\xi)} \right) \, d\xi
\]

\[
\leq \int_{(\xi_i - v_{wi}) n_i > 0} (\xi_i - v_{wi}) n_i f_w(\xi) \int_{(\xi_i - v_{wi}) n_i < 0} \frac{K_f(\xi, \xi_*) f_w(\xi_*)}{f_w(\xi)} F \left( \frac{f(\xi_*)}{f_w(\xi_*)} \right) \, d\xi_*, \, d\xi
\]

\[
= \int_{(\xi_i - v_{wi}) n_i < 0} f_w(\xi) F \left( \frac{f(\xi_*)}{f_w(\xi_*)} \right) \int_{(\xi_i - v_{wi}) n_i > 0} (\xi_i - v_{wi}) n_i K_f(\xi, \xi_*) d\xi d\xi_*
\]

\[
= - \int_{(\xi_i - v_{wi}) n_i < 0} \alpha(\xi_*) (\xi_i - v_{wi}) n_i f_w(\xi_*) F \left( \frac{f(\xi_*)}{f_w(\xi_*)} \right) \, d\xi_*, \quad (201)
\]

where \( 0 \leq \alpha(\xi_*) \leq 1 \) [the assumption (184)]. Thus, we obtain the extension of Eq. (M-A.267) to the case of an interface as follows:

\[
\int_{\text{all } \xi} (\xi_i - v_{wi}) n_i f_w(\xi) F \left( \frac{f(\xi)}{f_w(\xi)} \right) \, d\xi
\]

\[
\leq \int_{(\xi_i - v_{wi}) n_i > 0} \left[ 1 - \alpha(\xi_*) \right] (\xi_i - v_{wi}) n_i f_w(\xi_*) F \left( \frac{f(\xi_*)}{f_w(\xi_*)} \right) \, d\xi_* \leq 0. \quad (202)
\]

Obviously, the equal sign holds in the two inequalities of Eq. (202) when \( f(\xi) = f_w(\xi) \). Conversely, it is required for the equal sign to hold in the inequalities that \( f(\xi) = f_w(\xi) \) for all \( \xi \) when \( \text{R}_{\alpha=1} \) is empty.\(^{55}\) It should be noted that

\(^{54}\) To confirm the uniqueness condition of Eq. (181c) is simple. Note \( f(\xi) \) \( (\xi_i - v_{wi}) n_i > 0 \) for \( \xi \) in \( A_2 \).

\(^{55}\) The integration of a nonnegative function multiplied by a positive function does not change the equality condition. Thus, the equality condition of the inequality of Eq. (201) is the same as that of Eq. (194) \( B = A \iff f_a(\xi)|B(\xi) - A(\xi)| \, d\xi = 0 \) if \( A(\xi) \leq B(\xi) \) and \( a(\xi) > 0 \). Thus, the range where \( f(\xi_*) = f_w(\xi_*) \) is required is outside \( \text{R} \). For the equality of the Darrozes–Guiraud inequality, we have to examine the equality of the second inequality.
$F(x)$ is required to satisfy that $F(x) \geq 0$ and $F(1) = 0$ in addition to convexity. Here, we take

$$F(x) = x(\ln x - 1) + 1,$$

which is strictly convex, nonnegative, and zero at $x = 1$. Then,

$$\int_{\text{all } \xi} (\xi - v_{w_i}) n_i \left[ f(\xi) \left( \ln \frac{f(\xi)}{f_w(\xi)} - 1 \right) + f_w(\xi) \right] \, d\xi \leq 0,$$

or

$$\int_{\text{all } \xi} (\xi - v_{w_i}) n_i f(\xi) \ln \frac{f(\xi)}{f_w(\xi)} \, d\xi \leq \rho(v_i - v_{w_i}) n_i. \quad (203)$$

This is the extension of Eq. (M-A.262) for a simple boundary to an interface.

We try to express the inequality (203) in terms of macroscopic variables. It is simply transformed in the following form:

$$\int_{\text{all } \xi} (\xi - v_{w_i}) n_i f(\xi) \ln \frac{f(\xi)}{c_0} \, d\xi$$

$$\leq \int_{\text{all } \xi} (\xi - v_{w_i}) n_i f(\xi) \ln \frac{f_w(\xi)}{c_0} \, d\xi + \rho(v_i - v_{w_i}) n_i$$

$$= \frac{1}{RT_w} \left[ q_i n_i + (v_j - v_{w,j}) \tilde{p}_{ij} n_i + \rho(v_i - v_{w_i}) n_i \left( \frac{5}{2} RT + \frac{1}{2} (v_j - v_{w,j})^2 \right) \right]$$

$$+ \rho(v_i - v_{w_i}) n_i \left( \ln \frac{\rho_w}{(2\pi RT_w)^{3/2} c_0} + 1 \right),$$

where $c_0$ is a constant to make the argument of the logarithmic function dimensionless, and

$$\tilde{p}_{ij} = p_{ij} - p \delta_{ij}. \quad (204)$$

The $\tilde{p}_{ij}$ is the part of stress tensor with the pressure contribution subtracted. Only the tangential component of the stress $\tilde{p}_{ij} n_i$ contributes to $(v_j - v_{w,j}) \tilde{p}_{ij} n_i$ in Eq. (202). The second equal sign holds only when $F(f(\xi_i)/f_w(\xi_i)) = 0$ in $\mathbb{R}$ outside $R_{\alpha=1}$ because $f_w(\xi_i) > 0$ and $1 - \alpha(\xi_i) > 0$ there. Thus, $f(\xi_i)/f_w(\xi_i) = 1$ outside $R_{\alpha=1}$ in $(\xi_i - v_{w_i}) n_i < 0$ (see Footnote M-53 in Section 6.4.2). When $R_{\alpha=1}$ is empty, the integral $\int_{\text{all } \xi}$ on the left-most side reduces to $\int_{(\xi_i - v_{w_i}) n_i > 0}$. This vanishes only when $F(f(\xi)/f_w(\xi)) = 0$, i.e., $f(\xi) = f_w(\xi)$ for $(\xi_i - v_{w_i}) n_i > 0$. Thus, $f(\xi) = f_w(\xi)$ for all $\xi$ when $R_{\alpha=1}$ is empty. It may be noted that when $A_3$ is empty (or for the boundary condition [200]), $R_{\alpha=1}$ is the range of $\xi_i$ where $\alpha(\xi_i) = 1$ in $(\xi_i - v_{w_i}) n_i < 0$. Incidentally, $g_j(\xi)$ that is positive almost everywhere (Footnote M-5 in Section M-1.2) is classified positive, for which $A_1$ in the paragraph following to that of Eq. (199) is empty and Eq. (199) holds (that is, $R_{\alpha=1}$ is empty), and therefore the equal signs hold in Eq. (202) only when $f(\xi) = f_w(\xi)$ for all $\xi$.

(ii) If $\alpha(\xi_i)$ exceeds unity for some range of $\xi_i$ in $(\xi_i - v_{w_i}) n_i < 0$ and the assumption (184) is violated, but the integral

$$\int_{(\xi_i - v_{w_i}) n_i < 0} [1 - \alpha(\xi_i)] (\xi_i - v_{w_i}) n_i f_w(\xi_i) F \left( \frac{f(\xi_i)}{f_w(\xi_i)} \right) \, d\xi_i,$$

is nonpositive, the inequality holds.
when no flow to the boundary. Further, \( \ln \frac{\rho_w}{(2\pi RT_w)^{3/2}c_0} \) is related to the \( H \) function \( H_w \) for \( f(\xi) = f_w(\xi) \) as

\[
\frac{H_w}{\rho_w} = \ln \frac{\rho_w}{(2\pi RT_w)^{3/2}c_0} - \frac{3}{2},
\]

which is independent of \( v_i \). That is,

\[
H_w = \int_{\text{all } \xi} f_w(\xi) \ln \frac{f_w(\xi)}{c_0} d\xi = \int_{\text{all } \xi} f_w^{(v)}(\xi) \ln \frac{f_w^{(v)}(\xi)}{c_0} d\xi,
\]

where

\[
f_w^{(v)}(\xi) = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp \left( -\frac{(\xi - v_i)^2}{2RT_w} \right).
\]

On the other hand, by definition (see Section M-1.7),

\[
\int_{\text{all } \xi} (\xi_i - v_{wi}) n_i f(\xi) \ln[f(\xi)/c_0] d\xi = (H_i - Hv_{wi}) n_i.
\]

Therefore,

\[
(H_i - Hv_{wi}) n_i \leq -\frac{1}{RT_w} \left[ q_i n_i + (v_j - v_{wj}) \tilde{p}_{ij} n_i \right] + \rho (v_i - v_{wi}) n_i \left[ \frac{H_w}{\rho_w} - \frac{1}{RT_w} \left( \frac{5}{2} R(T - T_w) + \frac{1}{2} (v_j - v_{wj})^2 \right) \right]. \tag{206}
\]

When \( f = f_w \), both sides of the inequality vanish and the equal sign holds. Conversely, for the kernel \( K_f \) with \( R_{\gamma=1} \) empty, e.g., \( g_f \) that is positive almost everywhere, the equal sign holds only when \( f = f_w \).

Finally, we consider the variation of the integral \( \overline{H} \) of \( H \) over the domain \( D \). According to Eq. (M-1.36),

\[
\frac{d\overline{H}}{dt} = \int_{\partial D} (H_i - Hv_{wi}) n_i + \int_D G dX,
\]

where

\[
\overline{H} = \int_D H dX.
\]

With the aid of Eq. (206), the variation is bounded as

\[
\frac{d\overline{H}}{dt} \leq -\frac{1}{RT_w} \left[ q_i n_i + (v_j - v_{wj}) \tilde{p}_{ij} n_i \right] + \rho (v_i - v_{wi}) n_i \left[ \frac{H_w}{\rho_w} - \frac{1}{RT_w} \left( \frac{5}{2} R(T - T_w) + \frac{1}{2} (v_j - v_{wj})^2 \right) \right], \tag{207}
\]

because \( \int_D G dX \leq 0 \) [see Eq. (M-1.34b)].

(Section 6.4.2: Version 5-00)