Supplement to Molecular Gas Dynamics

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Update of bibliography

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59, (to be published) → **60**, 147–163

• [93] Ha, S.-Y., T.-P. Liu, and S.-H. Yu (2006). This paper is to be replaced by
Yu, S.-H. (2014), Initial and shock layer for Boltzmann equation, *Aech. Rational Mech. Anal.* **211**, 1–60.

• [136] Liu, T.-P. and S.-H. Yu (2006a):
(2006a) → (2006)

• [137] Liu, T.-P. and S.-H. Yu (2006b):
(2006b) → (2007)
59, (to be published) → **60**, 295–356

The corresponding corrections in the text

- p. 166, the line before the last in the second paragraph:

  Ha, Liu, & Yu [2006] $\rightarrow$ Yu [2014]

  See [93] in **Update of Bibliography**.
- p. 166, the first line in the third paragraph:

  [2004b, 2006a] $\rightarrow$ [2004b, 2006]
- p. 166, the last line in the third paragraph:

  [2006b] $\rightarrow$ [2007]
- p. 183, the second line:

  [2006] $\rightarrow$ [2007]
- p. 222, the 15th line in the second paragraph:

  Ha, Liu, & Yu [2006] $\rightarrow$ Yu [2014]

  See [93] in **Update of Bibliography**.

Errata

- p. xii, the 16th line in Preface:

  http://fd.kuaero.kyoto-u.ac.jp/member/sone $\rightarrow$
  https://hdl.handle.net/2433/66098 and https://hdl.handle.net/2433/66099

- p. 7, the 14th line:

  $\xi = |\xi_i| = (\xi_i)^{1/2} \quad \rightarrow \quad \xi = |\xi_i| = (\xi_i^2)^{1/2}$

- p. 9, the 7th line:

  specular condition $\rightarrow$ specular reflection

- p. 15, the 2nd line of Footnote 22 (This item is not corrections, but gives alternative expressions convenient to the discussion in Section 1.5, where the nondimensional form of the Boltzmann equation for an finite-range intermolecular potential is discussed.):

U_0: The dependence of U_0 on the mass m of a molecule is better to be explicit (see Sections 1.5.2 and 1.5.3). That is,

$U_0 \rightarrow m\overline{\mathcal{U}}_0$

Correspondingly, U_0/k_BT_0 on the the 3rd and 5th lines should be replaced by $\overline{\mathcal{U}}_0/RT_0$.

- p. 15, the 2nd and 4th lines of Footnote 22:

  $U \rightarrow \hat{U}$

- p. 27, the 3rd line of Footnote 26:

  Eq. (1.99) $\rightarrow$ a linear combination of Eqs. (1.99) and (1.101)

- p. 27, the 6th line of Footnote 26:

except for a common constant factor $\rightarrow$ except for a common constant factor and additive functions (say, f_a in $\hat{H}$ and f_{bi} in $\hat{H}_i$ in their second order) satisfying $\mathrm{Sh}\partial f_a/\partial\hat{t} + \partial f_{bi}/\partial x_i = 0$

- p. 48, the 21st line, p. 49, the 3rd line from below, and p. 488, the 4th line:
  solid angle element $\rightarrow$ solid-angle element
- p. 81, the 4th line in Footnote 7:
  u_{iGm} $\rightarrow$ $u_{iGm} - u_{jGm}n_jn_i$
  or ϕ_{eGm} $\rightarrow$ of ϕ_{eGm}
- p. 83, the first line in Footnote 14:
  u_{iGm} $\rightarrow$ $u_{iGm} - u_{jGm}n_jn_i$
- p. 235, Footnote M-5
  http://fd.kuaero.kyoto-u.ac.jp/member/sone $\rightarrow$
  https://hdl.handle.net/2433/120983
- p. 246, Footnote M-10
  http://fd.kuaero.kyoto-u.ac.jp/member/sone $\rightarrow$
  https://hdl.handle.net/2433/122357
- p. 502 – 505 and 508 (This item is not corrections, but gives alternative expressions convenient to the discussion in Section 1.5, where the nondimensional form of the Boltzmann equation for an finite-range intermolecular potential is discussed.):

The parameter U_0 expressing the strength of the intermolecular potential is introduced in Eq. (A.51) and on the first line of p.503. The dependence of U_0 on the mass m of a molecule is better to be explicit (see Sections 1.5.2 and 1.5.3). That is,

$U_0 \rightarrow m\overline{\mathcal{U}}_0$

Correspondingly, the following replacements with the new parameter $\overline{\mathcal{U}}_0$ should be made:

$U_0/m\mathcal{V}^2 \rightarrow \overline{\mathcal{U}}_0/\mathcal{V}^2$ (the 2rd and 4th lines and the 7th line from below in p. 503),

$U_0/m \rightarrow \overline{\mathcal{U}}_0$ (the 3rd line from below in p. 503 and the 4th, 5th, and 7th lines in p. 504),

$mRT_0/U_0 \rightarrow RT_0/\overline{\mathcal{U}}_0$ (the 12th and 14th – 16th lines in p. 508).

- p. 503, the 13th line from below:
  solid angle elements $\rightarrow$ solid-angle elements
- p. 504, the first line in Footnote 24:
  damin $\rightarrow$ domain

- p. 505, Eq. (A.60):

$$\left|\frac{1}{\sin\theta_c}\frac{\partial\sin^2\theta_e}{\partial\theta_c}\right| \quad\to\quad \left|\frac{1}{\sin\theta_c}\frac{\mathrm{d}\sin^2\theta_e}{\mathrm{d}\theta_c}\right|$$

$$\left|\frac{1}{\sin\theta_c}\frac{\partial b^2}{\partial\theta_c}\right| \quad\to\quad \left|\frac{1}{\sin\theta_c}\frac{\mathrm{d}b^2}{\mathrm{d}\theta_c}\right|$$

- p. 506, the 13th line [The line next to Eq. (A.63)]:
  with respect to θ_c $\to$ with respect to θ_α

- p. 617, the right-hand side of Eq. (C.2b):
  In order to avoid misunderstanding, $\frac{2(n+1)!}{\beta^{n+2}}\pi$ is better expressed as

$$\frac{2\pi(n+1)!}{\beta^{n+2}}.$$

- p. 628, Reference [110]:
  Reference [110] should be placed after Reference [112].

- p. 639, the 3rd line in Reference [262]:
  gs $\to$ gas

Supplementary Notes

In the present supplementary notes, the letter M is attached to the labels of sections, equations, etc. in the book *Molecular Gas Dynamics* and the letter K is attached to those in Y. Sone, *Kinetic Theory and Fluid Dynamics* (Sone [2002]) to avoid confusion. The two books, *Molecular Gas Dynamics* and *Kinetic Theory and Fluid Dynamics*, themselves are, respectively, referred to as MGD and KF.

1 Chapter M-1

1.1 Background of the Boltzmann equation (Sections M-1.1 and M-1.2)

The situation of a monatomic gas the description of which is the purpose of the Boltzmann equation is explained in more detail in Section 1.5.3 (the second half part of Section 1.5.2 before Version 13-00). This will serve as the supplement to Sections M-1.1 and M-1.2, though it is prepared for the discussion of the parameters in the nondimensional Boltzmann equation.

(Section 1.1: Version 9-00 and 13-00)

1.2 Supplement to Footnote M-9 in Section M-1.3

We will explicitly show the process of derivation of the conservation equations (M-1.12)–(M-1.14) by taking into account the discontinuity of the velocity distribution function $f(\boldsymbol{X}, \boldsymbol{\xi}, t)$ for a typical case.

Let $S(\boldsymbol{X})$ be a continuous and sectionally smooth function of $\boldsymbol{X}$, and let the surface in the $\boldsymbol{X}$ space consisting of the points $\boldsymbol{X}_0$ that satisfy $S(\boldsymbol{X}_0) = 0$ be indicated by S_0.[1] The surface S_0 may be an infinite surface or a bounded surface separating the space $\boldsymbol{X}$ into two regions. The velocity distribution function f at time t_0 is assumed to be discontinuous across the surface S_0 and to be smooth except on S_0. The discontinuity propagates along the characteristics of the Boltzmann equation (M-1.5), i.e., $X_i - \xi_i(t - t_0) = X_{0i}$, for each $\boldsymbol{\xi}$.[2] Take a point $(\boldsymbol{X}, t)$ in the space and time, where $t > t_0$. At this point or at $(\boldsymbol{X}, t)$, the discontinuity of f lies on the surface $S^{(\boldsymbol{\xi})}(\boldsymbol{X}, t)$ in the $\boldsymbol{\xi}$ space that consists of the points $\boldsymbol{\xi}_D$ satisfying

$$S(X_i - \xi_{Di}(t - t_0)) = 0, \text{ or } X_i - \xi_{Di}(t - t_0) = X_{0i}. \tag{1}$$

The point $\boldsymbol{\xi}_D$ is determined by $\boldsymbol{X}$, t, and $\boldsymbol{X}_0$, i.e., $\boldsymbol{\xi}_D(\boldsymbol{X}, t; \boldsymbol{X}_0)$. Let the side of the domain in the $\boldsymbol{\xi}$ space that satisfies $S(X_i - \xi_i(t - t_0)) > 0$ be indicated by V_+, and the other side of the domain by V_-; let the outward unit normal to the surface $S^{(\boldsymbol{\xi})}(\boldsymbol{X}, t)$ with respect to V_+ be indicated by $n_{Di}(\boldsymbol{\xi}_D; \boldsymbol{X}, t)$. Then,

$$n_{Di}(\boldsymbol{\xi}_D; \boldsymbol{X}, t) = -\left.\frac{\partial S(\boldsymbol{X} - \boldsymbol{\xi}(t - t_0))/\partial \xi_i}{|\partial S(\boldsymbol{X} - \boldsymbol{\xi}(t - t_0))/\partial \xi_j|}\right|_{\boldsymbol{\xi} = \boldsymbol{\xi}_D} = \left.\frac{\partial S(\boldsymbol{Y})/\partial Y_i}{|\partial S(\boldsymbol{Y})/\partial Y_j|}\right|_D, \tag{2}$$

where $|a_i| = (a_i^2)^{1/2}$ and the subscript D to $\partial S(\boldsymbol{Y})/\partial Y_j$ indicates $\boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{\xi}_D(t - t_0)$. The variations of $\boldsymbol{\xi}_D$ with respect to $\boldsymbol{X}$ or t for a given $\boldsymbol{X}_0$, i.e., $\partial \boldsymbol{\xi}_D/\partial X_i$ and $\partial \boldsymbol{\xi}_D/\partial t$, are determined from Eq. (1) as

$$\left.\frac{\partial S(\boldsymbol{Y})}{\partial Y_j}\right|_D \left(\delta_{ij} - \frac{\partial \xi_{Dj}}{\partial X_i}(t - t_0)\right) = 0, \qquad \left.\frac{\partial S(\boldsymbol{Y})}{\partial Y_j}\right|_D \left(\frac{\partial \xi_{Dj}}{\partial t}(t - t_0) + \xi_{Dj}\right) = 0.$$

Thus, with the aid of Eq. (2),

$$n_{Dj}\frac{\partial \xi_{Dj}}{\partial X_i} = \frac{n_{Di}}{t - t_0}, \qquad n_{Dj}\frac{\partial \xi_{Dj}}{\partial t} = -\frac{n_{Dj}\xi_{Dj}}{t - t_0}. \tag{3}$$

The integral of such a discontinuous function with respect to $\boldsymbol{\xi}$ over its whole space is split into two parts as

$$\int \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} = \int_{V_+} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi} + \int_{V_-} \psi(\boldsymbol{\xi}) f \mathbf{d}\boldsymbol{\xi},$$

where $\psi(\boldsymbol{\xi})$ is a smooth function of $\boldsymbol{\xi}$. Then, the integrand is smooth in each of V_+ and V_-. According to Lemma in page M-492, the following derivatives of

[1] It is assumed that $(\partial S/\partial X_i)^2 \neq 0$ on S_0. The normal to the surface S_0 is defined except at special points.

[2] For simplicity of explanation, we consider the case where $F_i = 0$ here.

integrals over the domain V_+ are transformed as[3]

$$\frac{\partial}{\partial t}\int_{V_+}\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}=\int_{V_+}\psi(\boldsymbol{\xi})\frac{\partial f}{\partial t}\mathbf{d}\boldsymbol{\xi}+\int_{S^{(\boldsymbol{\xi})}}\psi(\boldsymbol{\xi})f\frac{\partial\xi_{Dj}}{\partial t}n_{Dj}\mathrm{d}^2\boldsymbol{\xi},$$

$$\frac{\partial}{\partial X_i}\int_{V_+}\xi_i\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}=\int_{V_+}\xi_i\psi(\boldsymbol{\xi})\frac{\partial f}{\partial X_i}\mathbf{d}\boldsymbol{\xi}+\int_{S^{(\boldsymbol{\xi})}}\xi_i\psi(\boldsymbol{\xi})f\frac{\partial\xi_{Dj}}{\partial X_i}n_{Dj}\mathrm{d}^2\boldsymbol{\xi},$$

where the integral over the surface $S^{(\boldsymbol{\xi})}$ of the second term on the right-hand side of each equation is due to the variation of the domain V_+ with t or X_i. Summing the above two derivatives and noting Eq. (3), we have

$$\frac{\partial}{\partial t}\int_{V_+}\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}+\frac{\partial}{\partial X_i}\int_{V_+}\xi_i\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}=\int_{V_+}\psi(\boldsymbol{\xi})\frac{\partial f}{\partial t}\mathbf{d}\boldsymbol{\xi}+\int_{V_+}\xi_i\psi(\boldsymbol{\xi})\frac{\partial f}{\partial X_i}\mathbf{d}\boldsymbol{\xi},$$

where the surface integrals over $S^{(\boldsymbol{\xi})}$ are canceled. Similarly,

$$\frac{\partial}{\partial t}\int_{V_-}\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}+\frac{\partial}{\partial X_i}\int_{V_-}\xi_i\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}=\int_{V_-}\psi(\boldsymbol{\xi})\frac{\partial f}{\partial t}\mathbf{d}\boldsymbol{\xi}+\int_{V_-}\xi_i\psi(\boldsymbol{\xi})\frac{\partial f}{\partial X_i}\mathbf{d}\boldsymbol{\xi}.$$

Thus, we have

$$\frac{\partial}{\partial t}\int\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}+\frac{\partial}{\partial X_i}\int\xi_i\psi(\boldsymbol{\xi})f\mathbf{d}\boldsymbol{\xi}=\int\psi(\boldsymbol{\xi})\frac{\partial f}{\partial t}\mathbf{d}\boldsymbol{\xi}+\int\xi_i\psi(\boldsymbol{\xi})\frac{\partial f}{\partial X_i}\mathbf{d}\boldsymbol{\xi}. \quad (4)$$

It may be noted that the interchange of differentiation and integration is possible only for the above combination of the integrals. With this formula, the conservation equations are derived by choosing 1, ξ_i, and ξ_i^2 as $\psi(\boldsymbol{\xi})$.

When the surface S_0, i.e., $S(\boldsymbol{X})=0$, is a finite surface or semi-infinite surface which does not divide the $\boldsymbol{\xi}$ space into V_+ and V_-, we can take it as a special case where some part of S_0 joins to its other part and V_- degenerates empty. When there is a body in a gas, the discontinuity as shown in Section M-3.1.6 generally exists. The analysis can be carried out in a similar way; that is, determine the position of the discontinuity in the $\boldsymbol{\xi}$ space first, carry out the differentiations in each region where the velocity distribution function is smooth with the aid of the lemma in page M-492, and sum up the results.

(Section 1.2: Version 6-00)

1.3 Bulk viscosity (Section M-1.3)

The assumptions (M-1.15) and (M-1.16) for the stress tensor and heat-flow vector in classical gas dynamics are what is to be studied by kinetic theory (see Chapter M-3). For a monatomic gas, consisting of identical molecules whose

[3]The correspondence of the variables here and those in the lemma is as follows: $\boldsymbol{\xi}\leftrightarrow\boldsymbol{X}$, t or $X_i\leftrightarrow\vartheta$, $n_{Di}\leftrightarrow\boldsymbol{n}_w$, $\mathbf{d}\boldsymbol{\xi}\leftrightarrow\mathbf{d}\boldsymbol{X}$, $\mathrm{d}^2\boldsymbol{\xi}\leftrightarrow\mathrm{d}^2\boldsymbol{X}$, $V_+\leftrightarrow\mathcal{D}(\vartheta)$, $S^{(\boldsymbol{\xi})}\leftrightarrow\partial\mathcal{D}(\vartheta)$.

intermolecular potential is spherically symmetric, which is discussed in MGD, the bulk viscosity is easily seen to vanish. From Eqs. (M-1.2d) and (M-1.2f),[4]

$$p_{ii} = 3p. \tag{5}$$

On the other hand, the trace of the first relation of Eq. (M-1.16) is

$$p_{ii} = 3p - 3\mu_B \frac{\partial v_i}{\partial X_i}.$$

Thus, from the two relations, we have

$$\mu_B = 0. \tag{6}$$

(Section 1.3: Version 7-00)

1.4 Note on the equality condition of Eq. (M-1.38)

The statement of the equality condition of Eq. (M-1.38), i.e., "The equality in Eq. (1.38) holds when and only when f is the Maxwellian that satisfies the boundary condition (1.26)... ", needs supplementary explanation. Some condition is required of the scattering kernel K_B in the boundary condition (M-1.26) for f to be limited to the Maxwellian. For some K_B, the equality holds in Eq. (M-1.38) for f other than the Maxwellian. See Section 8.4.1 for more detailed discussion.

(Section 1.4: Version 5-00)

1.5 Nondimensional form of the Boltzmann equation for an infinite-range potential (Sections M-1.9 and M-A.2.4)

1.5.1 Preliminary

As explained in page M-505, the Boltzmann equation for an infinite-range intermolecular potential is conventionally introduced by taking the limit $d_m \to \infty$ with the impact parameter b fixed. For this B, the mean collision frequency

[4]For molecules with internal degree of freedom (e.g., rotational and vibrational freedoms), this freedom contributes to the integrands of Eqs. (M-1.2c)–(M-1.2g). Thus, Eq. (5) does not generally hold. (More precisely, the velocity distribution function f depends also on the variables of the internal degree of freedom of a molecule. The integration with respect to these variables in Eqs. (M-1.2a)–(M-1.2g) has to be carried out. The angular momentum due to the rotation of molecules of infinitesimal size per unit mass is negligible even when the energy of rotation is not negligible.) The density ρ and the specific internal energy e can be clearly defined whether the gas is in an equilibrium state or not. The specific internal energy e/i_f per unit freedom of a molecule is taken as $RT/2$, i.e., $e = i_f RT/2$, where i_f is the degree of freedom of a molecule; thus, the relation between e and T is independent of the state of the gas (equilibrium or nonequilibrium). The pressure is defined by the equation of state, i.e., the perfect gas relation $p = \rho RT$; thus, except for a monatomic gas without internal degree of freedom, the pressure differs generally from the isotropic part of stress tensor in a nonequilibrium state.

$\bar{\nu}_c$ and the reference quantity B_0 in Eq. (M-1.48d) become infinite. Thus, the mean free path ℓ, defined by Eq. (M-1.20), and the nondimensional form $\hat{B}$ of B, introduced in Eq. (M-1.48c), are useless. Thus, the proper nondimensional form of the Boltzmann equation for an infinite-range potential is not presented yet. We will give it here.

In the collision term (M-1.6), the change of the variables of integration is introduced from $\boldsymbol{\alpha}$ or (θ_c, φ) to (b, φ), where b is the impact parameter (Section M-A.2.4). Noting the relations (M-A.59) and (M-A.60) and the range $(0, \infty)$ of b for an infinite-range potential, we obtain the collision term for an infinite-range potential in the following form:

$$J(f, f) = \frac{1}{m} \int_{\text{all } \boldsymbol{\xi}_*} \int_0^{2\pi} \int_0^{\infty} |\boldsymbol{\xi}_* - \boldsymbol{\xi}| [f(\boldsymbol{\xi}') f(\boldsymbol{\xi}'_*) - f(\boldsymbol{\xi}) f(\boldsymbol{\xi}_*)] b \mathrm{d}b \mathrm{d}\varphi \mathbf{d}\boldsymbol{\xi}_*, \quad (7)$$

where

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + [\boldsymbol{\alpha} \cdot (\boldsymbol{\xi}_* - \boldsymbol{\xi})]\boldsymbol{\alpha}, \quad \boldsymbol{\xi}'_* = \boldsymbol{\xi}_* - [\boldsymbol{\alpha} \cdot (\boldsymbol{\xi}_* - \boldsymbol{\xi})]\boldsymbol{\alpha}. \quad (8)$$

The unit vector $\boldsymbol{\alpha}$ is determined by (b, φ) with the aid of the relation between θ_c and b:[5]

$$\theta_c = \int_0^{y_c} \frac{1}{(1 - C - y^2)^{1/2}} \mathrm{d}y, \quad C = \frac{4U(b/y)}{m(\boldsymbol{\xi}_* - \boldsymbol{\xi})^2}, \quad (9)$$

where y_c is the smallest solution of the equation

$$1 - \frac{4U(b/y)}{m(\boldsymbol{\xi}_* - \boldsymbol{\xi})^2} - y^2 = 0 \ \ (0 < y < b/d_K). \quad (10)$$

The potential $U(r)$ is assumed here to tend to zero as $r \to \infty$ and to increase indefinitely as $r \to d_K \ (\geq 0)$.[6] In Eq. (7), the function B disappears, but in turn its effect enters the relation between $(\boldsymbol{\xi}', \boldsymbol{\xi}'_*)$ and b through the relation (9).

1.5.2 General Case

Let the potential $U(r)$ be given. Choosing the characteristic extent d_M of the potential (or the size of a molecule) properly, we can express the potential $U(r)$ in the form

$$U(r) = m\mathcal{U}_0 \tilde{U}(r/d_M), \quad (11)$$

where $\tilde{U}(x)$ is a nondimensional function of a nondimensional variable x that takes the value unity at $x = 1$, tends to zero as $x \to \infty$, and increases indefinitely as $x \to \hat{d}_K \ (= d_K/d_M \leq 1)$; $\mathcal{U}_0$ is a constant of the order of RT_0.[7] Introducing the nondimensional impact parameter $\hat{b}$ by

[5] As explained in Section M-A.2.4, (θ_c, φ) is $\boldsymbol{\alpha}$ or $-\boldsymbol{\alpha}$. The relation between (θ_c, φ) and (θ_α, φ) under the convention $\boldsymbol{\alpha} \cdot (\boldsymbol{\xi}_* - \boldsymbol{\xi}) > 0$ introduced there, where $\boldsymbol{\alpha} = (\theta_\alpha, \varphi)$, is given in the second paragraph of page M-503.

[6] The case where U approaches a finite value as $r \to d_K$ and an infinitely high potential barrier lies at $r = d_K$ is included. A similar note applies to $\tilde{U}(x)$ in Eq. (11).

[7] (i) The symbols $\mathcal{U}_0$ and $\tilde{U}$ are chosen to avoid the confusion with U_0 and $\hat{U}$ introduced in Eq. (M-A.51).

$$\hat{b} = b/d_M, \tag{12}$$

we rewrite the collision-term formulas (7)–(10) in terms of the nondimensional variables $\tilde{U}$ and $\hat{b}$, those introduced in Eq. (M-1.43), and the corresponding reference quantities.[8] The result is

$$J(f,f) = \frac{\rho_0^2 d_M^2}{2RT_0 m} \int_{\substack{0\le\hat{b}<\infty \\ 0\le\varphi<2\pi \\ \text{all } \boldsymbol{\zeta}_*}} |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}| [\hat{f}(\boldsymbol{\zeta}')\hat{f}(\boldsymbol{\zeta}_*') - \hat{f}(\boldsymbol{\zeta})\hat{f}(\boldsymbol{\zeta}_*)] \hat{b}\mathrm{d}\hat{b}\mathrm{d}\varphi \mathbf{d}\boldsymbol{\zeta}_*, \tag{13}$$

where

$$\boldsymbol{\zeta}' = \boldsymbol{\zeta} + [\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})]\boldsymbol{\alpha}, \quad \boldsymbol{\zeta}_*' = \boldsymbol{\zeta}_* - [\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})]\boldsymbol{\alpha}. \tag{14}$$

The unit vector $\boldsymbol{\alpha}$ is determined by $\hat{b}$, φ, and $\boldsymbol{\zeta}_* - \boldsymbol{\zeta}$ with the aid of θ_c (see Footnote 5):

$$\theta_c = \int_0^{\bar{y}_c} \frac{1}{\left(1 - \bar{C} - \bar{y}^2\right)^{1/2}} \mathrm{d}\bar{y}, \tag{15}$$

where

$$\bar{C} = \frac{2\mathcal{U}_0 \tilde{U}(\hat{b}/\bar{y})}{RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2}, \tag{16}$$

and $\bar{y}_c$ is the smallest positive solution of the equation for $\bar{y}$:

$$1 - \bar{C} - \bar{y}^2 = 0 \quad \left(0 < \bar{y} < \hat{b}/\hat{d}_K\right). \tag{17}$$

Then, $\bar{y}_c$ is a function of $\hat{b}$ and $\mathcal{U}_0/RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2$; the integrand in Eq. (15) is also a function of the same variables. Thus, θ_c is a function of $\hat{b}$ and $\mathcal{U}_0/RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2$,

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(ii) From $U(r)$ and d_M, the function $\tilde{U}(x)$ and the constant $\mathcal{U}_0$ are determined as $\tilde{U}(x) = U(d_M x)/U(d_M)$ and $\mathcal{U}_0 = U(d_M)/m$. If we choose d_M in such a way that $U(d_M)/m$ is of the order of RT_0, the required properties of $\tilde{U}(x)$ and $\mathcal{U}_0$ are satisfied. Such a choice of d_M is possible owing to the behavior of $U(r)$.

(iii) The size d_M of a molecule is an important factor of $\tilde{k}$ defined by Eq. (21), which is chosen to indicate the magnitude of the collision term. Depending on the choice of d_M, the nondimentional collision integral (the integral part of the collision term) in Eq. (20) or (22) can be too large or too small. This happens when $\mathcal{U}_0/2RT_0$ in Eq. (16) is too large or too small (note that the case $\theta_c = \pi/2$, which occurs for $\bar{C} = 0$, corresponds to the case without interaction between molecules). Then, $\tilde{k}$ is not a good indicator of the magnitude of the collision term. Thus, d_M should be chosen so as for $\mathcal{U}_0/2RT_0$ to be of the order of unity (say, $\mathcal{U}_0/2RT_0 = \alpha_{\mathrm{pot}}$). For a given $U(r)$ and T_0, the size d_M is determined with an α_{pot} (for example, $\alpha_{\mathrm{pot}} = 1$); then $\mathcal{U}_0 = U(d_M)/m = 2\alpha_{\mathrm{pot}} RT_0$. For another reference temperature T_0^*, d_M is kept unchanged, and $\mathcal{U}_0$, accordingly, remains unchanged. Then, $\mathcal{U}_0/2RT_0^* = (T_0/T_0^*)\alpha_{\mathrm{pot}}$. This factor, $\mathcal{U}_0/2RT_0^*$, enters the collision integral through $\bar{C}$ in Eq. (15). For the reference state (ρ_0, T_0^*), the collision term is determined by the two parameters T_0^*/T_0 and $\tilde{k}$ based on d_M determined by T_0 as explained above though there is ambiguity due to T_0 or α_{pot}. The dependence of the collision term on T_0^*/T_0 is not widely mentioned.

(iv) The ambiguity of the size d_M due to the choice of T_0 or α_{pot} is of the same kind as that of a reference length and the thickness of shock wave or Knudsen layer, etc.

[8]The present way to obtain the nondimensional equation can be applied to a finite-range potential.

i.e.,

$$\theta_c = f_{\theta_c}(\hat{b}, \mathcal{U}_0/RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2), \tag{18a}$$

or

$$\hat{b} = f_{\hat{b}}(\theta_c, \mathcal{U}_0/RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2), \tag{18b}$$

where the functional forms of f_{θ_c} and $f_{\hat{b}}$ are determined only by $\tilde{U}(x)$.

The transport term (or the left-hand side) of the Boltzmann equation (M-1.5) is rewritten as

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial X_i} + \frac{\partial F_i f}{\partial \xi_i} = \frac{\rho_0}{2RT_0 L}\left(\mathrm{Sh}\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_i \frac{\partial \hat{f}}{\partial x_i} + \frac{\partial \hat{F}_i \hat{f}}{\partial \zeta_i}\right). \tag{19}$$

Comparing the two expressions (13) and (19), we obtain the following nondimensional form of the Boltzmann equation for an intermolecular potential of infinite range:

$$\mathrm{Sh}\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_i \frac{\partial \hat{f}}{\partial x_i} + \frac{\partial \hat{F}_i \hat{f}}{\partial \zeta_i} = \frac{1}{\tilde{k}}\int_{\substack{0\le \hat{b}<\infty \\ 0\le\varphi<2\pi \\ \text{all } \boldsymbol{\zeta}_*}} |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|[\hat{f}(\boldsymbol{\zeta}')\hat{f}(\boldsymbol{\zeta}'_*) - \hat{f}(\boldsymbol{\zeta})\hat{f}(\boldsymbol{\zeta}_*)]\hat{b}\mathrm{d}\hat{b}\mathrm{d}\varphi\mathbf{d}\boldsymbol{\zeta}_*, \tag{20}$$

where

$$\tilde{k} = 1/(\rho_0/m)d_M^2 L. \tag{21}$$

Changing the variables of integration from $(\hat{b}, \varphi)$ to $\boldsymbol{\alpha}$, we have another form of Eq. (20) with the B function in the collision term:

$$\mathrm{Sh}\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_i \frac{\partial \hat{f}}{\partial x_i} + \frac{\partial \hat{F}_i \hat{f}}{\partial \zeta_i} = \frac{1}{\tilde{k}}\int_{\text{all } \boldsymbol{\alpha},\, \text{all } \boldsymbol{\zeta}_*} [\hat{f}(\boldsymbol{\zeta}')\hat{f}(\boldsymbol{\zeta}'_*) - \hat{f}(\boldsymbol{\zeta})\hat{f}(\boldsymbol{\zeta}_*)]\tilde{B}\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_*, \tag{22}$$

where[9]

$$\tilde{B}(|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|/|\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|, \mathcal{U}_0/2RT_0) = \frac{|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|}{2}\left|\frac{f_{\hat{b}}}{\sin\theta_c}\frac{\partial f_{\hat{b}}(\theta_c, \mathcal{U}_0/RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2)}{\partial \theta_c}\right|. \tag{23}$$

The nondimensional form of the collision term contains the two parameters $\tilde{k}$ and $\mathcal{U}_0/2RT_0$,[10] which consist of macroscopic and molecular variables. For the correct handling of the molecular variables, some discussions are required, which will be given in Section 1.5.3.[11]

[9](i) See Footnote 5.

(ii) The range of integration with respect to $\boldsymbol{\alpha}$ in the integral on the right-hand side of Eq. (22), which is originally $\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta}) > 0$, is extended to the whole range of $\boldsymbol{\alpha}$ by putting the absolute-value sign on $\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})$ in the argument of $\tilde{B}$. Thus, $\tilde{B}$ is multiplied by 1/2 in Eq. (23).

[10]It should be noted that the parameter $\mathcal{U}_0/2RT_0$ enters Eq. (20) through the relation between $(\hat{b}, \varphi)$ and $\boldsymbol{\alpha}$ [see Eq. (18a)].

[11]It is generally said that the limit where the parameters m, d_M, and $U(d_M)$ [note: $\mathcal{U}_0 = U(d_M)/m$] tend to zero is taken in the derivation of the Boltzmann equation. Without paying attention to their relative speeds of approach to zero and putting them zero simply in the Boltzmann equation (20) or (22), we have a trouble. We have to review the background of the derivation of the Boltzmann equation.

1.5.3 Background of the Boltzmann equation and its parameters

Before discussing the parameters in the Boltzmann equation (20) or (22), it may be in order to review the situation of a monatomic gas the description of which is the purpose of the Boltzmann equation. A gas consists of very many molecules in a reference volume of our interest in discussing its behavior (and even in a very small volume in the scale of the reference volume), and its variables, such as density, flow velocity, and temperature, as a group of so many molecules are defined at a point (in the scale of our interest) in space and time. The reference quantities are set from the situation of our interest. Our interest is the behavior of a monatomic gas. The reference quantities are properly chosen for the description or analysis of it. Hereafter, the expression "G-reference" is used for this when the distinction with molecular quantities is preferable. In the situation of the present interest, the molecular size d_M and the molecular mass m are, respectively, very small compared with the G-reference length L and the mass $\rho_0 L^3$ in the G-reference volume L^3, i.e.,

$$d_M/L \ll 1, \tag{24a}$$

$$m/\rho_0 L^3 \ll 1, \tag{24b}$$

where ρ_0 is the reference density. Thus, very many molecules are in volume L^3 (in a unit volume),[12] i.e.,

$$n_0 L^3 \gg 1 \ \ (n_0 = \rho_0/m). \tag{25}$$

The mean value of the molecular velocities is the flow velocity of the gas, and their standard deviation is the sound speed or $(RT_0)^{1/2}$ except for a constant factor. We are interested in the situation where the flow speed is expressed in its Mach number or its scale is at the level of $(RT_0)^{1/2}$. Therefore, flow velocity and molecular velocity are expressed with $(RT_0)^{1/2}$ as their unit, or the G-reference scale and the molecular scale for velocity are commonly $(RT_0)^{1/2}$ in contrast to the mass and the linear dimension.[13]

Here, we are interested in the behavior of the above-mentioned gas in the case where the gas is in a state with $(\rho_0/m)d_M^2 L$ being at a nonzero finite value,[14] i.e.,

$$0 < (\rho_0/m)d_M^2 L < \infty. \tag{26}$$

Put it be C_L, i.e.,

$$C_L = (\rho_0/m)d_M^2 L, \tag{27}$$

[12]When we mention that the dimensional quantity n_0 is large, it is implicitly assumed that the unit volume is of the G-reference size. This kind of expression is common. For example, the mean free path is small. In this case, we compare it with the length under consideration or of our daily life. Adequate care is required when dealing with reference quantities of different scales.

[13]In the discussion of intermolecular collisions, only the relative velocity $\boldsymbol{\xi}_* - \boldsymbol{\xi}$ is important. Its characteristic size is at the level of $(RT_0)^{1/2}$, irrespective of the flow velocity of the gas.

[14]The $\pi(\rho_0/m)d_M^2 L$ is the number of molecules of a gas with density ρ_0 in a circular cylinder with radius d_M and length L. Thus, it is roughly the frequency of collision while a molecule travels distance L, or $1/\pi(\rho_0/m)d_M^2$ is roughly the mean free path, which is about 0.06 μm for air at the atmospheric condition (see Table M-C.1 in Section M-C.2).

where C_L is a nonzero finite value. In this situation, $(\rho_0/m)d_M^3 \ll 1$ because of Eq. (24a). The scale of velocity being common to the G-reference and the molecular reference scales, the time scales in the two view points are different. The time for a molecule to interact with another is of the order of $d_M/(RT_0)^{1/2}$, but the time to travel with speed $(RT_0)^{1/2}$ for the G-reference length L or the mean free path $1/(\rho_0/m)d_M^2$ is of the order of $L/(RT_0)^{1/2}$ or $1/(\rho_0/m)d_M^2(RT_0)^{1/2}$, the latter of which is the mean free time, i.e., the average time between two successive collisions of a molecule, and of the same order as the former because C_L is a nonzero finite value [note:$1/(\rho_0/m)d_M^2(RT_0)^{1/2} = L/C_L(RT_0)^{1/2}$]. The molecular-time scale [say, $t_{\mathrm{mol}}\,(= d_M/(RT_0)^{1/2})$] is very much smaller than the G-time scale [say, $t_L\,(= L/(RT_0)^{1/2}$] because of Eq. (24a), i.e.,

$$t_{\mathrm{mol}}/t_L \ll 1. \tag{28}$$

In the above discussion, another length scale L_δ and another time scale t_δ are implicitly introduced, which has the following conditions:

$$d_M/L_\delta \ll 1, \; L_\delta/L \ll 1, \tag{29a}$$

$$m/\rho_0 L_\delta^3 \ll 1 \;\; (n_0 L_\delta^3 \gg 1), \tag{29b}$$

$$t_{\mathrm{mol}}/t_\delta \ll 1, \; t_\delta/t_L \ll 1, \tag{29c}$$

where t_δ is the time to travel with speed $(RT_0)^{1/2}$ for distance L_δ, i.e., $t_\delta = L_\delta/(RT_0)^{1/2}$. This definition of t_δ is consistent with Eq. (29c) because of Eqs. (26) and (29a). By the introduction of L_δ and t_δ, we can define the local gas dynamic variables in space and time.[15] The length and time scales of variation of these variables are, respectively, L and t_L by their definition. Equation (29b) means that the average volume of the gas in which one molecule lies is much smaller than L_δ^3; that is, the distance d_{sp} between two neighboring molecules is much small than L_δ ($d_{\mathrm{sp}}/L_\delta \ll 1$). From Eq. (27) with nonzero finite C_L and Eq. (29a), we have

$$\frac{L_\delta}{1/(\rho_0/m)d_M^2} \ll 1. \tag{30}$$

That is, L_δ is much smaller than the mean free path $1/n_0 d_M^2$. In the time scale of t_δ, the molecules in a volume L_δ^3 stay in it and do not make collision because of the second relation of Eq. (29c), and none of them is in the process of interaction with another molecule because of the first relation of Eq. (29c). The molecules keep their velocity unchanged. Thus, the state in the volume L_δ^3 remains unchanged in that time scale. That is, the state at a point in space

[15](i) Condition (29b) is essential to this.
(ii) A point $\boldsymbol{X}$ and its neighborhood of the order of L_δ are taken as the point $\boldsymbol{X}$ in a gas (or in G scale). A time t and its neighborhood of the order of t_δ are taken as the time t in G scale. Accordingly, a molecular velocity $\boldsymbol{\xi}$ and its neighborhood of the order of $(RT_0)^{1/2}L_\delta/L$ are the molecular velocity $\boldsymbol{\xi}$ in G scale because all motions between two space-time points $(\boldsymbol{X}^{(0)}, t^{(0)})$ and $(\boldsymbol{X}^{(1)}, t^{(1)})$ in G scale with the above-mentioned allowance of the neighborhood are taken to have the same velocity in G scale. Local gas dynamic variables (G scale) are defined with the data over the above-mentioned neighborhood of the point under interest.

and time of the L_δ scale is well defined. For the convenience of the following discussion, we here introduce the notation:

$$\hat{d}_{\mathrm{re}} = d_M/L_\delta, \quad \hat{L}_\delta = L_\delta/L, \quad \hat{m}_{\mathrm{re}} = m/\rho_0 L_\delta^3. \tag{31}$$

To describe the behavior of the gas (or in the derivation of the Boltzmann equation), the limiting case where $\hat{d}_{\mathrm{re}}\,(= d_M/L_\delta) \to 0$, $\hat{L}_\delta\,(= L_\delta/L) \to 0$, and $\hat{m}_{\mathrm{re}}\,(= m/\rho_0 L_\delta^3) \to 0$ with $(\rho_0/m)d_M^2 L$ fixed at a nonzero finite value is considered (the Grad–Boltzmann limit), and the equation that determines the G-scale behavior of the limiting system is established.[16] First, the velocity distribution function that expresses the state of the gas (G-scale state) is introduced, and then the equation (the Boltzmann equation) that describes the variation of the velocity distribution function is derived. Obviously by definition, the velocity distribution function or the Boltzmann equation neither discriminates positions with difference of molecular size, nor describes the variation over that size. The transport term of the Boltzmann equation, the left-hand side of Eq. (M-1.5), is derived only by the discussion of the G-reference level. On the other hand, the collision term, the right-hand side of Eq. (M-1.5), is discussed by magnifying the scales of molecular parameters (mass, radius, position, intermolecular potential), and the frequency of intermolecular collision and the shift of molecular velocities by collision are calculated.[17] Thus, some quantities of molecular level are apparently included in the collision term. In Eq. (20) or (22), m and d_M appear in $\tilde{k}$ as the combination $(\rho_0/m)d_M^2 L$, which is fixed in the limiting process; thus, the real molecular data can be put in m and d_M. In addition to $\tilde{k}$, the collision term depends on the parameter $\mathcal{U}_0/2RT_0$,[18] which will be shown to be invariant in the limiting process in the next paragraph. Thus, the nondimensional Boltzmann equation (20) or (22) is expressed with the quantities invariant in the limiting process.

We discuss the dependence of $\mathcal{U}_0$ in the potential (11) on the scale factors $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$. Let the potential $U(r)$ be given. From the profile, we determine d_M, which has ambiguity (see Footnote 7), and rewrite $U(r)$ in the form

$$U = m\mathcal{U}_0 \tilde{U}(r/d_M), \tag{32}$$

where $\mathcal{U}_0$ has the dimension of RT_0. Take a given set of a molecule and a potential (or a given pair of molecules). Let the molecule be approaching the potential field with a relative velocity $(2RT_0)^{1/2}(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})$ and a relative position

[16](i) In this limit, $n_0 d_M^3 \to 0$ ($n_0 = \rho_0/m$), $(\rho_0/m)d_M^2 L_\delta \to 0$, $t_{\mathrm{mol}}/t_\delta \to 0$, and $t_\delta/t_L \to 0$. The first one shows that the volume of the molecules in a volume of a gas is negligible to the volume that the gas occupies.

(ii) The case where $(\rho_0/m)d_M^2 L$ is independent of $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$ is considered here.

(iii) In the gas under consideration, the scale parameters $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$ are so small that its behavior is well approximated by the solution of the equation obtained in the limit. This is the underlying assumption in the derivation of the Boltzmann equation.

[17]In the discussion of the collision term of the Boltzmann equation, the collision effect at each point of the gas is discussed with the binary collision of molecules in a volume of L_δ^3 scale.

[18]See Footnote 10

$(\hat{b},\varphi)$. Obviously, the reduced trajectory $\hat{r} = f_{\mathrm{Tr}}(\theta;\hat{b},\mathcal{U}_0/2RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2)$ of the binary collision, where $\hat{r} = r/d_M$, is independent of the G-reference scale L. So is θ_c. That is, these results are invariant in the limiting process that $\hat{d}_{\mathrm{re}} \to 0$, $\hat{L}_\delta \to 0$, and $\hat{m}_{\mathrm{re}} \to 0$. We examine the condition that Eq. (M-A.50) for the trajectory gives a solution that satisfies the above invariant condition, and easily find that $\mathcal{U}_0/2RT_0$ must be invariant in the limiting process.[19] From this invariant property of $\mathcal{U}_0/2RT_0$, we can choose the real molecular data for $\mathcal{U}_0$; that is, once we have chosen d_M for the real potential U, $\mathcal{U}_0$ is determined as

$$\mathcal{U}_0 = U(d_M)/m, \tag{33}$$

where the real molecular data of m and d_M are used. Thus, the nondimensional Boltzmann equation (20) or (22) is expressed with the parameters that are

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[19](i) With the relations $r = \hat{r}d_M$ and $b = \hat{b}d_M$ in Eq. (M-A.50), it is reduced to

$$\frac{\hat{b}^2}{\hat{r}^4}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta}\right)^2 = 1 - \frac{2\mathcal{U}_0\tilde{U}(\hat{r})}{RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2} - \frac{\hat{b}^2}{\hat{r}^2}.$$

Thus, the reduced trajectory $\hat{r} = f_{\mathrm{Tr}}(\theta;\hat{b},\mathcal{U}_0/2RT_0(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})^2)$ is required to be independent of $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$. This condition requires that $\mathcal{U}_0/RT_0$ is invariant in the limiting process.

(ii) The Boltzmann equation is not derived yet for an infinite-range potential which really extends up to infinity in the G-reference length . What is called the Boltzmann equation for an infinite-range potential is conventionally obtained as the limiting result of the corresponding finite-range potential confined in a L_δ^3 volume. The infinity is in the scale of d_M and the effect of the potential on the molecules outside the L_δ^3 volume is not counted. For an infinite-range potential $U(r)$, the corresponding cutoff potential $U_{\mathrm{cut}}(r)$ is defined by cutting off the tail of $U(r)$ for $r > d_m$, i.e., $U_{\mathrm{cut}} = U$ for $r \le d_m$ and $U_{\mathrm{cut}} = 0$ for $r > d_m$. Let the B function for the finite-range potential U_{cut} be $B_{d_m}^{\mathrm{cut}}$. Then, the limit of $B_{d_m}^{\mathrm{cut}}$ as $d_m/d_M \to \infty$ is taken under the condition that b/d_M is fixed at a finite value in the limiting process. Let the result be B_∞^*. The Boltzmann equation in which this B_∞^* is adopted as B is conventionally called the Boltzmann equation for the infinite-range potential $U(r)$. The term "conventionally" is used by the reason that the contribution of the case where $\lim_{d_m/d_M\to\infty} b/d_m > 0$ is not precisely estimated but is neglected, in addition to the note mentioned at the beginning. Let the potential for infinite range be given in the form

$$U(r) = m\mathcal{U}_0\tilde{U}(r/d_M).$$

Then, the potential U_{cut} is expressed as

$$U_{\mathrm{cut}} = m\mathcal{U}_0\tilde{U}_{\mathrm{cut}}(r/d_M),$$

where $\tilde{U}_{\mathrm{cut}}(x) = \tilde{U}(x)$ for $x \le d_m/d_M$ and $\tilde{U}_{\mathrm{cut}}(x) = 0$ for $x > d_m/d_M$. The $\mathcal{U}_0$ is common to the infinite-range potential and all the cutoff potentials. For each d_m/d_M, $\mathcal{U}_0/2RT_0$ is invariant with respect to $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$ from the trajectory discussion. Thus, $\mathcal{U}_0/2RT_0$ in Eq. (20) or (22) is invariant in the limiting process (see Footnote 10).

(iii) In the nondimensional form $\hat{B}$ given in Eq. (M-A.71) for a finite-range potential, U_0 corresponds to $m\mathcal{U}_0$ here. The U_0 in Section M-A.2.4 is better replaced by $m\overline{\mathcal{U}}_0$ because $\overline{\mathcal{U}}_0/RT_0$ is free from the scale factors $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$. The symbol $\overline{\mathcal{U}}_0$ different from $\mathcal{U}_0$ is used because of difference of the behavior of the nondimensional functions $\hat{U}(x)$ in Section M-A.2.4 and $\tilde{U}(x)$ here. As the result, the argument $(2mRT_0/U_0)^{1/2}|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|$ of $\hat{B}$ there is rewritten as $(2RT_0/\overline{\mathcal{U}}_0)^{1/2}|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|$. Owing to its invariance in the limiting process, $\overline{\mathcal{U}}_0$ is determined from the real molecular data of U. The U_0 (or $m\overline{\mathcal{U}}_0$) and $\hat{U}(x)$ in the potential $U(r)$ $[= U_0\hat{U}(r/d_m)]$ are determined as follows: First, d_M, $\mathcal{U}_0$, and $\tilde{U}(x)$ are determined from $U(r)$ $[= m\mathcal{U}_0\tilde{U}(r/d_M)]$ in the same way as for an infinite-range potential described in Footnote 7 (iii). From the result, $\overline{\mathcal{U}}_0$ and $\hat{U}(x)$ are determined as $\overline{\mathcal{U}}_0 = \mathcal{U}_0$ and $\hat{U}(x) = \tilde{U}(d_m x/d_M)$.

invariant in the limiting process. Finally, it should be noted that the potential or the molecule changes in the limiting process unless $\mathcal{U}_0$ is invariant.

1.5.4 Inverse-power potential

The collision term for the inverse-power potential is given by Eq. (M-A.64) as

$$J(f,f) = \frac{1}{m}\left(\frac{4a_0}{m}\right)^{\frac{2}{n-1}} \int_{\substack{0\le g<\infty \\ 0\le\varphi<2\pi \\ \text{all } \boldsymbol{\xi}_*}} (f'f'_* - ff_*)|\boldsymbol{\xi}_* - \boldsymbol{\xi}|^{\frac{n-5}{n-1}} g\mathrm{d}g\mathrm{d}\varphi\mathbf{d}\boldsymbol{\xi}_*, \quad (34)$$

where the intermolecular potential $U(r)$ [Eq. (M-A.49a)] is given by

$$U(r) = \frac{a_0}{r^{n-1}} \quad (a_0 > 0,\ n > 1), \quad (35)$$

and $\boldsymbol{\alpha}$ or (θ_c, φ) in f' and f'_* is determined only by g, φ, and n [see Eq. (M-A.62a) and (M-A.62b)].[20] With the use of the nondimensional variables introduced in Eq. (M-1.43), the collision term (34) is rewritten in the form

$$J(f,f) = \frac{1}{m}\left(\frac{4a_0}{m}\right)^{\frac{2}{n-1}} \frac{\rho_0^2}{(2RT_0)^{1+\frac{2}{n-1}}} \int_{\substack{0\le g<\infty \\ 0\le\varphi<2\pi \\ \text{all } \boldsymbol{\zeta}_*}} (\hat{f}'\hat{f}'_* - \hat{f}\hat{f}_*)|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|^{\frac{n-5}{n-1}} g\mathrm{d}g\mathrm{d}\varphi\mathbf{d}\boldsymbol{\zeta}_*. \quad (36)$$

The variables $\boldsymbol{\zeta}'$ and $\boldsymbol{\zeta}'_*$ in $\hat{f}'$ and $\hat{f}'_*$ are given by Eq. (14) with the aid of θ_c:

$$\theta_c = \int_0^{y_c(g)} \left[1 - \left(\frac{y}{g}\right)^{n-1} - y^2\right]^{-1/2} \mathrm{d}y, \quad (37)$$

where $y_c(g)$ is the positive solution, which is unique, of the equation

$$1 - (y/g)^{n-1} - y^2 = 0 \quad (0 < y < \infty). \quad (38)$$

The transport term (or the left-hand side) of the Boltzmann equation (M-1.5) is expressed as

$$\frac{\partial f}{\partial t} + \xi_i\frac{\partial f}{\partial X_i} + \frac{\partial F_i f}{\partial \xi_i} = \frac{\rho_0}{2RT_0 L}\left(\mathrm{Sh}\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_i\frac{\partial \hat{f}}{\partial x_i} + \frac{\partial \hat{F}_i\hat{f}}{\partial \zeta_i}\right). \quad (39)$$

From the two expressions (36) and (39), we have the following nondimensional form of the Boltzmann equation:

$$\mathrm{Sh}\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_i\frac{\partial \hat{f}}{\partial x_i} + \frac{\partial \hat{F}_i\hat{f}}{\partial \zeta_i} = \frac{1}{\bar{k}_{\mathrm{inv}}} \int_{\substack{0\le g<\infty \\ 0\le\varphi<2\pi \\ \text{all } \boldsymbol{\zeta}_*}} (\hat{f}'\hat{f}'_* - \hat{f}\hat{f}_*)|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|^{\frac{n-5}{n-1}} g\mathrm{d}g\mathrm{d}\varphi\mathbf{d}\boldsymbol{\zeta}_*, \quad (40)$$

[20]The variable b (the impact parameter) of integration is replaced by the nondimensional variable g defined by $g = (m/4a_0)^{1/(n-1)}|\boldsymbol{\xi}_* - \boldsymbol{\xi}|^{2/(n-1)}b$ [see Eq. (M-A61)].

where

$$\bar{k}_{\mathrm{inv}} = \frac{1}{(\rho_0/m)(2a_0/mRT_0)^{\frac{2}{n-1}}}\frac{1}{L}. \tag{41}$$

The integral on the right-hand side of Eq. (40), including the relation (37) between $(\boldsymbol{\zeta}', \boldsymbol{\zeta}'_*)$ and (g, φ), expressed in nondimensional variables does not contain parameters except n.[21] It is finite when $n > 3$ for a smooth $\hat{f}$ (Section M-A.2.4). Thus, $1/\bar{k}_{\mathrm{inv}}$ is the only parameter in the collision term and expresses the weight of the collision term in the Boltzmann equation (40). The constant $(2a_0/mRT_0)^{1/(n-1)}$ has the dimension of length. Let it be indicated by $\bar{d}_{\mathrm{inv}}$, i.e.

$$\bar{d}_{\mathrm{inv}} = \left(\frac{2a_0/m}{RT_0}\right)^{\frac{1}{n-1}}. \tag{42}$$

Then,

$$\bar{k}_{\mathrm{inv}} = \frac{1}{(\rho_0/m)\bar{d}_{\mathrm{inv}}^2 L}. \tag{43}$$

For a finite $\bar{k}_{\mathrm{inv}}$, $\bar{d}_{\mathrm{inv}}/L$ tends to zero in the limit $\rho_0 L^3/m \to \infty$.

In order to examine the invariance of $\bar{k}_{\mathrm{inv}}$ in the limiting process, we rewrite Eq. (35) in the form (11) as

$$U(r) = \frac{m\mathcal{U}_0}{(r/d_M)^{n-1}} \quad (n > 1), \tag{44}$$

where $\mathcal{U}_0/(RT_0)^{1/2}$ is independent of the scale factors $\hat{d}_{\mathrm{re}}$, $\hat{L}_\delta$, and $\hat{m}_{\mathrm{re}}$.[22] From Eqs. (35) and (44),

$$a_0 = m\mathcal{U}_0 d_M^{n-1}. \tag{45}$$

With this a_0 in Eq. (41), $\bar{k}_{\mathrm{inv}}$ is expressed as

$$\bar{k}_{\mathrm{inv}} = \frac{1}{(2\mathcal{U}_0/RT_0)^{\frac{2}{n-1}}(\rho_0/m)d_M^2 L} = \frac{\tilde{k}}{(2\mathcal{U}_0/RT_0)^{\frac{2}{n-1}}}. \tag{46}$$

In the limiting process, both $\tilde{k}$ and $2\mathcal{U}_0/RT_0$ are invariant. So is $\bar{k}_{\mathrm{inv}}$ from Eq. (46). From the invariance of $\bar{k}_{\mathrm{inv}}$ in the limiting process, $\bar{k}_{\mathrm{inv}}$ can be calculated by Eq. (41) with the real molecular data of m and a_0. The result is independent of the choice of d_M. For an inverse-power potential, the effects of the two parameters $\tilde{k}$ and $2\mathcal{U}_0/RT_0$ on the collision term are combined in the single parameter $\bar{k}_{\mathrm{inv}}$. In view of Eqs. (20), (21), (40), (41), and (43), the parameters $\bar{k}_{\mathrm{inv}}$ and $\bar{d}_{\mathrm{inv}}$ may be called, respectively, a reduced Knudsen number and a reduced molecular size.

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[21]The parameters a_0 and T_0 do not enter α in $\hat{f}'$ and $\hat{f}'_*$. They enter $\bar{k}_{\mathrm{inv}}$ combined in the form $2a_0/mRT_0$.

[22]The choice of d_M is arbitrary for the homogeneous potential, $U(br) = b^{-(n-1)}U(r)$, with a single parameter. The result will be seen to be independent of d_M.

1.6 Supplement to Footnote M-26 in Chapter M-1

Footnote M-26 is supplemented with more explicit mathematical expressions for the process given there. Take the non-dimensional form of the equation for the H function, i.e., Eq. (M-1.72):

$$\mathrm{Sh}\frac{\partial \hat{H}}{\partial \hat{t}} + \frac{\partial \hat{H}_i}{\partial x_i} = \frac{1}{k}\hat{G}, \tag{47}$$

where

$$\left.\begin{aligned} &\hat{H}(x_i,\hat{t}) = \int \hat{f}\ln(\hat{f}/\hat{c}_0)\mathbf{d}\boldsymbol{\zeta}, \quad \hat{H}_i(x_i,\hat{t}) = \int \zeta_i \hat{f}\ln(\hat{f}/\hat{c}_0)\mathbf{d}\boldsymbol{\zeta}, \\ &\hat{G} = -\frac{1}{4}\int (\hat{f}'\hat{f}'_* - \hat{f}\hat{f}_*)\ln\left(\frac{\hat{f}'\hat{f}'_*}{\hat{f}\hat{f}_*}\right)\widehat{B}\mathrm{d}\Omega\mathbf{d}\boldsymbol{\zeta}_*\mathbf{d}\boldsymbol{\zeta} \le 0, \end{aligned}\right\} \tag{48}$$

with $\hat{c}_0 = c_0(2RT_0)^{3/2}/\rho_0$. The perturbed form of the velocity distribution function $\hat{f}$ is defined by

$$\hat{f} = E(1+\phi), \tag{49}$$

where

$$E = \frac{1}{\pi^{3/2}}\exp(-\zeta^2).$$

Let ε be a small quantity. Here, we take the case in which ϕ is of the order of ε, and examine the terms of the order of ε^2 of Eq. (47). The perturbed function ϕ is expressed as

$$\phi = \phi_1\varepsilon + \phi_2\varepsilon^2 + \cdots . \tag{50}$$

Corresponding to the expansion, the macroscopic variables, i.e., ω, u_i, P, etc., $\hat{H}$, $\hat{H}_i$, and $\hat{G}$ are also expressed as

$$h = h_1\varepsilon + h_2\varepsilon^2 + \cdots , \tag{51a}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1\varepsilon + \hat{H}_2\varepsilon^2 + \cdots , \tag{51b}$$

$$\hat{H}_i = \hat{H}_{i0} + \hat{H}_{i1}\varepsilon + \hat{H}_{i2}\varepsilon^2 + \cdots , \tag{51c}$$

$$\hat{G} = \hat{G}_0 + \hat{G}_1\varepsilon + \hat{G}_2\varepsilon^2 + \cdots , \tag{51d}$$

where h represents the perturbed macroscopic variables, ω, u_i, P, etc., and the quantities ϕ_n, h_n, $\hat{H}_n$, $\hat{H}_{in}$, and $\hat{G}_n$ are of the order of unity. Then, with the aid of the expanded forms of Eqs. (M-1.78a)–(M-1.78f), $\hat{H}_n$, $\hat{H}_{in}$, and $\hat{G}_n$ are expressed as

$$\hat{H}_0 = -\frac{3}{2} - \ln \pi^{3/2}\hat{c}_0, \tag{52a}$$

$$\begin{aligned}\hat{H}_1 &= (1-\ln \pi^{3/2}\hat{c}_0)\int E\phi_1 \mathbf{d\zeta} - \int \zeta^2 E\phi_1 \mathbf{d\zeta} \\ &= (1-\ln \pi^{3/2}\hat{c}_0)\omega_1 - \frac{3}{2}P_1,\end{aligned} \tag{52b}$$

$$\begin{aligned}\hat{H}_2 &= (1-\ln \pi^{3/2}\hat{c}_0)\int E\phi_2 \mathbf{d\zeta} - \int \zeta^2 E\phi_2 \mathbf{d\zeta} + \frac{1}{2}\int E\phi_1^2 \mathbf{d\zeta} \\ &= (1-\ln \pi^{3/2}\hat{c}_0)\omega_2 - \left(\frac{3}{2}P_2 + u_{i1}^2\right) + \frac{1}{2}\int E\phi_1^2 \mathbf{d\zeta},\end{aligned} \tag{52c}$$

$$\hat{H}_{i0} = 0, \tag{53a}$$

$$\begin{aligned}\hat{H}_{i1} &= (1-\ln \pi^{3/2}\hat{c}_0)\int \zeta_i E\phi_1 \mathbf{d\zeta} - \int \zeta_i\zeta^2 E\phi_1 \mathbf{d\zeta} \\ &= (1-\ln \pi^{3/2}\hat{c}_0)u_{i1} - \left(Q_{i1} + \frac{5}{2}u_{i1}\right),\end{aligned} \tag{53b}$$

$$\begin{aligned}\hat{H}_{i2} &= (1-\ln \pi^{3/2}\hat{c}_0)\int \zeta_i E\phi_2 \mathbf{d\zeta} - \int \zeta_i\zeta^2 E\phi_2 \mathbf{d\zeta} + \frac{1}{2}\int \zeta_i E\phi_1^2 \mathbf{d\zeta} \\ &= (1-\ln \pi^{3/2}\hat{c}_0)(u_{i2}+\omega_1 u_{i1}) - \left(Q_{i2} + \frac{5}{2}u_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) \\ &\quad + \frac{1}{2}\int \zeta_i E\phi_1^2 \mathbf{d\zeta},\end{aligned} \tag{53c}$$

$$\hat{G}_0 = 0, \tag{54a}$$

$$\hat{G}_1 = 0, \tag{54b}$$

$$\hat{G}_2 = -\frac{1}{4}\int EE_*(\phi_1' + \phi_{1*}' - \phi_1 - \phi_{*_1})^2 \widehat{B}\mathrm{d}\Omega \mathbf{d\zeta}_* \mathbf{d\zeta} \leq 0. \tag{54c}$$

With the aid of these expressions, the ε and ε^2-order expressions of Eq (47) are given as

$$\begin{aligned}\mathrm{Sh}\frac{\partial \hat{H}_1}{\partial \hat{t}} + \frac{\partial \hat{H}_{i1}}{\partial x_i} &= (1-\ln \pi^{3/2}\hat{c}_0)\left(\mathrm{Sh}\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial u_{i1}}{\partial x_i}\right) \\ &\quad - \left[\frac{3}{2}\mathrm{Sh}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i1} + Q_{i1}\right)\right],\end{aligned} \tag{55a}$$

$$\begin{aligned}\mathrm{Sh}\frac{\partial \hat{H}_2}{\partial \hat{t}} + \frac{\partial \hat{H}_{i2}}{\partial x_i} &= (1-\ln \pi^{3/2}\hat{c}_0)\left(\mathrm{Sh}\frac{\partial \omega_2}{\partial \hat{t}} + \frac{\partial (u_{i2}+\omega_1 u_{i1})}{\partial x_i}\right) \\ &\quad - \mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left(\frac{3}{2}P_2 + u_{i1}^2\right) - \frac{\partial}{\partial x_i}\left(Q_{i2} + \frac{5}{2}u_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) \\ &\quad + \frac{1}{2}\left(\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d\zeta}\right).\end{aligned} \tag{55b}$$

Substituting the series expansion (51a) into the conservation equation (M-1.87), we have

$$\mathrm{Sh}\frac{\partial \omega_1}{\partial \hat{t}} + \frac{\partial u_{i1}}{\partial x_i} = 0, \tag{56a}$$

$$\mathrm{Sh}\frac{\partial \omega_2}{\partial \hat{t}} + \frac{\partial (u_{i2} + \omega_1 u_{i1})}{\partial x_i} = 0. \tag{56b}$$

Similarly, from the conservation equation (M-1.89), we have

$$\frac{3}{2}\mathrm{Sh}\frac{\partial P_1}{\partial \hat{t}} + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i1} + Q_{i1}\right) = 0, \tag{57a}$$

$$\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left(\frac{3}{2}P_2 + u_{i1}^2\right) + \frac{\partial}{\partial x_i}\left(\frac{5}{2}u_{i2} + Q_{i2} + u_{j1}P_{ij1} + \frac{3}{2}u_{i1}P_1\right) = 0. \tag{57b}$$

With the aid of the expanded forms (56a)–(57b) of the conservation equations (M-1.87) and (M-1.89), Eqs. (55a) and (55b) are reduced to, for the solution of the Boltzmann equation (M-1.47) or (M-1.75a),

$$\mathrm{Sh}\frac{\partial \hat{H}_1}{\partial \hat{t}} + \frac{\partial \hat{H}_{i1}}{\partial x_i} = 0, \tag{58a}$$

$$\mathrm{Sh}\frac{\partial \hat{H}_2}{\partial \hat{t}} + \frac{\partial \hat{H}_{i2}}{\partial x_i} = \frac{1}{2}\left(\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta}\right). \tag{58b}$$

Thus, the $o(\varepsilon^2)$ terms being neglected in Eq. (47), it is reduced to

$$\begin{aligned} &\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\int E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} + \frac{\partial}{\partial x_i}\int \zeta_i E\phi_1^2 \mathbf{d}\boldsymbol{\zeta} \\ &\qquad = -\frac{1}{2k}\int EE_*(\phi_1' + \phi_{1*}' - \phi_1 - \phi_{1*})^2 \widehat{B}\mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \leq 0. \end{aligned} \tag{59}$$

This expression does not contain ϕ_2.

(Section 1.6: Version 4-00)

2 Chapter M-2

2.1 Section M-2.5

2.1.1 Section M-2.5.1

The following form:

$$\sigma = -\frac{2}{\pi}\int_{0<\xi<\infty,\, l_i n_i<0} \xi^3 l_j n_j f(\boldsymbol{X}, \xi \boldsymbol{l}) \mathrm{d}\xi \mathrm{d}\Omega(\boldsymbol{l}),$$

is more appropriate as Eq. (M-2.39b) than the one in the book. Then, the explanation of $\mathrm{d}\Omega(\boldsymbol{l})$, i.e.,

$\mathrm{d}\Omega(\boldsymbol{l})$ is the solid-angle element in the direction of $\boldsymbol{l}$,

has to be inserted between ‘ where ’ and ‘ T_w ’ just after Eq. (M-2.39c).

(Section 2.1.1: Version 6-00)

3 Chapter M-3

3.1 Processes of solution of the systems in Section M-3.7.2 (July 2007)

The processes of solutions of the fluid-dynamic-type equations derived in Section M-3.7.1 are straightforward and may not need explanation. For the equations in Section M-3.7.2, some explanation may be better to be given. The discussion will be made on the basis of the boundary conditions in Section M-3.7.3 for a simple boundary where the shape of the boundary is invariant and its velocity component normal to it is zero.

3.1.1 "Incompressible Navier–Stokes set"

Consider the initial and boundary-value problem of Eqs. (M-3.265)–(M-3.268), i.e.,

$$\frac{\partial P_{S1}}{\partial x_i} = 0, \tag{60}$$

$$\frac{\partial u_{iS1}}{\partial x_i} = 0, \tag{61a}$$

$$\frac{\partial u_{iS1}}{\partial \tilde{t}} + u_{jS1}\frac{\partial u_{iS1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_{S2}}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{iS1}}{\partial x_j^2}, \tag{61b}$$

$$\frac{5}{2}\frac{\partial \tau_{S1}}{\partial \tilde{t}} - \frac{\partial P_{S1}}{\partial \tilde{t}} + \frac{5}{2}u_{jS1}\frac{\partial \tau_{S1}}{\partial x_j} = \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S1}}{\partial x_j^2}, \tag{61c}$$

$$\frac{\partial u_{iS2}}{\partial x_i} = -\frac{\partial \omega_{S1}}{\partial \tilde{t}} - \frac{\partial \omega_{S1} u_{iS1}}{\partial x_i}, \tag{62a}$$

$$\begin{aligned}
&\frac{\partial u_{iS2}}{\partial \tilde{t}} + u_{jS1}\frac{\partial u_{iS2}}{\partial x_j} + u_{jS2}\frac{\partial u_{iS1}}{\partial x_j} \\
&= -\frac{1}{2}\left(\frac{\partial P_{S3}}{\partial x_i} - \omega_{S1}\frac{\partial P_{S2}}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{iS2}}{\partial x_j} + \frac{\partial u_{jS2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{kS2}}{\partial x_k}\delta_{ij}\right) \\
&\quad - \frac{\gamma_1 \omega_{S1}}{2}\frac{\partial^2 u_{iS1}}{\partial x_j^2} + \frac{\gamma_4}{2}\frac{\partial}{\partial x_j}\left[\tau_{S1}\left(\frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i}\right)\right] - \frac{\gamma_3}{3}\frac{\partial}{\partial x_i}\frac{\partial^2 \tau_{S1}}{\partial x_j^2},
\end{aligned} \tag{62b}$$

$$\begin{aligned}
&\frac{3}{2}\frac{\partial P_{S2}}{\partial \tilde{t}} + \frac{3}{2}u_{jS1}\frac{\partial P_{S2}}{\partial x_j} + \frac{5}{2}\left(\frac{\partial P_{S1} u_{jS2}}{\partial x_j} - \frac{\partial \omega_{S2}}{\partial \tilde{t}} - \frac{\partial(\omega_{S2} u_{jS1} + \omega_{S1} u_{jS2})}{\partial x_j}\right) \\
&= \frac{5\gamma_2}{4}\frac{\partial^2 \tau_{S2}}{\partial x_j^2} + \frac{5\gamma_5}{4}\frac{\partial}{\partial x_j}\left(\tau_{S1}\frac{\partial \tau_{S1}}{\partial x_j}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i}\right)^2,
\end{aligned} \tag{62c}$$

where

$$P_{S1} = \omega_{S1} + \tau_{S1}, \quad P_{S2} = \omega_{S2} + \omega_{S1}\tau_{S1} + \tau_{S2}. \tag{63}$$

From Eq. (60), P_{S1} is a function of $\tilde{t}$, i.e.,

$$P_{S1} = f_1(\tilde{t}). \tag{64}$$

In an unbounded-domain problem where the pressure at infinity is specified (or the pressure is specified at some point), $P_{S1} = f_1(\tilde{t})$ is known, but in a bounded-domain problem consisting of simple boundaries, $f_1(\tilde{t})$ is unknown at this moment and is determined later. Let u_{iS1} and τ_{S1} as well as $f_1(\tilde{t})$ be given at time $\tilde{t}$ in such a way that u_{iS1} satisfies Eq. (61a). Taking the divergence of Eq. (61b) and using Eq. (61a), we have

$$\frac{\partial^2 P_{S2}}{\partial x_i^2} = -2\frac{\partial u_{jS1}}{\partial x_i}\frac{\partial u_{iS1}}{\partial x_j}. \tag{65}$$

On a simple boundary, the derivative of P_{S2} normal to it is found to be expressed with u_{iS1} and its space derivatives by multiplying Eq. (61b) by the normal vector to the boundary.[23] In the unbounded-domain problem, where $f_1(\tilde{t})$ is known, P_{S2} is determined by Eq. (65). In the bounded-domain problem, P_{S2} is determined by Eq. (65) except for an additive function of $\tilde{t}$ [say, $f_2(\tilde{t})$]. Anyway, $\partial P_{S2}/\partial x_i$ is independent of this ambiguity. From Eq. (61b), $\partial u_{iS1}/\partial \tilde{t}$ at $\tilde{t}$ is determined, irrespective of $f_2(\tilde{t})$, in such a way that $\partial(\partial u_{iS1}/\partial x_i)/\partial \tilde{t} = 0$ for the above choice of P_{S2}. Thus, the solution u_{iS1} of Eqs. (61a) and (61b) is determined by Eq. (61b) with the supplementary condition (65) instead of Eq. (61a). From Eq. (61c), $(5/2)\partial\tau_{S1}/\partial\tilde{t} - \partial P_{S1}/\partial\tilde{t}$ or $(5/2)\partial\tau_{S1}/\partial\tilde{t} - \mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ is determined, i.e.,

$$(5/2)\partial\tau_{S1}/\partial\tilde{t} - \mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t} = G(x_i, \tilde{t}), \tag{66}$$

where

$$G(x_i, \tilde{t}) = -\frac{5}{2}u_{jS1}\frac{\partial\tau_{S1}}{\partial x_j} + \frac{5\gamma_2}{4}\frac{\partial^2\tau_{S1}}{\partial x_j^2}. \tag{67}$$

Thus, τ_{S1} is determined in the unbounded-domain problem, but τ_{S1} has ambiguity owing to $f_1(\tilde{t})$ in the bounded-domain problem. The undetermined function $f_1(\tilde{t})$ is determined in the following way.

In the bounded-domain problem where the boundary consists of simple boundaries, the mass of the gas in the domain is invariant with respect to $\tilde{t}$. The condition at the leading order is

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \omega_{S1}\mathbf{d}\boldsymbol{x} = 0, \tag{68}$$

where V indicates the domain (or its volume in the later). With the aid of Eq. (63), we have

$$\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}V - \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x} = 0. \tag{69}$$

[23] The time-derivative term vanishes owing to the boundary condition mentioned in the first paragraph of Section 3.1.

On the other hand, from Eq. (66),

$$-\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}V+\frac{5}{2}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x}=\int_V G(x_i,\tilde{t})\mathbf{d}\boldsymbol{x}. \tag{70}$$

From Eqs. (69) and (70), we obtain $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ and $\mathrm{d}\int_V \tau_{S1}\mathbf{d}\boldsymbol{x}/\mathrm{d}\tilde{t}$ as

$$\frac{\mathrm{d}f_1(\tilde{t})}{\mathrm{d}\tilde{t}}=\frac{2}{3V}\int_V G(x_i,\tilde{t})\mathbf{d}\boldsymbol{x}. \tag{71}$$

This is the equation for $f_1(\tilde{t})$ in the bounded-domain problem. Thus, τ_{S1} and $f_1(\tilde{t})$ in the bounded-domain problem are determined together by Eqs. (61c) and (71).

The analysis of the higher-order equations is similar; for example, from Eqs. (62a)–(62c), u_{iS2}, τ_{S2}, and P_{S3} are determined in the unbounded-domain problem, but $f_2(\tilde{t})$, u_{iS2}, τ_{S2}, and P_{S3}, except for an additive function of $\tilde{t}$ in P_{S3}, are determined in the bounded-domain problem.[24] Let u_{iS2}, τ_{S2}, and $f_2(\tilde{t})$ be given at $\tilde{t}$ in such a way that Eq. (62a) is satisfied.[25] Taking the divergence of Eq. (62b) and using Eq. (62a) and the results obtained above, we find that P_{S3} is governed by the Poisson equation

$$\frac{\partial^2 P_{S3}}{\partial x_i^2}=\text{Inhomogeneous term}, \tag{72}$$

where the inhomogeneous term consists of u_{iS2}, P_{S2}, and the functions determined in the preceding analysis. On a simple boundary, the derivative of P_{S3} normal to it being known,[26] P_{S3} is determined by this equation, except for an additive function of $\tilde{t}$ [say, $f_3(\tilde{t})$] in the bounded-domain problem. Then, from Eq. (62b), $\partial u_{iS2}/\partial\tilde{t}$ at $\tilde{t}$ is determined irrespective of $f_3(\tilde{t})$. From Eq. (62c), $\partial(3P_{S2}-5\omega_{S2})/\partial\tilde{t}$ [or $\partial(5\tau_{S2}-2P_{S2})/\partial\tilde{t}$] at $\tilde{t}$ is determined. Thus, u_{iS2} and τ_{S2} (except for the additive function $2f_2/5$ in the bounded-domain problem) [thus, ω_{S2} (except for the additive function $3f_2/5$)] are determined. In the bounded-domain problem, where the boundary consists of simple boundaries, the condition of invariance of the mass of the gas in the domain at the corresponding order is[27]

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \omega_{S2}\mathbf{d}\boldsymbol{x}=0. \tag{73}$$

With the aid of Eq. (63), $\mathrm{d}f_2(\tilde{t})/\mathrm{d}\tilde{t}$ at $\tilde{t}$ is determined as $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$ is done.

To summarize, the solution $(u_{iS1},P_{S1},\tau_{S1},P_{S2})$ of the initial and boundary-value problem of Eqs. (60)–(61c) is determined, with an additive arbitrary function $f_2(\tilde{t})$ in P_{S2} in a bounded-domain problem consisting of simple boundaries,

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[24]Note that, with the aid of Eq. (63), the time-derivative term $\frac{3}{2}\partial P_{S2}/\partial\tilde{t}-\frac{5}{2}\partial\omega_{S2}/\partial\tilde{t}$ in Eq. (62c) is transformed into $\frac{5}{2}\partial\tau_{S2}/\partial\tilde{t}-\partial P_{S2}/\partial\tilde{t}+\frac{5}{2}\partial\omega_{S1}\tau_{S1}/\partial\tilde{t}$.

[25]The time derivative $\partial\omega_{S1}/\partial\tilde{t}$ is known from $\partial\tau_{S1}/\partial\tilde{t}$, $\mathrm{d}f_1(\tilde{t})/\mathrm{d}\tilde{t}$, and Eq. (63).

[26]Shift the discussion of the boundary condition for P_{S2} to the next order.

[27]The contribution of the Knudsen-layer correction to the mass in the domain is of a higher order, though it is required to ω_{S2}.

when the initial data of u_{iS1}, P_{S1}, τ_{S1}, and P_{S2} satisfy Eqs. (61a) and (65). The additive function $f_2(\tilde{t})$ does not affect the other variables. The function $f_2(\tilde{t})$ is determined in the next-order analysis. In other words, the solution $(u_{iS1}, P_{S1}, \tau_{S1})$ of Eqs. (60)–(61c) is determined consistently by Eqs. (60), (61b), and (61c) with the supplementary condition (65), instead of Eq. (61a), when the initial data of u_{iS1}, P_{S1}, and τ_{S1} satisfy Eq. (61a). Naturally, the initial P_{S2} is required to satisfy Eq. (65). This process is natural for numerical computation.

3.1.2 Ghost-effect equations (M-3.275)–(M-3.278b):

Consider the initial and boundary-value problem of Eqs. (M-3.275)–(M-3.278b), i.e.,

$$\hat{p}_{SB0} = \hat{p}_0(\tilde{t}), \tag{74}$$

$$\hat{p}_{SB1} = \hat{p}_1(\tilde{t}), \tag{75}$$

$$\frac{\partial \hat{\rho}_{SB0}}{\partial \tilde{t}} + \frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial x_i} = 0, \tag{76a}$$

$$\begin{aligned}
&\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}}{\partial \tilde{t}} + \frac{\partial \hat{\rho}_{SB0}\hat{v}_{jSB1}\hat{v}_{iSB1}}{\partial x_j} \\
&\quad = -\frac{1}{2}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[\Gamma_1(\hat{T}_{SB0})\left(\frac{\partial \hat{v}_{iSB1}}{\partial x_j} + \frac{\partial \hat{v}_{jSB1}}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{v}_{kSB1}}{\partial x_k}\delta_{ij}\right)\right] \\
&\qquad + \frac{1}{2\hat{p}_0}\frac{\partial}{\partial x_j}\left\{\Gamma_7(\hat{T}_{SB0})\left[\frac{\partial \hat{T}_{SB0}}{\partial x_i}\frac{\partial \hat{T}_{SB0}}{\partial x_j} - \frac{1}{3}\left(\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right)^2\delta_{ij}\right]\right\},
\end{aligned} \tag{76b}$$

$$\frac{3}{2}\frac{\partial \hat{\rho}_{SB0}\hat{T}_{SB0}}{\partial \tilde{t}} + \frac{5}{2}\frac{\partial \hat{\rho}_{SB0}\hat{v}_{iSB1}\hat{T}_{SB0}}{\partial x_i} = \frac{5}{4}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right), \tag{76c}$$

where $\hat{p}_0$ and $\hat{p}_1$ depend only on $\tilde{t}$, and

$$\left.\begin{aligned}
&\hat{p}_{SB0} = \hat{\rho}_{SB0}\hat{T}_{SB0},\ \hat{p}_{SB1} = \hat{\rho}_{SB1}\hat{T}_{SB0} + \hat{\rho}_{SB0}\hat{T}_{SB1}, \\
&\hat{p}_{SB2} = \hat{\rho}_{SB2}\hat{T}_{SB0} + \hat{\rho}_{SB1}\hat{T}_{SB1} + \hat{\rho}_{SB0}\hat{T}_{SB2},
\end{aligned}\right\} \tag{77}$$

$$\hat{p}^*_{SB2} = \hat{p}_{SB2} + \frac{2}{3\hat{p}_0}\frac{\partial}{\partial x_k}\left(\Gamma_3(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_k}\right). \tag{78}$$

Let $\hat{\rho}$, $\hat{v}_i$, and $\hat{T}$ (thus, $\hat{p} = \hat{\rho}\hat{T}$) at time $\tilde{t}$ be given; thus, $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$ ($\hat{p}_{SB0}$), etc., including $\hat{p}_{SB2}$, are given. Then $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{\rho}_{SB0}\hat{v}_{iSB1}/\partial\tilde{t}$, and $\partial\hat{T}_{SB0}/\partial\tilde{t}$ at $\tilde{t}$ are given by Eqs. (76a)–(76c); thus, the future $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{T}_{SB0}$ (also $\hat{p}_{SB0}$) are determined. However, the future $\hat{p}_{SB0}$, as well as $\hat{p}_{SB0}$ at $\tilde{t}$, is required to be independent of x_i owing to Eq. (74). Taking this point into account, we discuss how the solution is determined. For convenience of the discussion, transform Eq. (76c) in the form

$$\frac{\partial \hat{p}_{SB0}}{\partial \tilde{t}} = \mathcal{P}, \tag{79}$$

where

$$\mathcal{P} = -\frac{5}{3}\frac{\partial \hat{p}_{SB0}\hat{v}_{iSB1}}{\partial x_i} + \frac{5}{6}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right).$$

First, consider the case where $\hat{p}$ (thus, $\hat{p}_{SB0}$, $\hat{p}_{SB1}$, etc.) is specified at some point, e.g., at infinity. Then, from Eq. (74), $\hat{p}_0(\tilde{t})$ is a given function of $\tilde{t}$, and $\hat{p}_{SB0}$ is determined. The initial value of $\hat{p}_{SB0}$ is uniform, i.e., $\hat{p}_{SB0} = \hat{p}_0(0)$. On the other hand, from Eq. (79), the variation of $\partial \hat{p}_{SB0}/\partial \tilde{t}$ is also determined by the data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$, and their space derivatives at $\tilde{t}$. This must coincide with the corresponding data given by Eq. (74), i.e., $\partial \hat{p}_{SB0}/\partial \tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$. Substituting this relation into Eq. (79), we have

$$\frac{\partial}{\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}}, \tag{80}$$

which requires a relation among $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ for all $\tilde{t}$, because $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is given. This condition is equivalently replaced by the following two conditions: The initial data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ are required to satisfy Eq. (80), and the time derivative of Eq. (80) has to be satisfied for all $\tilde{t}$, i.e.,

$$\frac{\partial^2}{\partial \tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}^2\hat{p}_0}{\mathrm{d}\tilde{t}^2}. \tag{81}$$

With the aid of Eqs. (76a)–(76c) and (79), the left-hand side of Eq. (81) is expressed in the form without the time-derivative terms, i.e., $\partial \hat{p}_{SB0}/\partial \tilde{t}$, $\partial \hat{T}_{SB0}/\partial \tilde{t}$, and $\partial \hat{v}_{iSB1}/\partial \tilde{t}$, as follows:

$$\frac{\partial^2}{\partial \tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{1}{2}\hat{p}_{SB0}\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i}\right) + \mathrm{fn}_1,$$

where fn_1 is a given function of $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$, and their space derivatives. Thus, the condition (81) is reduced to an equation for $\hat{p}^*_{SB2}$, i.e.,

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial \hat{p}^*_{SB2}}{\partial x_i}\right) = \mathrm{Fn}, \tag{82}$$

where

$$\mathrm{Fn} = \frac{2}{\hat{p}_0}\left(\mathrm{fn}_1 + \frac{3}{5}\frac{\mathrm{d}^2\hat{p}_0}{\mathrm{d}\tilde{t}^2}\right).$$

The boundary condition for $\hat{p}^*_{SB2}$ in Eq. (82) on a simple boundary is derived by multiplying Eq. (76b) by the normal n_i to the boundary. In this process, the contribution of its time-derivative terms vanishes.[28] Thus, $\hat{p}^*_{SB2}$ (or $\hat{p}_{SB2}$) is determined in the present case, where $\hat{p}$ (thus, $\hat{p}_{SB2}$) is specified at some point. The solution $\hat{p}^*_{SB2}$ of Eq. (82) being substituted into Eq. (76b), Eqs. (76a)–(76c) with the first relation in Eq. (77) are reduced to the equations for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ which naturally determine $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$. Further, if the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ being chosen in such a way that $\hat{\rho}_{SB0}\hat{T}_{SB0}(=\hat{p}_{SB0}) = \hat{p}_0$ and that Eq. (80) is satisfied, the variation $\partial\hat{p}_{SB0}/\partial\tilde{t}$ of $\hat{p}_{SB0}(=\hat{\rho}_{SB0}\hat{T}_{SB0})$ given by these equations is consistent with Eq. (74), because Eq. (82) or (81) with the condition (80) at the initial state guarantees Eq. (80), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t} = \mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, for all $\tilde{t}$.

Equations (74) and (76a)–(76c) with Eqs. (77) and (78) determine $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{p}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$ consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. In the above system, $\hat{p}_{SB2}$ is determined. On the other hand, the variation $\partial\hat{p}_{SB2}/\partial\tilde{t}$ is determined independently by the counterpart of Eq. (79) at the order after next. The situation is similar to that at the leading order, where Eqs. (74), with a given $\hat{p}_0$, and (79) determine $\hat{p}_{SB0}$ independently. The analysis can be carried out in a similar way. Let $\hat{p}_{SB2}$ determined by Eq. (82) be indicated by $(\hat{p}_{SB2})_0$ and the equation for $\partial\hat{p}_{SB2}/\partial\tilde{t}$, or the counterpart of Eq. (79) at the order after next, be put in the form

$$\frac{\partial\hat{p}_{SB2}}{\partial\tilde{t}} = \mathcal{P}_2, \tag{83}$$

where $\mathcal{P}_2$ is a given function of $\hat{\rho}_{SBm}$, $\hat{v}_{iSBm+1}$ $\hat{T}_{SBm}$ $(m \le 2)$, and their space derivatives. For the consistency, $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ is substituted for $\partial\hat{p}_{SB2}/\partial\tilde{t}$ in Eq. (83), i.e.,

$$\mathcal{P}_2 = \frac{\partial(\hat{p}_{SB2})_0}{\partial\tilde{t}}, \tag{84}$$

where $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ is known. This requires a relation among $\hat{\rho}_{SBm}$, $\hat{v}_{iSBm+1}$ $\hat{T}_{SBm}$ $(m \le 2)$, and their space derivatives. This condition is equivalently replaced by the following two conditions: Equation (84) is applied only for the initial state, and the time derivative of Eq. (84), i.e.,

$$\frac{\partial\mathcal{P}_2}{\partial\tilde{t}} = \frac{\partial^2(\hat{p}_{SB2})_0}{\partial\tilde{t}^2},$$

has to be satisfied for all $\tilde{t}$. The $\partial\hat{\rho}_{SBm}/\partial\tilde{t}$, $\partial\hat{v}_{iSBm+1}/\partial\tilde{t}$, $\partial\hat{T}_{SBm}/\partial\tilde{t}$ $(m \le 2)$ in $\partial\mathcal{P}_2/\partial\tilde{t}$ being replaced by the counterparts of Eqs. (76a)–(76c) and (79) at the corresponding order, an equation for $\hat{p}_{SB4}$ for all $\tilde{t}$ is derived.[29] The conclusion is that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced and, instead, that the condition (82) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ (thus also $\hat{p}_{SB0}$).

[28] The discussion is similar to that in Footnote 23.

[29] The conditions on the odd-order $\hat{p}_{SB2n+1}$'s are derived by the analysis starting from the condition (75) that $\hat{p}_{SB1}$ is independent of x_i.

In this way, the solution of Eqs. (74), (76a)–(78) is determined consistently by Eqs. (76a)–(78) with the aid of the supplementary condition (82), instead of Eq. (74), when the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ satisfy Eqs. (74) and (80), where $\hat{p}_0(\tilde{t})$ is a known function of $\tilde{t}$ from the boundary condition.

Secondly, consider a bounded-domain problem where the boundary consists of simple boundaries. In contrast to the first case, $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is unknown because no condition is imposed on $\hat{p}_{SB0}$ on a simple boundary. However, in a bounded-domain problem consisting of simple boundaries, the mass of the gas in the domain is invariant with respect to $\tilde{t}$, i.e., at the leading order,

$$\frac{\mathrm{d}\displaystyle\int_V \hat{\rho}_{SB0}\mathbf{d}\boldsymbol{x}}{\mathrm{d}\tilde{t}} = 0, \tag{85}$$

where V indicates the domain under consideration. Using the first relation of Eq. (77), i.e., $\hat{\rho}_{SB0} = \hat{p}_0/\hat{T}_{SB0}$, in Eq. (85), we have

$$\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}}\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x} = \hat{p}_0\int_V \frac{1}{\hat{T}_{SB0}^2}\frac{\partial \hat{T}_{SB0}}{\partial \tilde{t}}\mathbf{d}\boldsymbol{x}. \tag{86}$$

Using Eq. (76c) for $\partial\hat{T}_{SB0}/\partial\tilde{t}$ in Eq. (86), we find that the variation $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is expressed with $\hat{p}_0$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ as follows:

$$\frac{\mathrm{d}\hat{p}_0}{\mathrm{d}\tilde{t}} = P(\tilde{t}), \tag{87}$$

where

$$\begin{aligned} P(\tilde{t}) = \hat{p}_0\int_V \frac{1}{\hat{T}_{SB0}^2}&\left[\frac{5}{6\hat{\rho}_{SB0}}\frac{\partial}{\partial x_i}\left(\Gamma_2(\hat{T}_{SB0})\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) - \frac{5}{3}\hat{v}_{iSB1}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right]\mathbf{d}\boldsymbol{x} \\ &\times\left(\int_V \frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x}\right)^{-1}. \end{aligned} \tag{88}$$

With this expression of $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, we can carry out the analysis in a similar way to that in the first case.

The variation $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ or $\partial\hat{p}_{SB0}/\partial\tilde{t}$ is also determined by Eq. (79). The two $\partial\hat{p}_{SB0}/\partial\tilde{t}$'s given by Eq. (87) with Eq. (88) and Eq. (79) have to be consistent. Thus, substituting Eq. (87) with Eq. (88) into $\partial\hat{p}_{SB0}/\partial\tilde{t}$ in Eq. (79), we have

$$\frac{\partial}{\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}P(\tilde{t}), \tag{89}$$

where $P(\tilde{t})$ is given by Eq. (88). This must hold for all $\tilde{t}$ for consistency. This condition is equivalently replaced by the following two conditions: The initial data of $\hat{p}_{SB0}$, $\hat{T}_{SB0}$, $\hat{v}_{iSB1}$ are required to satisfy Eq. (89), and the time derivative of Eq. (89) has to be satisfied for all $\tilde{t}$, i.e.,

$$\frac{\partial^2}{\partial\tilde{t}\partial x_i}\left(\hat{p}_{SB0}\hat{v}_{iSB1} - \frac{\Gamma_2(\hat{T}_{SB0})}{2}\frac{\partial \hat{T}_{SB0}}{\partial x_i}\right) = -\frac{3}{5}\frac{\mathrm{d}P(\tilde{t})}{\mathrm{d}\tilde{t}}. \tag{90}$$

Using Eqs. (76a), (76b), and (79) for the time derivatives $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{v}_{iSB1}/\partial\tilde{t}$, and $\partial\hat{p}_{SB0}/\partial\tilde{t}$ in Eq. (90), we find that $\hat{p}^*_{SB2}$ at $\tilde{t}$ is determined by the equation

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\hat{\rho}_{SB0}}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right)+\mathcal{L}\left(\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right)=\mathrm{Fn}, \tag{91}$$

where Fn is a given functional of $\hat{\rho}_{SB0}$, $\hat{v}_{iSB1}$, $\hat{T}_{SB0}$, and their space derivatives, and $\mathcal{L}(\partial\hat{p}^*_{SB2}/\partial x_i)$ is a given linear functional of $\partial\hat{p}^*_{SB2}/\partial x_i$, i.e.,

$$\mathcal{L}\left(\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\right)=-\frac{1}{\hat{p}_0}\int_V\frac{1}{\hat{T}_{SB0}}\frac{\partial\hat{T}_{SB0}}{\partial x_i}\frac{\partial\hat{p}^*_{SB2}}{\partial x_i}\mathbf{d}\boldsymbol{x}\left(\int_V\frac{1}{\hat{T}_{SB0}}\mathbf{d}\boldsymbol{x}\right)^{-1}.$$

On a simple boundary, the derivative of $\hat{p}^*_{SB2}$ normal to the boundary is specified. Thus, $\hat{p}^*_{SB2}$ is determined except for an additive function of $\tilde{t}$. The solution $\hat{p}^*_{SB2}$ of Eq. (91) being substituted into Eq. (76b), the result is independent of the additive function. Thus, Eqs. (76a)–(76c) with the first relation in Eq. (77) and the above $\hat{p}^*_{SB2}$ substituted are reduced to those for $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$, which naturally determine $\partial\hat{\rho}_{SB0}/\partial\tilde{t}$, $\partial\hat{T}_{SB0}/\partial\tilde{t}$, and $\partial\hat{v}_{iSB1}/\partial\tilde{t}$. Further, if the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ being chosen in such a way that $\hat{\rho}_{SB0}\hat{T}_{SB0}(=\hat{p}_{SB0})=\hat{p}_0$ and that Eq. (89) is satisfied, the variation $\partial\hat{p}_{SB0}/\partial\tilde{t}$ of $\hat{p}_{SB0}(=\hat{\rho}_{SB0}\hat{T}_{SB0})$ given by these equations is consistent with Eq. (74), because Eq. (91) or (90) with the condition (89) at the initial state guarantees Eq. (89), i.e., $\partial\hat{p}_{SB0}/\partial\tilde{t}=\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$, for all $\tilde{t}$.

Equations (74) and (76a)–(76c) with Eqs. (77) and (91) determine $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, $\hat{p}_{SB0}$, $\hat{v}_{iSB1}$, and $\hat{p}_{SB2}$, except for an additive function of $\tilde{t}$ in $\hat{p}_{SB2}$, consistently for appropriately chosen initial data. However, these equations are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. First, the undetermined additive function in $\hat{p}_{SB2}$ is determined by the condition of invariance of the mass of the gas in the domain at the order after next as $\mathrm{d}\hat{p}_0/\mathrm{d}\tilde{t}$ is determined.[30] The $\partial\hat{p}_{SB2}/\partial\tilde{t}$ or $\hat{p}_{SB2}$ determined in this way is indicated by $\partial(\hat{p}_{SB2})_0/\partial\tilde{t}$ or $(\hat{p}_{SB2})_0$. On the other hand, the variation $\partial\hat{p}_{SB2}/\partial\tilde{t}$ is determined independently by Eq. (83) or the counterpart of Eq. (79) at the order after next. The two results must coincide. The discussion from here is the same as that given from the sentence starting from Eq. (83) to the end of the paragraph. The results are that an additional initial condition and the condition for $\hat{p}_{SB4}$ are introduced, and that the condition (91) for $\hat{p}_{SB2}$ is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ (thus also $\hat{p}_{SB0}$).

In this way, the solution of Eqs. (74), (76a)–(76c) is determined consistently by Eqs. (76a)–(76c) with the aid of the supplementary condition (91), instead of Eq. (74), when the initial data of $\hat{\rho}_{SB0}$, $\hat{T}_{SB0}$, and $\hat{v}_{iSB1}$ satisfy Eqs. (74) and (89).

[30]The Knudsen-layer correction to $\hat{\rho}_{SB1}$, already determined (see Footnote 29), contributes to the mass at this order.

3.2 Notes on basic equations in classical fluid dynamics

3.2.1 Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

The mass, momentum, and energy-conservation equations of fluid flow are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0, \tag{92}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial X_j}(\rho v_i v_j + p_{ij}) = 0, \tag{93}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}v_i^2\right)\right] + \frac{\partial}{\partial X_j}\left[\rho v_j\left(e + \frac{1}{2}v_i^2\right) + v_i p_{ij} + q_j\right] = 0, \tag{94}$$

where ρ is the density, v_i is the flow velocity, e is the internal energy per unit mass, p_{ij}, which is symmetric with respect to i and j, is the stress tenor, and q_i is the heat-flow vector. The pressure p and the internal energy e are given by the equations of state as functions of T and ρ, i.e.,[31]

$$p = p(T, \rho), \quad e = e(T, \rho). \tag{95}$$

Especially, for a perfect gas,

$$p = R\rho T, \quad e = e(T). \tag{96}$$

Equations (93) and (94) are rewritten with the aid of Eq. (92) in the form

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} + \frac{\partial p_{ij}}{\partial X_j} = 0, \tag{97}$$

$$\rho\frac{\partial}{\partial t}\left(e + \frac{1}{2}v_i^2\right) + \rho v_j \frac{\partial}{\partial X_j}\left(e + \frac{1}{2}v_i^2\right) + \frac{\partial}{\partial X_j}(v_i p_{ij} + q_j) = 0. \tag{98}$$

The operator $\partial/\partial t + v_j \partial/\partial x_j$, which expresses the time variation along the fluid particle, is denoted by $\mathrm{D}/\mathrm{D}t$, i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}.$$

Multiplying Eq. (97) by v_i we obtain the equation for the variation of kinetic energy as

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v_i^2\right) = -v_i\frac{\partial p_{ij}}{\partial X_j}. \tag{99}$$

[31] The case where the first equation $p = p(T, \rho)$ cannot be solved with respect to ρ, or ρ is independent of p and T is called *incompressible*. The incompressible condition is a special case of the equation of state. In a system consisting of a single incompressible substance, the density is a constant.

Another form of Eq. (94), where Eq. (99) is subtracted from Eq. (98), is given as

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p_{ij} \frac{\partial v_i}{\partial X_j} - \frac{\partial q_j}{\partial X_j}. \tag{100}$$

Noting the thermodynamic relation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = T \frac{\mathrm{D}s}{\mathrm{D}t} + \frac{p}{\rho^2} \frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{101}$$

where s is the entropy per unit mass, and Eq. (92), Eq. (100) is rewritten as

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = -\frac{1}{T} \left[(p_{ij} - p\delta_{ij}) \frac{\partial v_i}{\partial X_j} + \frac{\partial q_j}{\partial X_j} \right]. \tag{102}$$

Equation (102) expresses the variation of the entropy of a fluid particle.

Equations (92)–(95) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor p_{ij} and the heat-flow vector q_i are assumed in some ways. The *Navier–Stokes set of equations* (or the *Navier–Stokes equations*) is Eqs. (92)–(95) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) - \mu_B \frac{\partial v_k}{\partial X_k} \delta_{ij}, \tag{103}$$

$$q_i = -\lambda \frac{\partial T}{\partial X_i}, \tag{104}$$

where μ, μ_B, and λ are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of T and ρ. The *Euler set of equations* (or the *Euler equations*) is Eqs. (92)–(95) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij}, \quad q_i = 0, \tag{105}$$

or the Navier–Stokes equations with $\mu = \mu_B = \lambda = 0$.

For the Navier–Stokes equations, in view of the relations (103) and (104), the entropy variation is expressed in the form[32]

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T} \left[\frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \mu_B \left(\frac{\partial v_k}{\partial X_k} \right)^2 + \frac{\partial}{\partial X_i} \left(\lambda \frac{\partial T}{\partial X_i} \right) \right]. \tag{106}$$

[32]Note the following transformation:

$$\begin{aligned} &\frac{\partial v_i}{\partial X_j} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) \\ &= \frac{1}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} + \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) \\ &= \frac{1}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \frac{1}{3} \frac{\partial v_l}{\partial X_l} \delta_{ij} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right). \end{aligned}$$

The second term in the last expression is easily seen to vanish.

For the Euler equations, for which p_{ij} and q_i are given by Eq.(105), the entropy of a fluid particle is invariant, i.e.,

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = 0. \tag{107}$$

The incompressible condition in Footnote 31 can be expressed as[33]

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{108}$$

Thus, from Eqs. (92) and (108),

$$\frac{\partial v_i}{\partial X_i} = 0. \tag{109}$$

Equation (103) for the Navier–Stokes-stress tensor reduces to

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right). \tag{110}$$

The first term on the right-hand side of Eq. (100) reduces to

$$\begin{aligned} -p_{ij} \frac{\partial v_i}{\partial X_j} &= - \left[p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right] \frac{\partial v_i}{\partial X_j} \\ &= \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2. \end{aligned}$$

Thus, Eq. (100) reduces to

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left(\lambda \frac{\partial T}{\partial X_j} \right). \tag{111}$$

To summarize, the Navier–Stokes equations for incompressible fluid are

$$\frac{\partial v_i}{\partial X_i} = 0, \tag{112a}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = - \frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j} \left[\mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right], \tag{112b}$$

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left(\lambda \frac{\partial T}{\partial X_j} \right), \tag{112c}$$

with the incompressible condition (108) being supplemented, i.e.,[34]

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{113}$$

[33](i) The density is invariant along fluid-particle paths. If ρ is of uniform value ρ_0 initially, it is a constant, i.e.,

$$\rho = \rho_0.$$

In a time-independent (or steady) problem, the density is constant along streamlines.

(ii) This condition is useful in the system consisting of non-mixing different incompressible substances (e.g., oil and water). Needless to say, e, μ, and λ depend on substances.

[34]The energy equation (112c) for incompressible fluid contains the heat-production term

3.2.2 Boundary condition for the Euler set

In Section M-3.5, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentum-conservation equations of the Euler set are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{114}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{115}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{116}$$

where ρ is the density, which is uniform, (u, v) is the flow velocity, and p is the pressure. Owing to Eq. (114), the stream function Ψ can be introduced as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{117}$$

Eliminating p from Eqs. (115) and (116), we have[35]

$$u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = 0, \tag{118}$$

due to the work done by viscous stress [the first term on its right-hand side] if we literally put Eq. (103) in the energy equation (100). The incompressible Navier-Stokes set of equations is generally used for the case where the flow velocity is not so large. That is, it is used for the case where the dynamic energy $v_i^2/2$ per unit mass is negligibly small compared with the internal energy e per unit mass ($v_i^2 \ll e$). The Mach number is a common reference scale of flow speed, but it is not useful for incompressible fluid because the speed of sound is not well defined. Let e_0 be e of the reference state. We take the speed $(2e_0)^{1/2}$ as the reference speed, which is comparable to the sound speed for perfect gas. Let the typical flow speed of the flow under consideration be U_0. With the nondimensiol small parameter $U_0/(2e_0)^{1/2}$, we compare the size of the terms in the energy equation (112c). The first term on the right-hand side of Eq. (112c) is seen to be of higher-order of smallness compared with the convection and conduction terms when the Reynolds number based on U_0 and the Prandtl number are of the order of unity. Thus, the neglected form is often given in literature, with the condition being implicitly assumed.

[35] The following equation is formed from them:

$$\partial\text{Eq.}\,(115)/\partial y - \partial\text{Eq.}\,(116)/\partial x = 0.$$

where Ω is the vorticity, i.e.,

$$\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}. \tag{119}$$

From Eqs. (117) and (118),

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Omega}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Omega}{\partial y} = 0. \tag{120}$$

This equation shows that Ω is a function of Ψ,[36] i.e.,

$$\Omega = F(\Psi). \tag{121}$$

This functional relation between Ω and Ψ is a local relation, and therefore F may be a multivalued function of Ψ. From Eqs. (119) and (121),

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = F(\Psi). \tag{122}$$

Consider a boundary-value problem in a simply-connected bounded domain, where Ψ is given on the boundary ($\Psi = \Psi_B$). Introduce a coordinate s ($0 \leq s <$

[36]This can be seen with the aid of theorems on implicit functions (see References M-[47, 48, 267]). The proof is outlined here. The Ω and Ψ are functions of x and y :

$$\Omega = \Omega(x, y), \quad \Psi = \Psi(x, y). \tag{$*$}$$

Solving the second equation with respect to x, we have

$$x = \hat{x}(\Psi, y). \tag{$**$}$$

With this relation into Eq. ($*$),

$$\Omega = \Omega(\hat{x}(\Psi, y), y) = \hat{\Omega}(\Psi, y), \tag{$\sharp$a}$$

$$\Psi = \Psi(\hat{x}(\Psi, y), y) = \hat{\Psi}(\Psi, y). \tag{$\sharp$b}$$

That is, Ω is expressed as a function of Ψ and y. From Eqs. ($\sharp$a) and ($\sharp$b),

$$\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = \frac{\partial \Omega(\hat{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Omega(x, y)}{\partial x}\frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Omega(x, y)}{\partial y}, \tag{$\sharp\sharp$a}$$

$$\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = 0. \tag{$\sharp\sharp$b}$$

On the other hand,

$$\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = \frac{\partial \Psi(\hat{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Psi(x, y)}{\partial x}\frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y}.$$

Thus,

$$\frac{\partial \Psi(x, y)}{\partial x}\frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y} = 0. \tag{$\ddagger$}$$

From Eqs. (120), ($\sharp\sharp$a) and ($\ddagger$), we have

$$\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = 0, \quad \text{or} \quad \Omega = \hat{\Omega}(\Psi).$$

S) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where $\partial\Psi_B/\partial s < 0$, and the fluid flows out from the domain on the boundary where $\partial\Psi_B/\partial s > 0$. When F is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose F on the part where $\partial\Psi_B/\partial s < 0$ or $\partial\Psi_B/\partial s > 0$. For example, take the case where $\partial\Psi_B/\partial s < 0$ for $0 < s < S_m$ and $\partial\Psi_B/\partial s > 0$ for $S_m < s < S$, and choose the distribution $\Omega_B(s)$ of Ω along the boundary for the part $0 < s < S_m$. By the choice of Ω_B, the function $F(\Psi)$ is determined in the following way. Inverting the relation $\Psi = \Psi_B(s)$ between Ψ and s on the part $0 < s < S_m$, i.e., $s(\Psi)$, and noting the relation (121), we find that F is given by

$$F(\Psi) = \Omega_B(s(\Psi)). \tag{123}$$

Then, the boundary-value problem is fixed. That is, Eq. (122) is fixed as[37]

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = \Omega_B(s(\Psi)), \tag{124}$$

and the boundary condition is given as $\Psi = \Psi_B(s)$. This system is a standard from the point of counting of the number of boundary conditions. Obviously, from Eq. (119), the solution of the above system automatically satisfies condition $\Omega = \Omega_B(s)$ along the boundary for $0 < s < S_m$. We cannot choose the distribution of Ω on the boundary for $S_m < s < S$.

The energy-conservation equation of the incompressible Euler set is given by Eq. (111) with $\mu = \lambda = 0$, i.e.,

$$u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} = 0, \quad \text{or} \quad \frac{\partial\Psi}{\partial y}\frac{\partial e}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial e}{\partial y} = 0, \tag{125}$$

where e is the internal energy. Thus, e is a function of Ψ, i.e.,

$$e = F_1(\Psi). \tag{126}$$

In the above boundary-value problem, therefore, e can be specified on the the part ($0 < s < S_m$) of the boundary, but no condition can be specified on other part ($S_m < s < S$) and vice versa.[38]

To summarize, we can specify three conditions for Ψ, Ω, and e on the part $\partial\Psi_B/\partial s < 0$ ($\partial\Psi_B/\partial s > 0$) of boundary but one condition for Ψ on the other part $\partial\Psi_B/\partial s > 0$ ($\partial\Psi_B/\partial s < 0$). The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

3.2.3 Ambiguity of pressure in the incompressible Navier–Stokes system

It may be better to note ambiguity of the solution of the initial and boundary-value problem of the incompressible Navier–Stokes equations in a bounded domain consisting of simple boundaries.

[37] There is still some ambiguity. The case where there is a region with closed stream lines $\Psi(x, y) =$ const inside the domain is not excluded.

[38] From the second relation on e of Eq. (95) and the uniform-density condition, the condition on e can be replaced by the condition on the temperature T.

Consider the Navier–Stokes equations for an incompressible fluid, i.e.,

$$\frac{\partial v_i}{\partial X_i} = 0, \tag{127a}$$

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j}\mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right), \tag{127b}$$

$$\rho\frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda\frac{\partial T}{\partial X_j}\right), \tag{127c}$$

$$\frac{\partial \rho}{\partial t} + v_i\frac{\partial \rho}{\partial X_i} = 0, \tag{127d}$$

where e, μ, and λ are functions of T and ρ.

Consider the initial and boundary-value problem of Eqs. (127a)–(127d) in a bounded domain D on the boundary ∂D of which v_i and T are specified as $v_i = v_{wi}$ and $T = T_w$ (v_{wi} and T_w are, respectively, the surface velocity and temperature of the boundary satisfying $\int_{\partial D} v_{wi} n_i \mathrm{d}S = 0$, n_i : the unit normal vector to the boundary) and no condition is imposed on ρ and p. Let $(v_i^{(s)}, \rho^{(s)}, T^{(s)}, p^{(s)})$ be a solution of the initial and boundary-value problem. Let $P^{(a)}$ be an arbitrary function of t, independent of x_i, that vanishes at initial time $t=0$, i.e., $P^{(a)} = f(t)$ with $f(0)=0$. Put

$$(v_i, \rho, T, p) = (v_i^{(s)}, \rho^{(s)}, T^{(s)}, p^{(s)} + P^{(a)}).$$

Then, e, μ, and λ corresponding to the new (v_i, ρ, T, p) are equal to $e^{(s)}$, $\mu^{(s)}$, and $\lambda^{(s)}$ respectively, because they are determined by ρ and T. The new (v_i, ρ, T, p) satisfy the equations (127a)–(127d) and the initial and boundary conditions.

3.2.4 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set although the difference is obvious from the set of equations derived, especially from the equation of state and the energy equation.[39] The difference is explained in the two books KF and MGD in connection with the set of fluid-dynamic type equations derived by the S expansion from the Boltzmann equation in Sections K-4.3, M-3.2, and M-3.7. To make difference clearer and to eliminate the misunderstanding, we will show the process of analysis from the compressible Navier–Stokes set with the boundary condition taken into account. Thus, it is made clear how the solution is constructed and how the behavior of the solution in the two systems is different in the initial and boundary-value problem. In the time-dependent case, the energy equation

[39] In a time-dependent case, we here consider the case where the variables vary in the diffusion time scale. In a shorter time scale, the sound wave propagates.

contains another time-derivative term, in addition to the difference in the time-independent case. An ambiguity of solution in the incompressible system in a bounded domain with the simple boundary is eliminated in the compressible system.

Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the nondimensional variables introduced by Eq. (M-1.74) in Section M-1.10 as follows:

$$\mathrm{Sh}\frac{\partial \omega}{\partial \hat{t}} + \frac{\partial (1+\omega)u_i}{\partial x_i} = 0, \tag{128}$$

$$\mathrm{Sh}\frac{\partial (1+\omega)u_i}{\partial \hat{t}} + \frac{\partial}{\partial x_j}\left((1+\omega)u_i u_j + \frac{1}{2}P_{ij}\right) = 0, \tag{129}$$

$$\begin{aligned} &\mathrm{Sh}\frac{\partial}{\partial \hat{t}}\left[(1+\omega)\left(\frac{3}{2}(1+\tau)+u_i^2\right)\right] \\ &+ \frac{\partial}{\partial x_j}\left[(1+\omega)u_j\left(\frac{3}{2}(1+\tau)+u_i^2\right) + u_i(\delta_{ij}+P_{ij}) + Q_j\right] = 0. \end{aligned} \tag{130}$$

The nondimensional stress tensor P_{ij}, and heat-flow vector Q_i are expressed as[40]

$$P_{ij} = P\delta_{ij} - \frac{\mu_0(2RT_0)^{1/2}}{p_0 L}(1+\bar{\mu})\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \tag{131a}$$

$$Q_i = -\frac{\lambda_0 T_0}{L p_0 (2RT_0)^{1/2}}(1+\bar{\lambda})\frac{\partial \tau}{\partial x_i}. \tag{131b}$$

Here, $\bar{\mu}$ and $\bar{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

$$\mu = \mu_0(1+\bar{\mu}), \quad \lambda = \lambda_0(1+\bar{\lambda}),$$

where μ_0 and λ_0 are, respectively, the values of the viscosity μ and the thermal conductivity λ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of τ and ω. The first relation of the equation of state [Eq. (96)] is expressed as

$$P = \omega + \tau + \omega\tau. \tag{132}$$

Take a small parameter ε, and consider the case where

$$u_i = O(\varepsilon), \quad \omega = O(\varepsilon), \quad \tau = O(\varepsilon), \quad \mathrm{Sh} = O(\varepsilon), \tag{133a}$$

$$\frac{\mu_0(2RT_0)^{1/2}}{p_0 L} = \gamma_1\varepsilon, \quad \frac{\lambda_0 T_0}{L p_0 (2RT_0)^{1/2}} = \frac{5}{4}\gamma_2\varepsilon, \tag{133b}$$

where γ_1 and γ_2 are constants of the order of unity. Thus,

$$P = O(\varepsilon), \quad \bar{\mu} = O(\varepsilon), \quad \bar{\lambda} = O(\varepsilon).$$

[40] For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B = 0$ (see Section 1.3).

Some notes on the conditions (133a) and (133b) may be in order.
(i-a) The first three relations in Eq. (133a) are the condition on the size of perturbation of gas dynamic variables from the background state ($v_i = 0$, $\rho = \rho_0$ and $T = T_0$) with respect to their reference values [$v_i = (2RT_0)^{1/2}$, $\rho = \rho_0$, and $T = T_0$]. The last relation is the condition of the time t_0 of appreciable variation of the perturbed quantities. This condition shows that the time scale t_0 of variation of the variables is chosen as

$$t_0 = \frac{L}{(2RT_0)^{1/2}\varepsilon}, \tag{134}$$

which is the time that the typical gas flow proceeds over the distance L. In other words, we are interested in the behavior of the gas, the perturbed quantities of which vary appreciably in time t_0. Naturally, the variation of boundary data is to be consistent with the above time scale.
(i-b) The discussion in this section (Section 3.2.4) is based on the Navier–Stokes equation, which is assumed to be valid without the restriction on the size of the parameters. We are going to derive the set of equations for small ε under the conditions (133a) and (133b) for the two kinds of fluid (perfect gas and incompressible fluid), and compare the difference between their results. Here we are interested in the leading nontrivial set. The equations at the higher-orders are obviously within the framework of the Navier-Stokes equations. This applies to the boundary condition. Nonslip or non-jump condition is used on the simple boundary irrespective of the size of ε. The boundary condition on the interface is borrowed from the leading-order results of the asymptotic analysis for small Knudsen numbers of the Boltzmann system for the case in (iii).
(ii) In the Navier–Stokes system, the viscosity μ and the thermal conductivity λ characterize the property of the fluid. The relation (133b) is the condition between these transport coefficients and the typical size ε of the perturbations. Form these relations, we have

$$\frac{\mu_0/\rho_0}{\varepsilon(2RT_0)^{1/2}L} = \frac{\gamma_1}{2}, \qquad \frac{2\lambda_0}{5R\mu_0} = \frac{\gamma_2}{\gamma_1}. \tag{135}$$

The quantities on the left-hand sides of the two relations consist of the parameters[41] of the flow under consideration. Thus, we are considering the case where the two combinations of the parameters are finite because γ_1 and γ_2 are constants.
(iii-a) In the S expansion in Chapter K-4 or in Section M-3.2, the parameter k characterizes the degree of rarefaction of the gas under consideration. The analysis there is carried out under the condition that

$$\varepsilon = k, \tag{136}$$

[41]Note that (a) $\varepsilon(2RT_0)^{1/2}$ is the typical flow speed, and $\varepsilon(2RT_0)^{1/2}L/(\mu_0/\rho_0)$ is the Reynolds number based on that flow speed and that (b) $3R/2$ is the heat capacity at constant volume because the internal energy per unit mass is $3RT_0/2$ for the gas under consideration (perfect gas).

where ε is the size of perturbation ϕ of the velocity distribution function from the background equilibrium state f_0 at rest with the density ρ_0 and the temperature T_0 corresponding to Eq. (133a), with the time-independent condition imposed. The time-dependent case is discussed for the two cases, $t_0 = O[L/(2RT_0)^{1/2}]$ and $t_0 = O[L/(2RT_0)^{1/2}\varepsilon]$, in Section K-4.9 and K-4.10.3 and Sections M-3.7.1 and M-3.7.2.
(iii-b) This is one of the several cases where the behavior of a gas for small Knudsen numbers (or $k \ll 1$) is discussed on the basis of the Boltzmann system (Boltzmann equation and its boundary condition), and the fluid-dynamic type equation and its boundary condition are derived for various limiting processes (see the two books KF and MGD). The nontrivial leading-order results of the case (136) corresponds to the set of equations of the case with Eqs. (133a) and (133b) of the Navier–Stokes equation for perfect gas. We will use the boundary condition on the interface obtained by the analysis of the Boltzmann system to the discussion in this section (Section 3.2.4). The higher-order results include higher-order effects of the Knudsen number, which is not covered by the Navier-Stokes system. The higher-order boundary condition does not contribute to the comparison. Thus, the leading-order boundary condition is used with adjustments as the higher one if necessary.
(iv) The expansion parameter ε can be chosen rather freely by a finite factor. So are the reference quantities. If we use a different reference velocity, e.g., $(2e_0)^{1/2}$, instead of $(2RT_0)^{1/2}$, with the same ε, the coefficients of the resulting equations are obviously different.

According to the definition of u_i in Eq. (M-1.74), ε is of the order of the Mach number of typical flow speed U_0 of the flow field. Here, we take ε as $\varepsilon = U_0/(2RT_0)^{1/2}$. The relation (133b) between ε and viscosity μ_0 or thermal conductivity λ_0 is taken from the result of analysis in Section M-3.2, where the situation that the Knudsen number and the perturbation scale is of the same of order of smallness, i.e., $k = \varepsilon$, is discussed. According to Eq. (M-1.48a), the condition $\mathrm{Sh} = O(\varepsilon)$ in Eq. (133a) means that the time scale t_0 of the variation of variables is of the order of $L/(2RT_0)^{1/2}\varepsilon$, which is of the order of time scale of viscous or thermal diffusion owing to the relation (133b). We can take $t_0 = L/(2RT_0)^{1/2}\varepsilon$ without loss of generality.[42] Then,

$$\mathrm{Sh} = \varepsilon. \tag{137}$$

The new symbol $\tilde{t}$, instead of $\hat{t}$ is introduced to make this time scale explicit:

$$\tilde{t} = \hat{t}. \tag{138}$$

Corresponding to the above situation, u_i, ω, P, and τ are expanded in power series of ε, i.e.,

$$u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots , \tag{139a}$$

$$\omega = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots , \tag{139b}$$

$$P = P_1\varepsilon + P_2\varepsilon^2 + \cdots , \tag{139c}$$

$$\tau = \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots , \tag{139d}$$

$$\bar{\mu} = \bar{\mu}_1\varepsilon + \bar{\mu}_2\varepsilon^2 + \cdots , \tag{139e}$$

$$\bar{\lambda} = \bar{\lambda}_1\varepsilon + \bar{\lambda}_2\varepsilon^2 + \cdots , \tag{139f}$$

$$P_{ij} = P_1\delta_{ij}\varepsilon + P_{ij2}\varepsilon^2 + \cdots , \tag{139g}$$

$$Q_i = Q_{i2}\varepsilon^2 + \cdots . \tag{139h}$$

Substituting Eqs. (139a)–(139h) with Eqs. (133b) and (137) into Eqs. (128)–(130) with Eqs. (131a) and (131b), and arranging the same-order terms of ε, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \qquad \frac{\partial P_1}{\partial x_i} = 0, \qquad \frac{\partial u_{i1}}{\partial x_i} = 0,$$

[42] This time scale, $t_0 = L/(2RT_0)^{1/2}\varepsilon$, is called diffusion time scale in view of the following situation. The viscous diffusion of the parallel flow in X_1 direction with nonuniform velocity v_1 is given by

$$\frac{\partial v_1}{\partial t} = \frac{\mu_0}{\rho_0}\frac{\partial^2 v_1}{\partial X_2^2},$$

where the variation with respect to x_3 is assumed to be absent. Let the length scale and the time scale of variation of v_1 be L and t_0 respectively. Putting $t = \tilde{t}\,t_0$ and $X_2 = x_2 L$, we have

$$\begin{aligned}\frac{\partial v_1}{\partial \tilde{t}} &= \frac{\mu_0 t_0}{\rho_0 L^2}\frac{\partial^2 v_1}{\partial x_2^2} \\ &= \frac{\gamma_1 t_0 (2RT_0)^{1/2}\varepsilon}{2L}\frac{\partial^2 v_1}{\partial x_2^2},\end{aligned}$$

where the first relation of Eq. (133b) is used in the final step. When $t_0 = L/(2RT_0)^{1/2}\varepsilon$, the coefficient is of the order of unity because $\gamma_2/2$ is so. Then, the time derivative $\partial v_1/\partial\tilde{t}$ and the space derivative $\partial^2 v_1/\partial x_2^2$ are of the same order $O(v_1)$. Thus, in the diffusion process, the time scale of variation is $t_0 = L/(2RT_0)^{1/2}\varepsilon$ when the length scale of variation is L.

The diffusion of heat can be discussed in the same way, with ρ_0 being replaced by $\alpha\rho_0 RT_0$ ($\alpha = 3/2$ or $5/2$ depending on the diffusion under constant volume or constant pressure), μ_0 by λ_0, and the velocity v_1 by temperature T. With the aid of the second relation of Eq. (133b),

$$\frac{\partial T}{\partial \tilde{t}} = \frac{5\gamma_2 t_0 (2RT_0)^{1/2}\varepsilon}{4\alpha L}\frac{\partial^2 T}{\partial x_2^2}.$$

The remaining discussion is the same because γ_2 is a constant of the order of unity.

$$\frac{\partial \omega_1}{\partial \tilde{t}} + \frac{\partial \omega_1 u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} = 0,$$
$$\frac{\partial u_{i1}}{\partial \tilde{t}} + \frac{\partial u_{i1} u_{j1}}{\partial x_j} + \frac{1}{2}\frac{\partial P_2}{\partial x_i} - \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right) = 0,$$
$$\frac{3}{2}\frac{\partial P_1}{\partial \tilde{t}} + \frac{\partial}{\partial x_j}\left(\frac{5}{2}u_{j2} + \frac{5}{2}P_1 u_{j1} - \frac{5}{4}\gamma_2 \frac{\partial \tau_1}{\partial x_j}\right) = 0,$$

and so on. At the leading order, the equations derived from Eqs. (128) and (130) degenerate into the same equation $\partial u_{i1}/\partial x_i = 0$. Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{140}$$

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{141a}$$
$$\frac{\partial u_{i1}}{\partial \tilde{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{141b}$$
$$\frac{5}{2}\frac{\partial \tau_1}{\partial \tilde{t}} - \frac{\partial P_1}{\partial \tilde{t}} + \frac{5}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2 \frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{141c}$$

$$\frac{\partial u_{i2}}{\partial x_i} = -\frac{\partial \omega_1}{\partial \tilde{t}} - \frac{\partial \omega_1 u_{i1}}{\partial x_i}, \tag{142a}$$
$$\begin{aligned}
&\frac{\partial u_{i2}}{\partial \tilde{t}} + u_{j1}\frac{\partial u_{i2}}{\partial x_j} + u_{j2}\frac{\partial u_{i1}}{\partial x_j} \\
&\quad = -\frac{1}{2}\left(\frac{\partial P_3}{\partial x_i} - \omega_1 \frac{\partial P_2}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right) \\
&\qquad - \frac{\gamma_1 \omega_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left[\bar{\mu}_1\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)\right],
\end{aligned} \tag{142b}$$
$$\begin{aligned}
&\frac{3}{2}\frac{\partial P_2}{\partial \tilde{t}} + \frac{3}{2}u_{j1}\frac{\partial P_2}{\partial x_j} + \frac{5}{2}\left(P_1 \frac{\partial u_{j2}}{\partial x_j} - \frac{\partial \omega_2}{\partial \hat{t}} - \frac{\partial(\omega_1 u_{j2} + \omega_2 u_{j1})}{\partial x_j}\right) \\
&\quad = \frac{5\gamma_2}{4}\frac{\partial}{\partial x_i}\left(\frac{\partial \tau_2}{\partial x_i} + \bar{\lambda}_1 \frac{\partial \tau_1}{\partial x_i}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)^2,
\end{aligned} \tag{142c}$$

where

$$P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1 \tau_1. \tag{143}$$

These equations are very similar to Eqs. (M-3.265)–(M-3.268) obtained by the S expansion of the Boltzmann equation in Section M-3.7.2.[43]

[43](i) Equations (140)–(142a) and (143) are of the same form as Eqs. (M-3.265)–(M-3.267a)

In order to compare Eqs. (140)–(141c) and the first relation of Eq. (143) with the incompressible Navier–Stokes equations (112a)–(113), we will rewrite the latter equations for the situation [Eqs. (133a) and (133b)] where the former equations are derived. The starting equations are Eqs. (128)–(131b)[44] and the nondimensional form of Eq. (108), i.e.,

$$\varepsilon \frac{\partial \omega}{\partial \tilde{t}} + u_i \frac{\partial \omega}{\partial x_i} = 0, \tag{144}$$

instead of Eq. (132). The analysis is carried out in a similar way[45] and the equations corresponding to Eqs. (140)–(141c) are[46]

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{145}$$

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and (M-3.268). Thus, the following discussion in this section (Section 3.2.4) applies to the two systems. Naturally, it applies to the S solution in Sections K-4.9 and K-4.10.3.

(ii) It may be noted that the second-order velocity u_{i2} is not solenoidal from Eq. (142a).

[44]As the internal energy e, $3RT/2$ [$= 3RT_0(1+\tau)/2$] is chosen for consistency.

[45](i) For incompressible fluid, the sound speed is not well defined. Thus, the speed $(2e_0)^{1/2}$ defined by the internal energy e_0 per unit mass is taken as the reference speed, which is comparable to the sound speed for perfect gas. Thus, the condition that the Mach number is small can be replaced by the condition that the flow speed $|v_i|/(2e_0)^{1/2}$ divided by that reference speed is small. This is the case where the work done by the viscous force is negligible in Eq. (112c) (see Footnote 34).

(ii) Different equation of state corresponds to different substance. Thus, various parameters differ in different systems. However, we are interested in the difference of solutions due to the change of the equation of state among the pressure, temperature and density (perfect gas or incompressible fluid). Thus, we take a state at rest ($v_i = 0$) with pressure p_0 and temperature T_0. Thus, the density $\rho_0 = p_0/RT_0$ for perfect gas. We imagine the incompressible fluid at rest with density ρ_0 and the other properties (internal energy e_0, viscosity μ_0, thermal conductivity λ_0) of the two kinds of fluid are taken to be the same. (This is not so real because the density differs much for gas and liquid, the latter of which is much closer to incompressible fluid. Here, we put aside the reality for the present purpose avoiding the similarity discussion. Incompressible fluid is not gas according to the classification of gas and liquid.) Taking the state at rest with pressure p_0 and temperature T_0 as the background state (thus, the density $\rho_0 = p_0/RT_0$; this relation holds only at the background state; the relation without subscript 0 applies only to perfect gas but not to incompressible fluid), we discuss the behavior of two kinds of fluid for the case where (a) the typical flow velocity, say U_0, is small compared with the reference speed $(2e_0)^{1/2}$, i.e., $U_0/(2e_0)^{1/2} \ll 1$, and (b) the relative variations P and τ of pressure and temperature is of the order of $U_0/(2e_0)^{1/2}$. In the main text, $(2RT_0)^{1/2}$ is used instead of $(2e_0)^{1/2}[= (3RT_0)^{1/2}]$ to define the expansion [note the definition $u_i = v_i/(2RT_0)^{1/2}$ and the discussion in the paragraph with Eq. (133a)].

(iii) In the preceding analysis, the equation of state, Eq. (132), is partially used in the middle of analysis. It is already used in Eq. (141c). Thus, the results, e.g., Eq. (141c), cannot directly be transferred to the case of incompressible fluid. The expansion, with P not related to ω and τ, should be done independently and apply the incompressible condition when necessary. It is much simpler to start with $\omega_n = 0$ for all n.

[46](i) From Eqs. (128) and (144), we have $\partial u_i/\partial x_i = 0$. Obviously, one of Eqs. (128) and (144) can be replaced by $\partial u_i/\partial x_i = 0$, but both cannot be replaced by it. Some confusion about the incompressibility is due to the misunderstanding of the statement.

(ii) It should be noted that Eqs. (141a) and (146a) are derived from Eq. (128) under the condition (133a) without the help of the equation of state. Incompressibility cannot be judged only by Eq. (141a) or (146a).

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{146a}$$

$$\frac{\partial u_{i1}}{\partial \tilde{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{146b}$$

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \tilde{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{146c}$$

and the equation corresponding to the first relation of Eq. (143) is obtained from the incompressible condition (144) as[47]

$$\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i} = 0. \tag{147}$$

Now the basic equations, the behavior of solutions of which we are going to compare, are prepared [Eqs. (140)–(141c), and (143) for perfect gas and Eqs. (145)–(147) for incompressible fluid]. For the comparison, the initial and boundary condition have to be chosen commonly. The diffusion time scale being natural time scale of the solution of the incompressible Navier–Stokes equation, this scale solution is the subject of comparison. Incidentally, the boundary data must be consistent with the diffusion time scale. The form of the two sets of equations requires two conditions on the initial data. That is, the initial values of u_{i1} and P_1 must be solenoidal and uniform respectively in the domain under consideration, i.e., Eqs. (141a) and (140) for perfect gas and Eqs. (146a) and (145) for incompressible fluid). The initial condition common to perfect gas and incompressible fluid is determined in the following way: From the request of incompressible fluid, the density is uniform, i.e., $\omega_1 = 0$; from common request of uniformity of pressure, i.e., $P_1 = 0$; on the plane $\omega_1 = 0$, the temperature $\tau_1 = 0$ from Eq. (143) for perfect gas[48] (see also Footnote 45). Only the uniform state with respect to density, pressure and temperature can be the common initial condition to perfect gas and incompressible fluid. The velocity can be chosen freely under the solenoidal condition. When comparing solutions in the two kinds of fluid, we have to choose their initial condition that satisfies the above condition. In this case, the time-variation of the boundary data make the difference clear. An example of comparison of this kind is shown in K-4.10.3, where nontrivial difference of their temperature fields are shown.[49] Next, we examine the two sets of equations (perfect gas and incompressible fluid) and make clear the effect or mechanism that makes the time development of the two

[47] From the choice of the background state in Footnote 45, $\omega = 0$ and $\partial u_i/\partial x_i = 0$ or $\omega_n = 0$ and $\partial u_{in}/\partial x_i = 0$ for any n.

[48] (i) On the surface $\omega_1 = 0$, the pressure P_1 and temperature τ_1 can be chosen freely for incompressible fluid.

(ii) We have chosen zero for the constant values for ω_1, P_1, and τ_1 without loss of generality. It is a problem of choosing the background state.

[49] The example in Section K-4.10.3 is a simple problem with the simple boundary and the interface. In the example, the S solution of the Boltzmann equation (note Footnote 43) and the corresponding one of the incompressible fluid are shown. In the former, the density varies with time, and further, the temperature field is quite different from that of the latter owing to the time-dependent boundary condition on P_{S1}, corresponding to P_1 here.

sets different. Then, we discuss the process of solution (or how the solution is constructed) for each set with its boundary condition taken into account and show an important difference between them.

Equations (145), (146a), and (146b) are, respectively, of the same form as Eqs. (140), (141a), and (141b). Equation (141c) is rewritten with the aid of Eqs. (140) and (143) as

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \tilde{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} - \left(\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right) = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{148}$$

The difference of Eq. (141c) or (148) from Eq. (146c) is

$$\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}, \tag{149}$$

which vanishes for an incompressible fluid. The work W done per unit time on unit volume of fluid by pressure, given by $-p_0(2RT_0)^{1/2}L^{-1}\partial(1+P)u_i/\partial x_i$, is transformed with the aid of Eqs. (140), (141a), and (142a) in the following way:

$$\begin{aligned}
\frac{W}{p_0(2RT_0)^{1/2}L^{-1}} &= -\frac{\partial(1+P)u_i}{\partial x_i} \\
&= -\frac{\partial u_{i1}}{\partial x_i}\varepsilon - \left(P_1\frac{\partial u_{i1}}{\partial x_i} + u_{i1}\frac{\partial P_1}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i}\right)\varepsilon^2 + \cdots \\
&= -\frac{\partial u_{i2}}{\partial x_i}\varepsilon^2 + \cdots \\
&= \left(\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right)\varepsilon^2 + \cdots .
\end{aligned} \tag{150}$$

The work vanishes up to the order considered here for incompressible fluid, because $\partial u_i/\partial x_i = 0$ and $\partial P_1/\partial x_i = 0$ [see Footnotes 46 and Eq. (145)]. That is, Eq. (141c) differs from Eq. (146c) by the amount of the work done by pressure. Thus, naturally, the temperature τ_1 fields in the two cases are different owing to this difference. Thus, naturally, the temperature τ_1 fields in the two cases are different owing to this difference.

The variation (149) of the density ω_1 along a fluid-particle path is expressed with u_{i1}, τ_1, and P_1. That is, ω_1 in Eq. (149) is replaced by $P_1 - \tau_1$ with the aide of Eq. (143), and Eqs. (140) and (141c) are applied to the result successively. Then, we have

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i} &= \frac{\partial P_1}{\partial \tilde{t}} - \frac{\partial \tau_1}{\partial \tilde{t}} - u_{i1}\frac{\partial \tau_1}{\partial x_i} \\
&= \frac{3}{5}\frac{\partial P_1}{\partial \tilde{t}} - \frac{1}{2}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}.
\end{aligned} \tag{151}$$

Therefore, the density ω_1 generally varies along a fluid-particle path.

Here, we will explain the process of solution (how the solution is determined from the basic equation and boundary condition) for the two systems (perfect gas and incompressible fluid). In the two systems, the difference is the energy equation among the conservation equations [Eqs. (140)–(141c) and Eqs. (145)–(146c)] and the equation of state [Eqs. (143) and (147)].[50] When the pressure P_1 is independent of $\tilde{t}$, the difference of the energy equations (141c) and (146c) in the two systems is the ratio of the convection term to the heat-conduction term. That is, the heat-conduction is of the same form, but the convection term differs only by numerical factor ($5/2$ and $3/2$). Thus, we can say the difference is the convection term by the ratio ($5:3$) or the thermal conductivity by the ratio ($3:5$). The source of this difference is due to the work done by pressure in perfect gas. The pressure P_1 being constant, the variation of the flow along a fluid particle path takes place under constant pressure during its motion, and the work done by pressure can be incorporated into the variation of enthalpy defined by $e+p/\rho$, which is equal to $5RT/2$ for perfect gas.[51] Thus, the change during the motion is expressed by the variation of the enthalpy. That is, the energy equation expresses that the enthalpy variation along a fluid particle path is equal to the energy supplied by heat conduction. On the other hands, when P_1 depends on $\tilde{t}$, the variation along a fluid particle path is neither under constant pressure nor under constant volume, and thus, the extra term $\partial P_1/\partial \tilde{t}$ enters Eq. (141c) in addition to the above difference of the coefficients. To find the solution, the boundary condition is required, which depends on the kind of the boundary. The present discussion is for the Navier–Stokes system apart from the kinetic theory except that sizes of viscosity and thermal conductivity, compared with the parameter ε, are borrowed from its result. The boundary conditions is the one used in the Navier–Stokes system. That is, we take the non-slip condition [Eq. (K-4.61a) and (K-4.61b) or Eq. (M-3.113a) and (M-3.113b)] on the simple boundary and the leading-order result of the kinetic theory [Eq. (K-4.68a) and (K-4.68c) or Eq. (M-3.119a) and (M-3.119c)] on the interface extended up to higher order in ε.[52] The same boundary condition is used for incompressible fluid. The process of solution of a similar system, the S solution in Chapter K-4 or Section M-3.2, is given in Section 3.1.1 for the simple boundary. This discussion applies to the present case.

In an unbounded domain where the pressure at infinity is imposed, $P_1(\tilde{t})$ is determined, but in a bounded domain with the simple boundary, it is undetermined. According to the above mentioned process, the velocity field u_{i1} is

[50] The difference of the equation of state is often treated carelessly. Equation (141a) or (146a) is mistaken to be incompressible condition with discarding Eqs. (143) and (147).

[51] (i) The factor $5R/2$ is the heat capacity at constant pressure for perfect gas.

(ii) In incompressible fluid, the pressure produces no work as noted just after Eq. (150).

[52] (i) The formulas for the interface is the leading-order result of S expansion of kinetic theory analysis. The non-slip condition is also confirmed by it.

(ii) The formulas quoted above are derived for time-independent problems. The results are shown to be applicable to the time-dependent problem with the time scale under discussion in Section M-3.7.3.

(iii) In the formulas in the two books, the subscript S is to be neglected. The subscript 1 showing the order is extended to $2, \cdots$. The formulas with subscript K is to be discarded.

(iv) Keeping the fundamental form of the condition on the interface, we generalize the formula formulas allowing the coefficients, C_4^* and d_4^*, to be functions of position and time, and the discussion is made under the generalized boundary condition.

determined, which is independent of P_1, for a simple boundary. On the other hand, when the boundary is the interface or a part of it is the interface, P_1 is determined by the boundary condition, as shown in Footnote 53 below, owing to the presence of P_1 in the boundary condition on $u_{i1}n_i$.[53] From P_1 determined, the boundary value of $u_{i1}n_i$ on the interface is determined by one of the relations in Eq. (K-4.68c) or Eq. (M-3.119c). From $u_{i1}n_i$ just determined, the boundary value of τ_1 on the interface is determined by another relation in Eq. (K-4.68c) or Eq. (M-3.119c).[54] Thus, all kinds of boundary data are prepared. The velocity field u_{i1} is determined under the boundary data just obtained as in Section 3.1.1 with the aid of Footnote 55 below[55]. Then, we can determine the temperature

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[53](i) For the case of the interface, $P(\tilde{t})$ enters the boundary condition on the velocity normal to the boundary. On the other hand, the integral of $u_{i1}n_i$ over the boundary vanishes owing to Eq. (141a), i.e.,

$$\int_S u_{i1}n_i\mathrm{d}S = -\int_V \frac{\partial u_{i1}}{\partial x_i}\mathbf{d}\boldsymbol{X} = 0.$$

This determines the relation of P_1 and the integral of the boundary data P_{w1}, i.e.

$$P_1(\tilde{t})\int_S (1/C_4^*)\mathrm{d}S - \int_S (P_{w1}/C_4^*)\mathrm{d}S = \int_S u_{i1}n_i\mathrm{d}S, = 0.$$

Thus,

$$P_1(\tilde{t}) = \frac{\int_S (P_{w1}/C_4^*)\mathrm{d}S}{\int_S (1/C_4^*)\mathrm{d}S},$$

where S is the surface of the boundary [see Eq. (K-4.68c) or Eq. (M-3.119c)]. It is noted that C_4^* is a constant in the formula by the kinetic theory where the complete condensation boundary is considered. Here, C_4^* is allowed to be a given function of x_i and t to express more general condition of the interface ($0 < -1/C_4^* \leq c_{\mathrm{M1}}$; c_{M1} : const). Thus, P_1 is determined. With this $P_1(\tilde{t})$, the boundary data of $u_{i1}n_i$ is specified by Eq. (K-4.68c) or Eq. (M-3.119c). Thus, the boundary value of u_i is given by this $u_{i1}n_i$ and Eq. (K-4.68a) or (M-3.119a).

(ii) When the boundary consists of the two kinds of boundaries, By the same reason, the integral $u_{i1}n_i$ over the boundary vanishes, which is divided into the contributions of the two kinds of boundaries. That is,

$$\int_{S_{\mathrm{S}}} u_{i1}n_i\mathrm{d}S + \int_{S_{\mathrm{I}}} u_{i1}n_i\mathrm{d}S = 0,$$

where S_{S} and S_{I} indicate, respectively, the simple boundary and the interface. The first integral vanishes because $u_{i1}n_i = 0$ on the simple boundary. Thus, the second integral also vanishes. Then, similarly to note (i), we have

$$P_1(\tilde{t}) = \frac{\int_{S_{\mathrm{I}}} (P_{w1}/C_4^*)\mathrm{d}S}{\int_{S_{\mathrm{I}}} (1/C_4^*)\mathrm{d}S}.$$

Thus, $P_1(\tilde{t})$ is determined, from which $u_{i1}n_i$ on the interface is given by Eq. (K-4.68c) or Eq. (M-3.119c). With this data, the boundary value of u_{i1} is specified on the interface. Together with the condition on the simple boundary, the boundary value of u_{i1} is determined.

[54]The constant d_4^* in the formula is allowed be a given function of x_i and t as C_4^* is allowed to be so in Footnote 53 ($0 < -d_4^* \leq c_{\mathrm{M2}}$; c_{M2} : const).

[55]In the process solving the velocity field u_{iS1} in the simple boundary problem in Section 3.1.1, P_{S2} is required to satisfy the relation

$$\frac{\partial^2 P_{S2}}{\partial x_i^2} = -2\frac{\partial u_{jS1}}{\partial x_i}\frac{\partial u_{iS1}}{\partial x_j},$$

in order to ensure the subsequent (or future) solenoidal condition of u_{iS1}. According to Section 3.1.1, to determine the boundary value of $n_i\partial P_{S2}/\partial x_i$, the time-derivative of the boundary

τ_1 from Eq. (141c) except for the bounded domain with the simple boundary. With the determined P_1 and τ_1, the density ω_1 is determined as

$$\omega_1 = P_1 - \tau_1 \tag{152}$$

by the first relation of Eq. (143).

In the exceptional case, Eq. (141c) contains two unknown functions τ_1 and P_1. We need another condition to determine τ_1 and P_1. When the temperature of the boundary is time-dependent and nonuniform, so is the solution τ_1 in the domain, irrespective of $\partial P_1/\partial\tilde{t}$. The density ω_1 is given by $P_1 - \tau_1$, which is time-dependent and nonuniform and includes undetermined P_1. In a bounded domain of the simple boundary, the mass of the fluid in the domain is invariant. This has to be confirmed.[56] The condition that the mass in the domain V is invariant is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \omega_1 \mathbf{d}\boldsymbol{x} = \frac{\mathrm{d}P_1(\tilde{t})}{\mathrm{d}\tilde{t}}V - \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_1 \mathbf{d}\boldsymbol{x} = 0, \tag{153}$$

where V also indicates its volume. On the other hand, the integral form of Eq. (141c) is

$$\frac{\mathrm{d}P_1(\tilde{t})}{\mathrm{d}\tilde{t}}V - \frac{5}{2}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_V \tau_1 \mathbf{d}\boldsymbol{x} = \int_V \left(\frac{5}{2}u_{i1}\frac{\partial\tau_1}{\partial x_i} - \frac{5}{4}\gamma_2\frac{\partial^2\tau_1}{\partial x_j^2}\right)\mathbf{d}\boldsymbol{x}. \tag{154}$$

From these two equations, the equation for P_1 is obtained as

$$\frac{\mathrm{d}P_1(\tilde{t})}{\mathrm{d}\tilde{t}} = -\frac{5}{3V}\int_V \left(u_{i1}\frac{\partial\tau_1}{\partial x_i} - \frac{1}{2}\gamma_2\frac{\partial^2\tau_1}{\partial x_j^2}\right)\mathbf{d}\boldsymbol{x}. \tag{155}$$

From two equations (141c) and (155), The temperature $\tau(x_i,\tilde{t})$ and the pressure $P_1(\tilde{t})$ are determined. Then, ω_1 is determined by Eq. (152). Thus, we have

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data of $u_{iS1}n_i$ is required (u_{iS1} and P_{S2} corresponds to u_{i1} and P_2 here). It vanishes because $u_{iS1}n_i = 0$ on the simple boundary. In the interface problem, the boundary value of $u_{i1}n_i$ is given by the condition (K-4.68c) or (M-3.119c) and expressed with known data $P_1(\tilde{t})$ and P_{w1} [see Footnote 53 (i)]. Thus, $\partial u_{i1}n_i/\partial\tilde{t}$ on the interface is known. This is used in constructing the boundary data $n_i\partial P_2/\partial x_i$ as explained in Section 3.1.1. Thus, P_2 is determined with an arbitrary additive function $f_2(\tilde{t})$ of $\tilde{t}$. For the combined boundary in (ii) of Footnote 53, we can obtain P_2 by the combination of $n_i\partial P_2/\partial x_i$ of the two kinds of boundaries, with an arbitrary additive function of $\tilde{t}$. From P_2 thus obtained, we can obtain the solution u_{i1} of Eq (141b) with the boundary condition on u_{i1} for the cases (i) and (ii) of Footnote 53 in the same way as Section 3.1.1. This u_{i1} satisfies the solenoidal condition in subsequent $\tilde{t}$. The additive function in P_2 does not influence the solution u_{i1}.

[56](i) It has been made clear by the analysis up to now that the condition $u_{i1}n_i = 0$ on the boundary does not guarantee that mass flow in the diffusion time scale $\tilde{t} = O(1)$ or $t = O[L/(2RT_0)^{1/2}\varepsilon]$ is negligible compared with the quantity of $O(\varepsilon)$ under concern.

(ii) Here, the order of variation of density by inflow to or outflow from a volume $O(L^3)$ by the higher-order velocity $(2RT_0)^{1/2}\varepsilon^2$ in time $L/(2RT_0)^{1/2}\varepsilon$ is estimated. The inflow or outflow of the mass of fluid over the surface $O(L^2)$ of the volume during the time is of the order of $\rho_0 \times (2RT_0)^{1/2}\varepsilon^2 \times L/(2RT_0)^{1/2}\varepsilon \times L^2$, which is $\rho_0\varepsilon L^3$. Thus, the density varies by the order of $\rho_0\varepsilon$, which is of the same order as the term ω_1 of the expansion of ρ in ε.

obtained the required information for perfect gas, up to the order of ε under concern.

At the final stage of the discussion of the process of solution, we briefly describe the process for incompressible fluid, where $\omega = 0$. The process up to the step to obtain the velocity u_{i1} is the same as for perfect gas. That is, the velocity field u_{i1} is determined independently of undetermined $P_1(\tilde{t})$ for a bounded domain with the simple boundary. In the other cases, the velocity field u_{i1} is determined together with $P_1(\tilde{t})$. The energy equation (146c) contains only τ_1 and determined u_{i1} without $P_1(\tilde{t})$ term. It is similar to the equation for perfect gas with difference of numerical coefficient and the absence of $P_1(\tilde{t})$ term. Thus, the solution τ_1 is determined. The pressure $P_1(\tilde{t})$, however, is left undetermined for a bounded domain with the simple boundary. This situation corresponds to the situation described in Section 3.2.3. Obviously, the mass is invariant in a bounded domain because the density is invariant, which does not work to determine $P_1(\tilde{t})$. This ambiguity is due to the combination of the two limiting characters of the incompressible fluid and the simple boundary.[57]

[57] Incompressible fluid is the extreme (or limiting) case of fluid very hard to compress. Suppose that the equation of state is given by

$$\omega = \left(\frac{P-\tau}{1+\tau}\right)^{2m+1} \qquad (m = 0, 1, 2, \cdots), \tag{†1}$$

where $m = 0$ corresponds to perfect gas. Then,

$$\omega_n = 0 \quad \text{for } n \leq 2m, \tag{†2a}$$

$$\omega_{2m+1} = (P_1 - \tau_1)^{2m+1}. \tag{†2b}$$

The equation of state (†1) is reduced to incompressible fluid in the limit $m \to \infty$ because $\omega = 0$ in the limit irrespective of the perturbations P and τ of the order of ε. For $m \geq 1$ (fluid hard to compress; harder for larger m), the conservation equations are the same as those for incompressible fluid up to $n = 2m$, That is, Eqs. (145)–(146c), where the contribution of work done by pressure is absent, apply to fluid hard to compress commonly. In a bounded domain with the simple boundary, the velocity u_{i1} and temperature τ_1 are determined irrespective of P_1, but P_1 cannot be specified within the above set of equations, and the mass in V is invariant up to the level ω_{2m}. The undetermined pressure P_1 is determined by Eq. (†2b) and the mass conservation at the level ω_{2m+1} in the volume V, i.e., $\mathrm{d}\int_V \omega_{2m;+1}\mathbf{d}\boldsymbol{x}/\mathrm{d}\tilde{t} = 0$. From them, we obtain the condition

$$\int_V (P_1 - \tau_1)^{2m} \left(\frac{\mathrm{d}P_1}{\mathrm{d}\tilde{t}} - \frac{\partial \tau_1}{\partial \tilde{t}}\right) \mathbf{d}\boldsymbol{x} = 0.$$

With the aid of Eq. (146c),

$$\frac{\mathrm{d}P_1}{\mathrm{d}\tilde{t}} = \int_V (P_1 - \tau_1)^{2m} \left(-u_{i1}\frac{\partial \tau_1}{\partial x_i} + \frac{5}{6}\gamma_2 \frac{\partial^2 \tau_1}{\partial x_i^2}\right) \mathbf{d}\boldsymbol{x} \bigg/ \int_V (P_1 - \tau_1)^{2m} \mathbf{d}\boldsymbol{x}\,, \tag{†3}$$

where $\mathrm{d}P_1/\mathrm{d}\tilde{t}$ is expressed with the data of the present state (u_{i1}, τ_1, P_1). Thus, the future P_1, thus (u_{i1}, τ_1, P_1), is determined. In incompressible fluid, the limit $m \to \infty$ is taken first. After the limit $m \to \infty$, the number n showing the level of expansion of the solution in ε is smaller than $2m + 1$ $(=\infty)$, and therefore $\omega_n = 0$ for any n, and P_1 remains undetermined. This is the ambiguity mentioned in Section 3.2.3.

To summarize, for any finite positive m (≥ 1), the solution (u_{i1}, τ_1, P_1) in a bounded domain with the simple boundary is determined by the conservation equations (145)–(146c) with the mass conservation condition (†3) in V, and the velocity u_{i1} and temperature τ_1 are the same as those of incompressible fluid. This solution is distinct from the solution for perfect gas

To summarize, the mass and momentum equations (141a) and (141b) of the set derived from the compressible Navier–Stokes set [Eqs. (128)–(131b) and (132)] under the situation given by Eqs. (133a) and (133b) with small ε are of the same form as the corresponding equations (146a) and (146b) of the set derived from the incompressible Navier–Stokes set [Eqs. (128)–(131b) and (144)], but the energy equations (141c) and (146c) of the two sets differ by the work done by pressure. The density ω_1 obtained from u_{i1}, τ_1, and P_1 by the first relation of Eq. (143) does not generally satisfy the incompressible condition (144). Both the density and temperature fields (ω_1, τ_1) are different in the two sets. The variation of the density ω_1 along a particle path is given by Eq. (151). Even if the temperature τ_1 varies according to Eq. (146c), the density ω_1 determined by Eq. (151) does not generally satisfy the incompressible condition. Further, the two systems have a decisive difference in bounded domain problems with the simple boundary. That is, the pressure P_1 is undetermined in the incompressible fluid system, but no such ambiguity exists in the perfect gas system, where the pressure P_1 is determined.

Finally, it may be repeated that under the situation (133a), the solenoidal condition for u_{i1}, i.e., Eq. (141a) or (146a), is derived only from the mass conservation equation (128) without the help of the equation of state [perfect gas condition (132) or the incompressible condition (144)]. That is, the mass conservation equation at the level of $O(\varepsilon)$ are common to perfect gas and incompressible fluid, i.e.,

$$\frac{\partial u_{i1}}{\partial x_i} = 0,$$

but the equation of state is different for the two kinds of fluid, i.e.,

$$\begin{aligned} &\omega_1 = P_1 - \tau_1 \quad \text{(perfect gas)}, \\ &\omega_1 = 0 \quad \text{(incompressible fluid)}. \end{aligned}$$

---

because the work done by pressure is absent in the energy equation (146c). The solution for incompressible fluid is different from both the solutions, perfect gas ($m = 0$) or fluid hard to compress ($m \geq 1$), on the point of the ambiguity of P_1. The solution for perfect gas is totally different from the solution for incompressible fluid. The solution for fluid hard to compress partially agrees with the solution for incompressible fluid. In this sense, the solution hard to compress is on the side of incompressible fluid. The incompressible condition is too strongly simplified to approximate the solution for fluid hard to compress (note that any fluid is not exactly incompressible). It sacrifices the determinacy of P_1, which is the qualitative difference from perfect gas and fluid hard to compress.

This kind of situation is often seen when some small parameter is approximated by its extreme value. Here, we have two parameters taken to their limits first (incompressible fluid and the simple boundary, the latter of which can be taken, for example, as the limiting case with the coefficient $1/C_4^*$ in the interface condition being taken to zero). Another well-known example is the approximation of fluid with small viscosity (Navier–Stokes equation with large Reynolds number) by ideal fluid (Euler equation). The ghost effect of infinitesimal curvature in References [3]–[5] and Section 7.3 (see also Chapter M-9 and Section 7.2) is another aspect of this kind of behavior, where we see that a plane wall or straight pipe is too strong simplification (any plane wall or straight pipe to be discussed is not exactly so). The ghost effect of infinitesimal curvature is shown by analysis with the limiting processes as above taken into consideration.

From the above discussion, it is clear that the two systems under consideration (perfect gas and incompressible fluid) are distinct.

The solenoidal condition (141a) or (146a), i.e., $\partial u_{i1}/\partial x_i = 0$, does not guarantee the invariance of the density ω_1 in the diffusion time scale $\tilde{t} = O(1)$ (see Footnote 56).[58]

3.2.5 Equations derived from the compressible Euler set when the Mach number and the temperature variation are small

Take the Euler set of equations, Eqs. (M-3.250a)–(M-3.250c) and the equation of state, in the nondimensional form derived from the Boltzmann equation in the limit $k \to 0$:

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial \hat{\rho}\hat{v}_j}{\partial x_j} = 0, \tag{156a}$$

$$\frac{\partial \hat{\rho}\hat{v}_i}{\partial \hat{t}} + \frac{\partial \hat{\rho}\hat{v}_j\hat{v}_i}{\partial x_j} + \frac{1}{2}\frac{\partial \hat{p}}{\partial x_i} = 0, \tag{156b}$$

$$\frac{\partial}{\partial \hat{t}}\left[\hat{\rho}\left(\hat{v}_i^2 + \frac{3}{2}\hat{T}\right)\right] + \frac{\partial}{\partial x_j}\left[\hat{\rho}\hat{v}_j\left(\hat{v}_i^2 + \frac{5}{2}\hat{T}\right)\right] = 0, \tag{156c}$$

$$\hat{p} = \hat{\rho}\hat{T}, \tag{156d}$$

where the subscript $H0$ is eliminated for simpleness of notation. We consider the situation where the state of the gas deviates slightly from a uniform equilibrium state at rest. That is,

$$\hat{\rho} = 1 + \hat{\omega},\ \hat{p} = 1 + \hat{P},\ T = 1 + \hat{\tau},\ \hat{v}_i = \hat{u}_i, \tag{157}$$

where the perturbed quantities $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ are small, say of the order of ε. They are expanded as

$$\hat{h} = \hat{h}_1\varepsilon + \hat{h}_2\varepsilon^2 + \cdots, \tag{158}$$

where $\hat{h} = \hat{\omega}$, $\hat{P}$, $\hat{\tau}$, or $\hat{u}_i$.

We discuss the two cases with different time scale. The first case is

$$\frac{\partial \hat{h}}{\partial \hat{t}} = O(\hat{h}). \tag{159}$$

Substituting the expansions (158) of the variables $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ into the Euler equations (156a)–(156d) and arranging the same-order terms with Eq. (159) in mind, we find that the leading-order variables are governed by the following set

[58] Note the size ε of the variation of quantities, the diffusion time-scale $[L/(2RT_0)^{1/2}\varepsilon]$ under consideration, and the nonlinearity in the mass conservation equation (128). Owing to these situations, the density variation, i.e,

$$\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i},$$

along the fluid particle path is obtained from u_i, τ_1, and P_1, and is generally finite ($\neq 0$), affecting $\partial u_{i2}/\partial x_i$ in Eq. (142a).

of equations:

$$\frac{\partial \hat{\omega}_1}{\partial \hat{t}} + \frac{\partial \hat{u}_{j1}}{\partial x_j} = 0, \tag{160a}$$

$$\frac{\partial \hat{u}_{i1}}{\partial \hat{t}} + \frac{1}{2}\frac{\partial \hat{P}_1}{\partial x_i} = 0, \tag{160b}$$

$$\frac{\partial \hat{P}_1}{\partial \hat{t}} + \frac{5}{3}\frac{\partial \hat{u}_j}{\partial x_j} = 0, \tag{160c}$$

$$\hat{P}_1 = \hat{\omega}_1 + \hat{\tau}_1. \tag{160d}$$

This set is the well-known acoustic equations (see Section M-3.7.1), which are explained in a standard textbook of gas dynamics, e.g., M-Liepmann & Roshko [1957].

The second case is the case where the variables are slowly varying or the time scale of variation of the variables is long and of the order $1/\varepsilon$:

$$\frac{\partial \hat{h}}{\partial \hat{t}} = \varepsilon O(\hat{h}). \tag{161}$$

Here, we introduce the shrunk time $\hat{t}_\varepsilon$:

$$\hat{t}_\varepsilon = \varepsilon \hat{t}. \tag{162}$$

Then,

$$\frac{\partial \hat{h}}{\partial \hat{t}_\varepsilon} = O(\hat{h}). \tag{163}$$

Substituting the expansion (158) of the variables $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ into the Euler equations (156a)–(156d) and arranging the same-order terms with Eq. (163) in mind, we obtain the equations that determine the leading-order variables as[59]

$$\frac{\partial \hat{P}_1}{\partial x_i} = 0, \tag{164a}$$

$$\frac{\partial \hat{u}_{j1}}{\partial x_j} = 0, \tag{164b}$$

$$\frac{\partial \hat{u}_{i1}}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{u}_{i1}}{\partial x_j} + \frac{1}{2}\frac{\partial \hat{P}_2}{\partial x_i} = 0, \tag{164c}$$

$$\frac{5}{2}\frac{\partial \hat{\tau}_1}{\partial \hat{t}_\varepsilon} - \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \frac{5}{2}\hat{u}_{j1}\frac{\partial \hat{\tau}_1}{\partial x_j} = 0, \tag{164d}$$

$$\hat{P}_1 = \hat{\omega}_1 + \hat{\tau}_1. \tag{164e}$$

[59] Under the assumptions (157) and (161) or (163), the solenoidal condition (164b) for $\hat{u}_{i1}$ is derived solely from the mass conservation equation (156a). It should not be confused with the incompressible condition.

From Eq. (164a), $\hat{P}_1$ is a function of $\hat{t}_\varepsilon$ only, and thus is determined by the boundary condition.[60] The relation

$$\frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} = \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{P}_1}{\partial x_j}, \tag{165}$$

obvious from Eq. (164a), is conveniently used in the following discussion. The energy equation (164d) is transformed as

$$\frac{3}{2}\left(\frac{\partial \hat{\tau}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{\tau}_1}{\partial x_j}\right) - \frac{\partial \hat{\omega}_1}{\partial \hat{t}_\varepsilon} - \hat{u}_{j1}\frac{\partial \hat{\omega}_1}{\partial x_j} = 0, \tag{166}$$

by using Eqs. (164e) and (165) for $\partial \hat{P}_1/\partial \hat{t}_\varepsilon$. From Eqs. (166) and (164e), the variation of $\hat{\omega}_1$ along the fluid-particle path is expressed as follows:

$$\frac{\partial \hat{\omega}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{\omega}_1}{\partial x_j} = \frac{3}{5}\left(\frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{P}_1}{\partial x_j}\right) = \frac{3}{5}\frac{\mathrm{d}\hat{P}_1}{\mathrm{d}\hat{t}_\varepsilon}. \tag{167}$$

Eqs. (166) and (167) are the linearized forms of the isentropic variations of $\hat{\omega}_1$ versus $\hat{\tau}_1$ and $\hat{P}_1$ along the fluid-particle path. The energy equation (166) is conveniently compared with the energy equation of incompressible fluid. For the latter, the last two terms are absent and the temperature is invariant along the fluid-particle path. The difference is the work done by pressure, which can be shown as is done in Section 3.2.4.

The behavior of the gas governed by Eqs. (164a)–(164e) is summarized as follows:
(1): Equations (164b) and (164c) for the velocity field are of the same form as those of incompressible fluid.
(2): Depending on the condition of the boundary, $\hat{P}_1$ can be time dependent or independent. (i) If $\hat{P}_1$ is time dependent, the density $\hat{\omega}_1$ varies along the fluid-particle path owing to Eq. (167). (ii) If $\hat{P}_1$ is time independent, the temperature $\hat{\tau}_1$ and the density $\hat{\omega}_1$ are invariant along the fluid-particle path owing to Eqs. (164d) and (167).

(Section 3.2.5: Version 8-00)

4 Chapter M-4

4.1 Notes on application of the solution in Section M-4.3

In the application of the quasi-unidirectional solution in Section M-4.3, some cares are required. For some ranges from the entrance and from the exit of the pipe, the assumptions made in the first paragraph of Section M-4.3 and the assumption[61] just after Eq. (M-4.64) are not generally satisfied. For a long

[60] For example, the pressure is specified at infinity in an unbounded problem.
[61] This condition is consistent with the preceding assumptions.

pipe,[62] it is expected that the three regions, entrance, central, and exit regions, can be analyzed separately and that the results can be smoothly connected. In the central region, we try to use the solution of Section M-4.3. This process being successfully done, and the contributions of the entrance and exit regions to the pressure and temperature variations being estimated to be much smaller than the contributions of the central region, the solution in Section M-4.3 gives the global behavior of flow through a long pipe. It is often applied without confirmation that the end effects are so small as to be neglected. The solution of the central region, for which the solution of Section M-4.3 is used, has to be confirmed that it satisfies the above assumptions made in the analysis. For example, if the vacuum condition (or the vanishing density condition) is directly applied to the exit when exit is connected to a large vacuum chamber, one finds the average flow velocity on the cross section of the exit is infinite owing to the mass flow conservation through the pipe. This obviously violates the latter assumption. The solution cannot be applied up to such low density (or pressure) region. The contribution of the region where the assumption is violated has to be investigated in more complete formulation.

In a pipe problem, the temperature is controlled locally by the temperature of the pipe, but the pressure is controlled only at the entrance and the exit. The local pressure in between is determined by the mass conservation condition as shown in Section M-4.3. Thus, the nondimensional local pressure gradient cannot be specified at our disposal. For example, let us examine how the solution on the basis of the local linear theory breaks down in a straight pipe with a uniform cross section and a uniform temperature. According to Eq. (M-4.77) required by the mass flow conservation through the pipe, the quantities at the cross section A and the cross section B are related as follows:

$$p_{\mathrm{A}}\left(\frac{L}{p}\frac{\mathrm{d}p}{\mathrm{d}X_1}\right)_{\mathrm{A}}\hat{M}_{\mathrm{P}}(k_{\mathrm{A}})=p_{\mathrm{B}}\left(\frac{L}{p}\frac{\mathrm{d}p}{\mathrm{d}X_1}\right)_{\mathrm{B}}\hat{M}_{\mathrm{P}}(k_{\mathrm{B}}),$$

where the subscripts A and B indicate the values at the cross sections A and B respectively, X_1 is the coordinate along the pipe, and the unnecessary common factors T_w and L are eliminated from the formula (M-4.77). The nondimensional pressure gradient at the cross section B is expressed with that at A as

$$\left(\frac{L}{p}\frac{\mathrm{d}p}{\mathrm{d}X_1}\right)_{\mathrm{B}}=\frac{p_{\mathrm{A}}\hat{M}_{\mathrm{P}}(k_{\mathrm{A}})}{p_{\mathrm{B}}\hat{M}_{\mathrm{P}}(k_{\mathrm{B}})}\left(\frac{L}{p}\frac{\mathrm{d}p}{\mathrm{d}X_1}\right)_{\mathrm{A}},$$

where the ratio $\hat{M}_{\mathrm{P}}(k_{\mathrm{A}})/\hat{M}_{\mathrm{P}}(k_{\mathrm{B}})$ is bounded from below by a positive constant when $k_{\mathrm{B}}>k_{\mathrm{A}}$ and becomes infinite as $k_{\mathrm{A}}\to 0$ (see, for example, Table M-5.3, M-Sone & Yamamoto [1968]).[63] Thus, $|(L/p_{\mathrm{B}})(\mathrm{d}p/\mathrm{d}X_1)_{\mathrm{B}}|$ becomes very large or infinite even when $|(L/p_{\mathrm{A}})(\mathrm{d}p/\mathrm{d}X_1)_{\mathrm{A}}|$ is small if $p_{\mathrm{A}}/p_{\mathrm{B}}\gg 1$ or $k_{\mathrm{A}}\ll 1$,

[62] This means that the length of the pipe is much larger than the linear dimension of its cross section.

[63] Generally, $\hat{M}_{\mathrm{P}}(k)$ first decreases from infinity as k increases from zero, reaching the minimum value at some k (around $k=\sqrt{\pi}$ or $\mathrm{Kn}=2$ in Table M-5.3) (Knudsen minimum) and increases to the finite value at $k=\infty$.

and the slowly varying assumption $|(L/p)(\mathrm{d}p/\mathrm{d}X_1)| \ll 1$ is violated at the cross section B. Thus, the solution in Section M-4.3 is generally no longer valid there, and more complete analysis is required. In practical applications, the quantities that are assumed to be small may be small but are not very small. Thus, they may easily reach non-small values in another cross section.

(Section 4.1: Version 8-00)

4.2 Gas over a plane interface: Supplement to M-4.4

Here, the discussion of the half-space problem under the boundary condition (M-1.26) for a simple boundary in Section M-4.4 is extended to that under the boundary condition (M-1.30) or (285) for an interface of a gas and its condensed phase. That is, a plane simple boundary is replaced by a plane condensed phase of the gas, and the possible solution including the possible state at infinity is discussed in the situation when no evaporation or condensation is taking place on the condensed phase. This is the problem first discussed by Golse under the complete condensation condition (M-Bardos, Golse & Sone [2006]), which is a special case of the boundary condition (M-1.30). The analysis goes parallel to that in Section M-4.4. The full explanation is given with the difference being shown with footnotes, though it may be redundant.

Consider a semi-infinite expanse of a gas ($X_1 > 0$) bounded by its stationary plane condensed phase with a uniform temperature T_w at $X_1 = 0$. There is no external force acting on the gas. The state of the gas is time-independent and uniform with respect to X_2 and X_3, i.e., $f = f(X_1, \boldsymbol{\xi})$, and it approaches an equilibrium state as $X_1 \to \infty$, i.e.,

$$f \to \frac{\rho_\infty}{(2\pi RT_\infty)^{3/2}} \exp\left(-\frac{(\xi_i - v_{i\infty})^2}{2RT_\infty}\right) \quad \text{as } X_1 \to \infty, \tag{168}$$

where ρ_∞, $v_{i\infty}$, and T_∞ are bounded. The boundary condition on the interface is given by Eq. (285) with the conditions (286a)–(286c) and (289), i.e.,

$$f(0, \boldsymbol{\xi}) = g_I + \int_{\xi_{1*}<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(0, \boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_* \quad (\xi_1 > 0). \tag{169}$$

Here, we are interested in the case where no evaporation or condensation is taking place on the condensed phase,[64] i.e.,

$$\rho v_1 = \int \xi_1 f \mathbf{d}\boldsymbol{\xi} = 0 \quad \text{at } X_1 = 0. \tag{170}$$

We will show that the solution of the Boltzmann equation (M-1.5), i.e.,

$$\xi_1 \frac{\partial f}{\partial X_1} = J(f, f), \tag{171}$$

[64] No mass flux across the boundary irrespective of a situation is the definition of a simple boundary.

describing the above situation exists only when

$$v_{i\infty} = 0, \quad \rho_\infty = \rho_w, \quad T_\infty = T_w,$$

where ρ_w is the saturation gas density at temperature T_w, and that the solution is uniquely given by the Maxwellian

$$f = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{\xi_i^2}{2RT_w}\right). \tag{172}$$

From the integral of the Boltzmann equation (171) over the whole space of $\boldsymbol{\xi}$ [or the conservation equation (M-1.12)], i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}X_1}\left(\int \xi_1 f \mathbf{d}\boldsymbol{\xi}\right) = 0,$$

and Eq. (170), we find that the mass flux vanishes for $X_1 \geq 0$, i.e.,

$$\int \xi_1 f \mathbf{d}\boldsymbol{\xi} = 0 \quad (0 \leq X_1 < \infty). \tag{173}$$

With this result in the condition (168) at infinity, we have

$$\int \xi_1 \xi_i^2 f \mathbf{d}\boldsymbol{\xi} = 0 \quad \text{at infinity}. \tag{174}$$

The integral of the Boltzmann equation (171) multiplied by ξ_j^2 over the whole space of $\boldsymbol{\xi}$ [or the conservation equation (M-1.14)] gives

$$\frac{\mathrm{d}}{\mathrm{d}X_1}\left(\int \xi_1 \xi_j^2 f \mathbf{d}\boldsymbol{\xi}\right) = 0. \tag{175}$$

Thus, from Eqs. (174) and (175), we have

$$\int \xi_1 \xi_j^2 f \mathbf{d}\boldsymbol{\xi} = 0 \quad (0 \leq X_1 < \infty). \tag{176}$$

For the boundary condition (285) with the conditions (286a)–(286c) and (289), the following inequality holds at $X_1 = 0$ [Eq. (308) with $\rho v_1 = 0$, $v_{wi} = 0$, $n_i = (1, 0, 0)$]:[65]

$$\int \xi_1 f \ln(f/f_w) \mathbf{d}\boldsymbol{\xi} \leq 0, \tag{177}$$

where f_w is the Maxwellian with the temperature T_w and velocity v_{wi} $(= 0)$ of the condensed phase and the saturation gas density ρ_w at temperature T_w, i.e.,

$$f_w = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{\xi_i^2}{2RT_w}\right). \tag{178}$$

[65]The same equality holds for a simple boundary except that ρ_w in f_w is a free parameter for this case (see Section M-4.4).

With the aid of Eqs. (173) and (176),

$$\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi} \leq \int \xi_1 f \ln(f_w/c_0)\mathbf{d}\boldsymbol{\xi}$$
$$= -\frac{1}{2RT_w}\int \xi_1 \xi_i^2 f \mathbf{d}\boldsymbol{\xi} = 0 \quad \text{at } X_1 = 0, \tag{179}$$

where c_0 is a constant to make the argument of the logarithmic function dimensionless, whose choice does not influence the result.

On the other hand, from the H theorem, i.e., Eq. (M-1.36), in a time-independent one-dimensional case,

$$-\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\bigg|_{X_1=0} + \int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\bigg|_{X_1=\infty} = \int_0^\infty G \mathrm{d}X_1 \leq 0, \tag{180}$$

where

$$G = -\frac{1}{4m}\int (f'f'_* - ff_*)\ln\left(\frac{f'f'_*}{ff_*}\right) B \mathrm{d}\Omega \mathbf{d}\boldsymbol{\xi}_* \mathbf{d}\boldsymbol{\xi} \leq 0.$$

From Eqs. (168), (173), and (174), the second term on the left-hand side of Eq. (180) vanishes, that is,

$$-\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\bigg|_{X_1=0} = \int_0^\infty G \mathrm{d}X_1 \leq 0. \tag{181}$$

Combining the two inequalities (179) and (181), we have

$$0 \leq -\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\bigg|_{X_1=0} = \int_0^\infty G \mathrm{d}X_1 \leq 0.$$

Therefore, we have

$$\int_0^\infty G \mathrm{d}X_1 = 0, \quad \text{thus,} \quad G = 0, \tag{182}$$

and

$$\int \xi_1 f \ln(f/c_0)\mathbf{d}\boldsymbol{\xi}\bigg|_{X_1=0} = 0.$$

From Eq. (182), f is Maxwellian in $0 < X_1 < \infty$, and Eq. (171) is reduced to $\xi_1 \partial f/\partial X_1 = 0$. That is, f is a uniform Maxwellian. From the condition (168) at infinity and Eq. (173), the solution is to be in the form

$$f = \frac{\rho_\infty}{(2\pi RT_\infty)^{3/2}}\exp\left(-\frac{\xi_1^2 + (\xi_2 - v_{2\infty})^2 + (\xi_3 - v_{3\infty})^2}{2RT_\infty}\right) \quad (0 < X_1 < \infty). \tag{183}$$

From the uniqueness condition of Eq. (286c), the Maxwellian that satisfies the boundary condition (286c) is given by Eq. (178). Thus, the parameters in Eq. (183) have to be[66]

$$v_{2\infty} = v_{3\infty} = 0, \quad \rho_\infty = \rho_w, \quad T_\infty = T_w,$$

[66]For a simple boundary, we can choose ρ_∞ at our disposal, because ρ in Eq. (M-1.27c) is arbitrary.

and the solution is given by Eq. (172).

The same statement holds for the linearized Boltzmann equation with the corresponding general boundary condition (M-1.112) on an interface of the gas and its condensed phase. The temperature T_w of the condensed phase and the saturation gas density ρ_w at temperature T_w are, respectively, taken here as the reference temperature T_0 or $\tau_w = 0$ and the reference density ρ_0 or $\omega_w = 0$.[67] The linearized Boltzmann equation is given in the form

$$\zeta_1 \frac{\partial \phi}{\partial \eta} = \mathcal{L}(\phi) \quad (0 < \eta < \infty). \tag{184}$$

The boundary condition on the interface is given by Eq. (M-1.112) with the supplementary conditions (i), (ii-a), and (ii-b) as

$$E(\zeta)\phi(\eta, \boldsymbol{\zeta}) = \int_{\zeta_{1*}<0} \hat{K}_{I0}(\boldsymbol{\zeta}, \boldsymbol{\zeta}_*)\phi(\eta, \boldsymbol{\zeta}_*)E(\zeta_*)\mathbf{d}\boldsymbol{\zeta}_* \quad (\zeta_1 > 0) \quad \text{at } \eta = 0. \tag{185}$$

The condition at infinity is

$$\phi(\eta, \boldsymbol{\zeta}) \to \omega_\infty + 2\zeta_i u_{i\infty} + \left(\zeta_i^2 - \frac{3}{2}\right)\tau_\infty \quad \text{as } \eta \to \infty, \tag{186}$$

where ω_∞, $u_{i\infty}$ and τ_∞ are some constants and $\eta = x_1/k$ $(= 2X_1/\sqrt{\pi}\ell_0)$. Then, the solution of the boundary-value problem (184)–(186) exists when and only when

$$\omega_\infty = 0, \quad u_{i\infty} = 0, \quad \tau_\infty = 0, \tag{187}$$

and the unique solution is given by

$$\phi = 0. \tag{188}$$

The proof can be given in the same way as the preceding proof for the nonlinear case. From the conservation equation (M-1.99), i.e., $\partial u_1/\partial\eta = 0$, and the condition of absence of evaporation or condensation on the condensed phase ($u_1 = \int \zeta_1 \phi E \mathbf{d}\boldsymbol{\zeta} = 0$ at $\eta = 0$[68]), we have

$$u_1 = \int \zeta_1 \phi E \mathbf{d}\boldsymbol{\zeta} = 0 \qquad (0 \leq \eta < \infty). \tag{189}$$

Thus,

$$u_{i\infty} = 0. \tag{190}$$

From Eqs. (186) and (190),

$$\int \zeta_1 \phi^2 E \mathbf{d}\boldsymbol{\zeta} = 0 \quad \text{at infinity.} \tag{191}$$

[67] We take the reference density ρ_w in contrast with the case of a simple boundary. This is only for convenience of explanation. For this choice, ω_w term disappears in Eq. (185) but ω_∞ term appears in Eq. (186)

[68] The boundary where this equality holds irrespective of a situation is the definition of a simple boundary.

According to the second part of Section M-A.10,[69]

$$\int \zeta_1 \phi^2 E \mathbf{d}\boldsymbol{\zeta} \leq 0 \quad \text{at } \eta = 0. \tag{192}$$

The linearized-Boltzmann-equation version of the equation for the H function given by Eq. (M-1.115) is expressed as

$$\frac{\partial}{\partial \eta} \int \zeta_1 \phi^2 E \mathbf{d}\boldsymbol{\zeta} = LG, \tag{193}$$

where

$$LG = -\frac{1}{2} \int E E_* (\phi' + \phi'_* - \phi - \phi_*)^2 \widehat{B} \mathrm{d}\Omega \mathbf{d}\boldsymbol{\zeta}_* \mathbf{d}\boldsymbol{\zeta} \leq 0. \tag{194}$$

From Eqs. (191), (192), and (193) with Eq. (194), we find that LG is to be zero and that ϕ is a summational invariant or the linearized form of a Maxwellian, i.e.,

$$\phi = \omega + 2(\zeta_2 u_2 + \zeta_3 u_3) + \left(\zeta_i^2 - \frac{3}{2}\right)\tau,$$

where Eq. (189) is used. Then, Eq. (184) reduces to $\zeta_1 \partial\phi/\partial\eta = 0$, and therefore, ω, u_2, u_3, and τ are constant. In view of Eq. (186), the constants ω, u_2, u_3, τ, and ϕ are given as

$$\omega = \omega_\infty, \quad u_2 = u_{2\infty}, \quad u_3 = u_{3\infty}, \quad \tau = \tau_\infty,$$
$$\phi = \omega_\infty + 2(\zeta_2 u_{2\infty} + \zeta_3 u_{3\infty}) + \left(\zeta_i^2 - \frac{3}{2}\right)\tau_\infty.$$

Owing to the supplementary condition (ii-b) to the boundary condition (M-1.112) together with Eq. (190), we have[70]

$$\omega_\infty = 0, \quad u_{1\infty} = 0, \quad u_{2\infty} = 0, \quad u_{3\infty} = 0, \quad \tau_\infty = 0,$$
$$\phi = 0.$$

(Section 4.2: Version 5-00)

4.3 Onsager relation (Section M-4.5)

In the last paragraph of Section M-4.5, a short comment on the Onsager relation for the solution of the Boltzmann equation is given. Recently, comprehensive discussion of the symmetry of solutions of the linearized Boltzmann system and the Onsager relation in the system were given by Takata [2009a,b]. Making

[69]This is the linearized-Boltzmann-equation version of the inequality (308) and valid for both types of boundaries, a simple boundary and an interface. For the case of an interface, an additional condition (M-A.271), which corresponds to Eq. (289) in the nonlinear case, is imposed on the kernel $\hat{K}_{I0}$ (see also Footnote 108 in Section 8.4.2).

[70]Owing to the difference of the supplementary condition (ii-b) of Eq. (M-1.112) [or Eq. (185)] for an interface from the condition (iii) of Eq. (M-1.107) for a simple boundary, ω is determined for an interface. For a simple boundary, ω_∞ can be chosen at our disposal.

use of the property of the linearized kinetic boundary condition (see Sections M-1.11 and M-A.9), Takata considered three kinds of the Green function of the time-independent linearized Boltzmann equation, and showed symmetric relations among them. On the basis of this symmetric property, various symmetric relations of solutions of the time-independent linearized Boltzmann system were derived. Then, he proceeded to the discussion of the Onsager relation of the Boltzmann system. The incompleteness of M-Sharipov [1994a,b] was also mentioned there. Further, he tried to extend his works to time-dependent problems (Takata [2010]).

(Section 4.3: Version 10-00)

5 Chapter M-5

5.1 Flows induced by temperature fields and video files of their experiments

The addresses of the videos of the experiments on flows induced by temperature fields in Kyoto University Research Information Repository, which are permanent, are listed here.

Thermal creep flow

The Web address in Footnote M-5 in Section M-5.11 should be made by

https://hdl.handle.net/2433/120983,

which is the video file of a part of the experiments in Sone [1991].

(Section 5.1: Version 11-02)

The thermal creep flow [see Eq. (M-5.1)], i.e.,

$$v_1 = -K_1 \left(\frac{\pi R T_0}{2}\right)^{1/2} \frac{\ell_0}{T_0} \frac{\mathrm{d}T_w}{\mathrm{d}X_1}, \tag{195}$$

vanishes in the limit that the mean free path tends to zero ($\ell_0 \to 0$).[71] At the standard state, the mean free path is small but finite, and therefore, the thermal creep flow does not vanish. The mean free path is related to the thermal conductivity λ by Eq. (M-3.71), i.e.,

$$\ell_0 = \frac{4(2RT_0)^{1/2}}{5\sqrt{\pi}\gamma_2 R p_0}\lambda. \tag{196}$$

The above-mentioned formula of the thermal creep flow where the mean free path ℓ_0 replaced by the thermal conductivity λ with the above relation, i.e.,

$$v_1 = -\frac{4K_1\lambda}{5\gamma_2 p_0}\frac{\mathrm{d}T_w}{\mathrm{d}X_1}, \tag{197}$$

makes the thermal creep flow more accessible. This kind of replacement can be made between mean free path and viscosity [see Eq. (M-3.71)].

[71] Except in the thin layer adjacent to the boundary, Y_1 is negligible.

Thermal edge flow

The Web address in Footnote M-13 in Section M-5.14 should be replaced by

https://hdl.handle.net/2433/122357,

which is the video file of a part of the experiments in Sone & Yoshimoto [1997].

(Section 5.1: Version 11-02)

6 Chapter M-7

6.1 New reference

A mathematical work related to the subject in Chapter M-7 was published recently:

Liu, T.-P. and S.-H. Yu (2013), Invariant manifolds for steady Boltzmann flows and applications, *Arch. Rational Mech. Anal.* **209**, 869–997.

(Version 12-00)

7 Chapter M-9

7.1 Processes of solution of the equations with the ghost effect of infinitesimal curvature (July 2007)

The way in which Eqs. (M-9.33)–(M-9.39b) or Eqs. (M-9.49a)–(M-9.50e), including the time-dependent case with the additional time-derivative terms given by Eq. (M-9.42) or the mathematical expressions next to Eq. (M-9.59), contain the pressure terms, $(\hat{p}_{\mathfrak{S}0},\hat{p}_{\mathfrak{S}2})$ or (P_{01}, P_{02}, P_{20}), is different from the way in which the Navier–Stokes equations (M-3.265)–(M-3.266c) do the pressure terms, (P_{S1}, P_{S2}). In Section M-9.4, we consider the time-independent solution of Eqs. (M-9.49a)–(M-9.50e) [Eqs. (M-9.56)–(M-9.57d)] that is uniform with respect to $\bar{\chi}$. Here, it may be better to explain how a solution of Eqs. (M-9.33)–(M-9.39b) or Eqs. (M-9.49a)–(M-9.50e) in a general case or a time-dependent solution that depends on χ or $\bar{\chi}$ is obtained. Incidentally, the boundary conditions for the time-dependent case are derived in the same way as in Section M-3.7.3. Naturally from the derivation of the equations, the domain of a gas is in a straight pipe or channel of infinite length whose axis is in the x or χ direction.

7.1.1 Equations (M-9.33)–(M-9.39b):

Take Eqs. (M-9.33)–(M-9.39b) with the additional time-derivative terms given by Eq. (M-9.42), i.e.,[72]

$$\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial y} = \frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial z} = 0, \tag{198}$$

[72] Equation (M-9.33) is replaced by its equivalent form (198).

$$\frac{\partial \hat{\rho}_{\mathfrak{S}0}}{\partial \hat{t}} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial \hat{\rho}_{\mathfrak{S}0}\hat{v}_{z\mathfrak{S}1}}{\partial z} = 0, \tag{199}$$

$$\begin{aligned}\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \hat{t}} &+ \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) \\ &= -\frac{1}{2}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi} + \frac{1}{2}\frac{\partial}{\partial y}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right),\end{aligned} \tag{200}$$

$$\begin{aligned}\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial \hat{t}} &+ \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} - \frac{1}{c^2}\hat{v}_{x\mathfrak{S}0}^2\right) \\ = &-\frac{1}{2}\frac{\partial \hat{p}^c_{\mathfrak{S}2}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial \chi}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right) \\ &+ \frac{\partial}{\partial y}\left(\Gamma_1\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left[\Gamma_1\left(\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y}\right)\right] \\ &+ \frac{1}{2\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left[\Gamma_7\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)^2\right] + \frac{\partial}{\partial z}\left(\Gamma_7\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right\} \\ &+ \frac{1}{\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left[\Gamma_8\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2\right] + \frac{\partial}{\partial z}\left(\Gamma_8\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)\right\},\end{aligned} \tag{201}$$

$$\begin{aligned}\hat{\rho}_{\mathfrak{S}0}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial \hat{t}} &+ \hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial \chi} + \hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y} + \hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) \\ = &-\frac{1}{2}\frac{\partial \hat{p}^c_{\mathfrak{S}2}}{\partial z} + \frac{1}{2}\frac{\partial}{\partial \chi}\left(\Gamma_1\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) \\ &+ \frac{1}{2}\frac{\partial}{\partial y}\left[\Gamma_1\left(\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial z} + \frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(\Gamma_1\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right) \\ &+ \frac{1}{2\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left(\Gamma_7\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right) + \frac{\partial}{\partial z}\left[\Gamma_7\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)^2\right]\right\} \\ &+ \frac{1}{\hat{p}_{\mathfrak{S}0}}\left\{\frac{\partial}{\partial y}\left(\Gamma_8\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right) + \frac{\partial}{\partial z}\left[\Gamma_8\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]\right\},\end{aligned} \tag{202}$$

$$
\begin{aligned}
&\frac{5\hat{\rho}_{\mathfrak{S}0}}{2}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial \hat{t}}+\frac{5}{2}\hat{\rho}_{\mathfrak{S}0}\left(\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial \chi}+\hat{v}_{y\mathfrak{S}1}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}+\hat{v}_{z\mathfrak{S}1}\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\\
&-\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \hat{t}}-\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi}\\
&\qquad=\frac{5}{4}\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)+\frac{5}{4}\frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)+\Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2+\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right],
\end{aligned}
\tag{203}
$$

and the subsidiary relations

$$
\hat{p}_{\mathfrak{S}0}(\chi,\hat{t})=\hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0}, \tag{204a}
$$

$$
\begin{aligned}
\hat{p}^{c}_{\mathfrak{S}2}=\hat{p}_{\mathfrak{S}2}&+\frac{2\Gamma_1}{3}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi}+\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y}+\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right)+\frac{\Gamma_7}{3\hat{p}_{\mathfrak{S}0}}\left[\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)^2+\left(\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)^2\right]\\
&+\frac{2}{3\hat{p}_{\mathfrak{S}0}}\left[\frac{\partial}{\partial y}\left(\Gamma_3\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\Gamma_3\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]\\
&-\frac{2\Gamma_9}{3\hat{p}_{\mathfrak{S}0}}\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2+\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right],
\end{aligned}
\tag{204b}
$$

where Γ_1, Γ_2, Γ_3, Γ_7, Γ_8, and Γ_9 are short forms of the functions $\Gamma_1(\hat{T}_{\mathfrak{S}0})$, $\Gamma_2(\hat{T}_{\mathfrak{S}0}),\dots,\Gamma_9(\hat{T}_{\mathfrak{S}0})$ of $\hat{T}_{\mathfrak{S}0}$ defined in Section M-A.2.9.

Consider the solution of the initial and boundary-value problem of Eqs. (198)–(204b).

Let $\hat{\rho}$, $\hat{v}_i$, and $\hat{T}$ (thus, $\hat{p}=\hat{\rho}\hat{T}$) at time $\hat{t}$ be given; thus, $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{T}_{\mathfrak{S}0}$ ($\hat{p}_{\mathfrak{S}0}$), etc., including $\hat{p}_{\mathfrak{S}2}$, are given. Then $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{T}_{\mathfrak{S}0}/\partial\hat{t}$ at $\hat{t}$ are given by Eqs. (199)–(204b); thus, the future $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ (also $\hat{p}_{\mathfrak{S}0}$) are determined. However, the future $\hat{p}_{\mathfrak{S}0}$ is required to be independent of y and z, as well as $\hat{p}_{\mathfrak{S}0}$ at $\hat{t}$, owing to Eq. (198). Taking this into account, we will discuss how the solution is obtained by this system consistently.

First, transform Eq. (203) with the aid of Eqs. (199) and (204a) in the following form:

$$
\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \hat{t}}=\mathcal{P}, \tag{205}
$$

where

$$
\begin{aligned}
\mathcal{P}=&-\frac{5}{3}\hat{p}_{\mathfrak{S}0}\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial \chi}+\frac{\partial \hat{v}_{y\mathfrak{S}1}}{\partial y}+\frac{\partial \hat{v}_{z\mathfrak{S}1}}{\partial z}\right)-\hat{v}_{x\mathfrak{S}0}\frac{\partial \hat{p}_{\mathfrak{S}0}}{\partial \chi}\\
&+\frac{5}{6}\left[\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial y}\right)+\frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial \hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]+\frac{2}{3}\Gamma_1\left[\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2+\left(\frac{\partial \hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right].
\end{aligned}
\tag{206}
$$

For $\hat{p}_{\mathfrak{S}0}$ to be independent of y and z [see Eq. (198)], $\mathcal{P}$ as well as the initial data of $\hat{p}_{\mathfrak{S}0}$ is required to be independent of y and z. Noting that $\hat{p}_{\mathfrak{S}0}$ is independent of y and z, and taking the average of Eq. (206) over the cross section S of the pipe or channel,[73] we have another expression $\mathfrak{P}$ of $\mathcal{P}$, explicitly uniform with respect to y and z, i.e.,

$$\begin{aligned}\mathfrak{P} = &-\frac{5}{3}\frac{\partial\overline{\hat{v}_{x\mathfrak{S}0}}}{\partial\chi}\hat{p}_{\mathfrak{S}0} - \overline{\hat{v}_{x\mathfrak{S}0}}\frac{\partial\hat{p}_{\mathfrak{S}0}}{\partial\chi} + \frac{5}{6}\overline{\left[\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right]}\\ &+\frac{2}{3}\overline{\Gamma_1\left[\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]},\end{aligned} \tag{207}$$

where

$$\overline{A} = \int_S A\mathrm{d}y\mathrm{d}z \bigg/ \int_S \mathrm{d}y\mathrm{d}z.$$

The expression (207) is noted to be independent of $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$. The two expressions (206) and (207) must give the same result, i.e.,

$$\mathcal{P} = \mathfrak{P},$$

or

$$\begin{aligned}&-\frac{5}{3}\hat{p}_{\mathfrak{S}0}\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial\chi} + \frac{\partial\hat{v}_{y\mathfrak{S}1}}{\partial y} + \frac{\partial\hat{v}_{z\mathfrak{S}1}}{\partial z}\right) - \hat{v}_{x\mathfrak{S}0}\frac{\partial\hat{p}_{\mathfrak{S}0}}{\partial\chi}\\ &+\frac{5}{6}\left[\frac{\partial}{\partial y}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_2\frac{\partial\hat{T}_{\mathfrak{S}0}}{\partial z}\right)\right] + \frac{2}{3}\Gamma_1\left[\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial y}\right)^2 + \left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial z}\right)^2\right]\\ &= \mathfrak{P},\end{aligned} \tag{208}$$

when Eq. (198) holds, and vice versa. The condition (208) for all $\hat{t}$ is equivalently replaced by the two conditions that the initial data of $\hat{p}_{\mathfrak{S}0}$, $\hat{T}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ satisfy Eqs. (198) and (208) and that the time derivative of Eq. (208) holds for all $\hat{t}$, i.e.,

$$\frac{\partial\mathcal{P}}{\partial\hat{t}} = \frac{\partial\mathfrak{P}}{\partial\hat{t}}. \tag{209}$$

Using Eqs. (199)–(202) and (205) for $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}$ ($\hat{\rho}_{\mathfrak{S}0}\partial\hat{T}_{\mathfrak{S}0}/\partial\hat{t} = \partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t} - \hat{T}_{\mathfrak{S}0}\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$) in $\partial\mathcal{P}/\partial\hat{t}$ derived from Eq. (206), we find that $\partial\mathcal{P}/\partial\hat{t}$ is expressed with $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and $\hat{p}^c_{\mathfrak{S}2}$ in the form

$$\frac{\partial\mathcal{P}}{\partial\hat{t}} = \frac{5}{6}\hat{p}_{\mathfrak{S}0}\left[\frac{\partial}{\partial y}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial\hat{p}^c_{\mathfrak{S}2}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial\hat{p}^c_{\mathfrak{S}2}}{\partial z}\right)\right] + \mathrm{Fn}_1, \tag{210}$$

[73](i) In a channel, where the gas extends from $z = -\infty$ to $z = \infty$, the integral $\int_S A\mathrm{d}y\mathrm{d}z$ per unit length in z, per a period in z, etc. should be considered. Otherwise, it can be infinite.

(ii) Note that $\hat{v}_{y\mathfrak{S}1}n_y + \hat{v}_{z\mathfrak{S}1}n_z = 0$ on a simple boundary where $n_i = (0, n_y, n_z)$ is the normal to the boundary.

where Fn_1 is a given function of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives. The expression (207) of $\mathfrak{P}$ being independent of $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$, its time derivative $\partial\mathfrak{P}/\partial\hat{t}$ does not contain $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$ and $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$. Therefore, with the aid of Eqs. (199), (200), and (203), $\partial\mathfrak{P}/\partial\hat{t}$ is expressed with $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives, i.e.,

$$\frac{\partial\mathfrak{P}}{\partial\hat{t}} = \mathrm{Fn}_2(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{p}_{\mathfrak{S}0}, \text{and their space derivatives}), \tag{211}$$

where Fn_2 is a given functional of its arguments. From Eqs. (209), (210), and (211), we have

$$\frac{\partial}{\partial y}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\hat{\rho}_{\mathfrak{S}0}}\frac{\partial\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}}{\partial z}\right) = \mathrm{Fn}, \tag{212}$$

where $\mathrm{Fn} = 6(\mathrm{Fn}_2 - \mathrm{Fn}_1)/5\hat{p}_{\mathfrak{S}0}$, and therefore, Fn is a given functional of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, $\hat{p}_{\mathfrak{S}0}$, and their space derivatives. This is the equation for $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ over a cross section of the pipe or channel.

The boundary condition for $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ on a simple boundary is obtained by multiplying Eqs. (200)–(202) by the normal $n_i = (0, n_y, n_z)$ to the boundary; In this process, the contribution of their time-derivative terms vanishes because $\hat{v}_{y\mathfrak{S}1}n_y + \hat{v}_{z\mathfrak{S}1}n_z = 0$; Then, the $n_y\partial\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}/\partial y + n_z\partial\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}/\partial z$ is imposed as the boundary condition. Thus, $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ is determined by Eq. (212) except for an additive function of $\hat{t}$ and χ. With this $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ substituted into Eqs. (201) and (202), $\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}$, $\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}$, $\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}$, and $\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}$ are determined by Eqs. (199)–(204b) independently of the additive function in $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ in such a way that $\partial(\partial\hat{p}_{\mathfrak{S}02}/\partial y)/\partial\hat{t} = \partial(\partial\hat{p}_{\mathfrak{S}0}/\partial z)/\partial\hat{t} = 0$ and $\partial(\partial\mathcal{P}/\partial y)/\partial\hat{t} = \partial(\partial\mathcal{P}/\partial z)/\partial\hat{t} = 0$. That is, the solution $(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{T}_{\mathfrak{S}0})$ of Eqs. (198)–(204b) is determined by Eqs. (199)–(204b) with the aid of the supplementary condition (212), instead of Eq. (198), when the initial condition for $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ is given in such a way that $\hat{p}_{\mathfrak{S}0}$ $(= \hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0})$ and $\mathcal{P}$ are independent of y and z.

Equations (198)–(204b) are the leading-order set of equations derived by the asymptotic analysis of the Boltzmann equation. The analysis of the higher-order equations not shown here is carried out in a similar way. The equation for $\partial\hat{p}_{\mathfrak{S}2}/\partial\hat{t}$, corresponding to Eq. (205), is derived at the order after next. However, owing to the consistency of $\hat{p}_{\mathfrak{S}0}$, $\hat{p}_{\mathfrak{S}2}$ is already determined by Eq. (212) except for an additive function of χ and $\hat{t}$. The situation is similar to that at the leading order. That is, $\hat{p}_{\mathfrak{S}0}$ and $\hat{p}_{\mathfrak{S}2}$ are, respectively, determined by Eqs. (198) and (212), each with an additive function of χ and $\hat{t}$ and also by Eqs. (205) and the counterpart of Eq. (205) at the order after next. Thus, the higher-order analysis can be carried out in a similar way. The results are that an additional initial condition and an equation for $\hat{p}_{\mathfrak{S}4}$, the counter part of Eq. (212), are introduced and that the condition (212) is required only for the initial data. The higher-order consideration does not affect the determination of the solution $\hat{\rho}_{\mathfrak{S}0}$, $\hat{T}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ (thus also $\hat{p}_{\mathfrak{S}0}$).

To summarize, the solution $(\hat{\rho}_{\mathfrak{S}0}, \hat{v}_{x\mathfrak{S}0}, \hat{v}_{y\mathfrak{S}1}, \hat{v}_{z\mathfrak{S}1}, \hat{T}_{\mathfrak{S}0})$ of Eqs. (198)–(204b) is determined by Eqs. (199)–(204b) with the aid of the supplementary condition

(212), instead of Eq. (198), when the initial data of $\hat{\rho}_{\mathfrak{S}0}$, $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, $\hat{v}_{z\mathfrak{S}1}$, and $\hat{T}_{\mathfrak{S}0}$ are given in such a way that $\hat{p}_{\mathfrak{S}0}$ $(= \hat{\rho}_{\mathfrak{S}0}\hat{T}_{\mathfrak{S}0})$ and $\mathcal{P}$ are independent of y and z.[74] The results are not affected by the higher-order analysis.

7.1.2 Equations (M-9.49a)–(M-9.50e):

Take Eqs. (M-9.49a)–(M-9.50e) with the additional time-derivative terms given in the first mathematical expressions after Eq. (M-9.59), i.e.,

$$\frac{\partial P_{01}}{\partial \tilde{\chi}} = \frac{\partial P_{01}}{\partial y} = \frac{\partial P_{01}}{\partial z} = 0, \quad P_{01} = \omega + \tau, \tag{213a}$$

$$\frac{\partial P_{02}}{\partial y} = \frac{\partial P_{02}}{\partial z} = 0, \tag{213b}$$

$$\frac{\partial u_x}{\partial \tilde{\chi}} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \tag{214a}$$

$$\frac{\partial u_x}{\partial \hat{t}} + u_x \frac{\partial u_x}{\partial \tilde{\chi}} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{2}\frac{\partial P_{02}}{\partial \tilde{\chi}} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right), \tag{214b}$$

$$\frac{\partial u_y}{\partial \hat{t}} + u_x \frac{\partial u_y}{\partial \tilde{\chi}} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} - \frac{u_x^2}{C^2} = -\frac{1}{2}\frac{\partial P_{20}}{\partial y} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right), \tag{214c}$$

$$\frac{\partial u_z}{\partial \hat{t}} + u_x \frac{\partial u_z}{\partial \tilde{\chi}} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{2}\frac{\partial P_{20}}{\partial z} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right), \tag{214d}$$

$$\frac{\partial \tau}{\partial \hat{t}} - \frac{2}{5}\frac{\partial P_{01}}{\partial \hat{t}} + u_x \frac{\partial \tau}{\partial \tilde{\chi}} + u_y \frac{\partial \tau}{\partial y} + u_z \frac{\partial \tau}{\partial z} = \frac{\gamma_2}{2}\left(\frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2}\right). \tag{214e}$$

The qualitative difference of this set of equations from the set (198)–(204b) is the absence of the time-derivative term in Eq. (214a) that corresponds to Eq. (199).

Consider the solution of the initial and boundary-value problem of Eqs. (213a)–(214e). Let u_x, u_y, u_z, and τ at $\hat{t}$ be given in such a way that Eq. (214a) is satisfied. Integrating Eq. (214a) over the cross section of the channel or pipe $\left[\int_S \text{Eq. (214a)}\mathrm{d}y\mathrm{d}z\right]$, we find that $\int_S u_x \mathrm{d}y\mathrm{d}z$ depends only on $\hat{t}$,[75] i.e.,

$$\int_S (\partial u_x/\partial \tilde{\chi})\mathrm{d}y\mathrm{d}z = 0, \tag{215}$$

where S indicates the cross section. Applying Eqs. (213b), (214a), and (215) to the equation $\partial \int \text{Eq. (214b)}\mathrm{d}y\mathrm{d}z/\partial \tilde{\chi}$, we have $\partial^2 P_{02}/\partial \tilde{\chi}^2$ as

$$\frac{\partial^2 P_{02}}{\partial \tilde{\chi}^2} = \frac{\partial}{\partial \tilde{\chi}}\overline{\left[-2\frac{\partial u_x^2}{\partial \tilde{\chi}} + \gamma_1\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\right]}, \tag{216}$$

[74] If $\mathcal{P}$ is independent of y and z, $\mathcal{P} = \mathfrak{P}$ by definition.

[75] See Footnote 73, with $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$ being replaced by u_y and u_z.

where

$$\overline{A} = \int_S A\mathrm{d}y\mathrm{d}z \bigg/ \int_S \mathrm{d}y\mathrm{d}z.$$

Thus, $\partial P_{02}/\partial\tilde{\chi}$ and P_{02} are determined if they are specified at a point in the gas. Here, we consider this case.[76] Using Eq. (214a) in the sum of ∂[Eq. (214b)]$/\partial\tilde{\chi}$, ∂[Eq. (214c)]$/\partial y$, and ∂[Eq. (214d)]$/\partial z$, we obtain the equation for P_{20} in the form

$$\frac{\partial^2 P_{20}}{\partial y^2} + \frac{\partial^2 P_{20}}{\partial z^2} = \mathrm{Fn}(u_x, u_y, u_z, \text{and their space derivatives}), \tag{217}$$

where Fn is a given functional of the variables in the parentheses, and the time derivatives are absent owing to Eq. (214a). Thus, the right-hand side of Eq. (217) is known. This equation is the Poisson equation for P_{20} over the cross section S. Its boundary condition is obtained in a way similar to how the condition for $\hat{p}^{\mathrm{c}}_{\mathfrak{S}2}$ in Eq. (212) is derived. Thus, P_{20} over each cross section is determined except for an additive function of $\hat{t}$ and $\tilde{\chi}$. This ambiguity does not influence $\partial P_{20}/\partial y$ and $\partial P_{20}/\partial z$.

With P_{02} and P_{20} prepared above into Eqs. (214b)–(214e), the time derivatives $\partial u_x/\partial\hat{t}$, $\partial u_y/\partial\hat{t}$, $\partial u_z/\partial\hat{t}$, and $\partial\tau/\partial\hat{t}$ are determined in such a way that $\partial(\partial u_x/\partial\tilde{\chi} + \partial u_y/\partial y + \partial u_z/\partial z)/\partial\hat{t} = 0$ owing to the above choice of P_{20}.[77] Thus, the solution (u_x, u_y, u_z, τ) of Eqs. (213b), (214a)–(214e) is determined by Eqs. (214b)–(214e) with the aid of the supplementary conditions (216) and (217) for P_{02} and P_{20}, instead of Eqs. (213b) and (214a). This process is natural for numerical computation. The undetermined additive function of $\tilde{\chi}$ and $\hat{t}$ in P_{20}, which does not affect the solution (u_x, u_y, u_z, τ), is determined by the higher-order equation derived from that for $\partial\hat{v}_{x\mathfrak{S}2}/\partial\hat{t}$ (see Section 7.1.1), in a way similar to that in which P_{02} is determined by Eq. (214b). In the higher-order equation, P_{20} plays the same role as P_{02} in Eq. (214b); Equation (217) corresponds to Eq. (213b), and P_{20} and P_{02} are determined by these equations, each with an additive function of $\tilde{\chi}$ and $\hat{t}$.

7.2 Notes on the equations with the ghost effect of infinitesimal curvature, Eqs. (M-9.33)–(M-9.39b)

Here, the process of analysis where the curvilinear coordinates x, y, and z in Eqs. (M-9.33)–(M-9.39b)[78] are identified with rectangular ones is explained in more detail.

[76](i) Imagine the case of the Poiseuille flow.
(ii) Here, P (thus, P_{01}) is specified at some point. Then, P_{01} is a given function of $\hat{t}$.

[77]Note that P_{01} is known (Footnote 76).

[78]Equations (M-9.33)–(M-9.39b) are those for time-independent states. The corresponding equations for time-dependent states are given by adding the time-derivative terms (M-9.42) to them or by Eqs. (198)–(204b). When Eqs. (M-9.33)–(M-9.39b) are mentioned in this section, they mean the equations with the time-dependent terms.

7.2.1 The curvilinear system (x, y, z)

The coordinate system (x, y, z), introduced in Eq. (M-9.4a), is practically a cylindrical one, and is related to a rectangular one (x_1, x_2, x_3) as

$$x_1 = (\hat{L} + y)\sin\frac{x}{\hat{L}}, \quad x_2 = (\hat{L} + y)\cos\frac{x}{\hat{L}} - \hat{L}, \quad x_3 = z, \tag{218}$$

where $\hat{L} = L_A/D$. It obviously reduces to the rectangular one (x_1, x_2, x_3) in the limit $\hat{L} \to \infty$ for any finite range of (x, y, z). In Sections M-9.1 and M-9.2, we studied the asymptotic behavior of the Boltzmann system in the limit that $k \to 0$ and $\hat{L} \to \infty$ simultaneously under the condition

$$\hat{L}k^2 = c^2, \tag{219}$$

where c (> 0) is a constant. In this process, we consider the range of (x, y, z) when the range of θ satisfies the conditions

$$-\infty < \hat{L}\theta < \infty, \tag{220a}$$

$$\hat{L}\theta^2 \to 0, \tag{220b}$$

where

$$\theta = -x/\hat{L}. \tag{221}$$

The three conditions (219), (220a), and (220*b*) are satisfied if we take the range of θ to be

$$|\theta| \leq \theta_0, \tag{222}$$

where θ_0 tends to zero as $k \to 0$ under the two conditions

$$\theta_0/k^2 \to \infty \ \text{ as } k \to 0, \tag{223a}$$

$$\theta_0 = o(k^\alpha) \quad (1 \leq \alpha < 2), \tag{223b}$$

for some α in the above range. In the limit $k \to 0$, the variable x covers $(-\infty, \infty)$ for the above range of θ, and the system (x, y, z) reduces to the rectangular system (x_1, x_2, x_3), i.e., $(x, y, z) = (x_1, x_2, x_3)$.[79] In the analysis in Section 7.2, we further limit the bound θ_0 of the range of θ to

$$\theta_0 = o(k^\alpha) \quad (3/2 \leq \alpha < 2), \tag{224}$$

instead of Eq. (223b). Under the condition, the system (x, y, z) converges faster to the rectangular system (x_1, x_2, x_3) as will be seen below.

From Eq. (218), we have

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial x} & \dfrac{\partial x_1}{\partial y} \\ \dfrac{\partial x_2}{\partial x} & \dfrac{\partial x_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\hat{L} + y}{\hat{L}}\cos\dfrac{x}{\hat{L}} & \sin\dfrac{x}{\hat{L}} \\ -\dfrac{\hat{L} + y}{\hat{L}}\sin\dfrac{x}{\hat{L}} & \cos\dfrac{x}{\hat{L}} \end{pmatrix}, \tag{225a}$$

[79] When $\theta = \pm k$, $y = 0$ corresponds to $x_2 = -c^2/2$ in the limit $k \to 0$. When $\theta = \pm k^2$, x corresponds to $x_1 = \mp c^2$ in the limit. The inequalities (223a) and (223b) are required for the system (x, y, z) to approach the rectangular system.

$$\begin{pmatrix} \dfrac{\partial^2 x_1}{\partial x^2} & \dfrac{\partial^2 x_1}{\partial x\partial y} & \dfrac{\partial^2 x_1}{\partial y^2} \\ \dfrac{\partial^2 x_2}{\partial x^2} & \dfrac{\partial^2 x_2}{\partial x\partial y} & \dfrac{\partial^2 x_2}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{\hat{L}+y}{\hat{L}^2}\sin\dfrac{x}{\hat{L}} & \dfrac{1}{\hat{L}}\cos\dfrac{x}{\hat{L}} & 0 \\ -\dfrac{\hat{L}+y}{\hat{L}^2}\cos\dfrac{x}{\hat{L}} & -\dfrac{1}{\hat{L}}\sin\dfrac{x}{\hat{L}} & 0 \end{pmatrix}. \tag{225b}$$

From Eqs. (218), (225a), and (225b), noting the relations (219), (221), (222), and (224), we obtain the following uniform bounds for small k of the difference between the two systems (x, y, z) and (x_1, x_2, x_3) in $-\infty < x < \infty$ and $|y| \leq a_0$ (a_0 : a constant independent of k):[80]

$$0 \leq |x - x_1| \leq o(k^\alpha), \qquad 0 \leq y - x_2 \leq c^2 o(k^{2(\alpha-1)}), \tag{226a}$$

$$0 \leq \left|\frac{\partial x_1}{\partial x} - 1\right| \leq \frac{O(k^2)}{c^2}, \qquad 0 \leq \left|\frac{\partial x_1}{\partial y}\right| \leq o(k^\alpha), \tag{226b}$$

$$0 \leq \left|\frac{\partial x_2}{\partial x}\right| \leq o(k^\alpha), \qquad 0 \leq 1 - \frac{\partial x_2}{\partial y} \leq o(k^{2\alpha}), \tag{226c}$$

$$\left|\frac{\partial^2 x_1}{\partial x^2}\right| \leq \frac{o(k^{2+\alpha})}{c^2}, \quad 0 < \frac{\partial^2 x_1}{\partial x\partial y} \leq \frac{O(k^2)}{c^2}, \quad \frac{\partial^2 x_1}{\partial y^2} = 0, \tag{226d}$$

$$0 < -\frac{\partial^2 x_2}{\partial x^2} \leq \frac{O(k^2)}{c^2}, \quad \left|\frac{\partial^2 x_2}{\partial x\partial y}\right| \leq \frac{o(k^{2+\alpha})}{c^2}, \quad \frac{\partial^2 x_2}{\partial y^2} = 0. \tag{226e}$$

7.2.2 Process to identify (x, y, z) in Eqs. (M-9.33)–(M-9.39b) with (x_1, x_2, x_3)

The flow velocity components $(\hat{v}_{x\mathfrak{S}0}, 0, 0)$ in Section M-9.2 coincide with those $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ of the rectangular system, i.e., $(\hat{v}_1, \hat{v}_2, \hat{v}_3) = (\hat{v}_{x\mathfrak{S}0}, 0, 0)$ in the limit $k \to 0$ described in Section 7.2.1. In the higher orders in k, differences between the two systems, coordinates and velocity components, are introduced. For a nearly parallel flow considered here, some of the series of the conservation equations in the expansion in k degenerate. Owing to the degeneracy, the series of solutions in the expansion is obtained by staggered combinations of equations. That is, the limiting velocity field $(\hat{v}_{x\mathfrak{S}0}, 0, 0)$ is determined together with the next-order components $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$ owing to the degeneracy of the momentum conservation equations by the equations (M-9.33)–(M-9.39b), where the variables (x, y, z) are identified with (x_1, x_2, x_3). Some notes should be given

[80]i) Dependence of the bounds on the constant c^2 in Eq. (219) is made explicit for the convenience to the discussion in Section M-9.3.

ii) For any finite x, or $|x| \leq C_0$, the bound is tighter; for example, $0 \leq |x - x_1| \leq C_0 O(k^2)/c^2$, $0 \leq y - x_2 \leq C_0^2 O(k^2)/c^2$.

to identify (x, y, z) with (x_1, x_2, x_3).[81] The set of Eqs. (M-9.33)–(M-9.38) is the combination of the component equations at different levels of expansion in k of the conservation equations. For the momentum conservation equations (M-9.33), (M-9.35)–(M-9.37), equations of three different levels appear: Eq. (M-9.33) is at the level of the order of unity, Eq. (M-9.35) is at the level of the order of k, and Eqs. (M-9.36) and (M-9.37) are at the level of the order of k^2. The deviation (x, y, z) from (x_1, x_2, x_3) for $k \neq 0$, including those in the arguments of functions, introduces residual contributions to equations at higher-order levels. In the mass and energy conservation equations (M-9.34) and (M-9.38), the variables (x, y, z) can be identified with (x_1, x_2, x_3) because they appear as the nontrivial leading-order equations. The momentum conservation equations are vector equations. Their x_1, x_2, and x_3 components are related to their x, y, and z components by the relation

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \tag{227a}$$

$$a_3 = a_z, \tag{227b}$$

where a_1, a_2, and a_3 are, respectively, the x_1, x_2, and x_3 components of a vector, and a_x, a_y, and a_z are its x, y, and z components.

For the further analysis, we prepare the expressions of $\partial\hat{p}_{\mathfrak{S}0}(x, y, z)/\partial y$ and $\partial\hat{p}_{\mathfrak{S}0}(x, y, z)/\partial z$ in the rectangular system. Owing to the chain rule of differentiation,

$$\frac{\partial\hat{p}_{\mathfrak{S}0}(x, y, z)}{\partial y} = k\frac{\partial\check{p}_{\mathfrak{S}0}(x_i(x, y, z))}{\partial\chi_1}\frac{\partial x_1}{\partial y} + \frac{\partial\check{p}_{\mathfrak{S}0}(x_i(x, y, z))}{\partial x_2}\frac{\partial x_2}{\partial y}, \tag{228a}$$

$$\frac{\partial\hat{p}_{\mathfrak{S}0}(x, y, z)}{\partial z} = \frac{\partial\check{p}_{\mathfrak{S}0}(x_i(x, y, z))}{\partial x_3}, \tag{228b}$$

where

$$\chi_1 = kx_1, \tag{229}$$

and $\check{p}_{\mathfrak{S}0}$ is the function $\hat{p}_{\mathfrak{S}0}$ expressed with the rectangular variables (x_1, x_2, x_3). In Eqs. (228a) and (228b), it should be noted that x_1 and x_2 are independent of z, and x_3 depends only on z. In view of the bound of $\partial x_1/\partial y$ in Eq. (226b), the first term on the right-hand side of Eq. (228a) is bounded by $o(k^{\alpha+1})$.[82]

[81] Equations (M-9.33)–(M-9.39b) are derived from the Boltzmann equation (M-9.5) with (x, y, z) as its independent space variables. In this process, the relation between (x, y, z) and (x_1, x_2, x_3) is not taken into account until the last step. Their relation depends on k as shown in Section 7.2.1. With this relation, we have to rewrite the equations expressed with (x, y, z) into the equations expressed with (x_1, x_2, x_3). After this process, it is seen that (x, y, z) can be identified with (x_1, x_2, x_3) in Eqs. (M-9.33)–(M-9.39b). This process is explained in more detail.

[82] The derivative $\partial/\partial\chi_1$ agrees with $\partial/\partial\chi$ at the leading order in k. In fact,

$$\frac{\partial}{\partial\chi_1} = \frac{\partial\chi}{\partial\chi_1}\frac{\partial}{\partial\chi} + \frac{\partial y}{\partial\chi_1}\frac{\partial}{\partial y},$$

and from Eqs. (225a), (226b), and (226c), the estimates of $\partial\chi/\partial\chi_1$ and $\partial y/\partial\chi_1$ are obtained

Thus, it does not contribute to the result up to the level of the order of k^2, and can be neglected in the present discussion, where the momentum conservation equations up to the level of the order of k^2 are considered. For the evaluation of the second term of Eq. (228a) and Eq. (228b), we put

$$x_1 = x + X(x,y),\ x_2 = y + Y(x,y), \tag{230}$$

where the bounds of X and Y for small k are given by Eq. (226a). Then, the derivatives of $\check{p}_{\mathfrak{S}0}$ with respect to x_2 or x_3 at (x,y,z) in Eqs. (228a) and (228b) are

$$\begin{aligned}
&\frac{\partial \check{p}_{\mathfrak{S}0}(x_i(x,y,z))}{\partial x_2} = \frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_2}\bigg|_{x_1=x+X,\, x_2=y+Y,\, x_3=z} \\
&\quad = \frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_2}\bigg|_{(x_1,x_2,x_3)=(x,y,z)} + \left(X\frac{\partial}{\partial x_1} + Y\frac{\partial}{\partial x_2}\right)\frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_2}\bigg|_{(x_1,x_2,x_3)=(x,y,z)} \\
&\quad + \frac{1}{2}\left(X^2\frac{\partial^2}{\partial x_1^2} + 2XY\frac{\partial^2}{\partial x_1 \partial x_2} + Y^2\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_2}\bigg|_{(x_1,x_2,x_3)=(x,y,z)} + \cdots,
\end{aligned} \tag{231a}$$

$$\begin{aligned}
&\frac{\partial \check{p}_{\mathfrak{S}0}(x_i(x,y,z))}{\partial x_3} = \frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_3}\bigg|_{x_1=x+X,\, x_2=y+Y,\, x_3=z} \\
&\quad = \frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_3}\bigg|_{(x_1,x_2,x_3)=(x,y,z)} + \left(X\frac{\partial}{\partial x_1} + Y\frac{\partial}{\partial x_2}\right)\frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_3}\bigg|_{(x_1,x_2,x_3)=(x,y,z)} \\
&\quad + \frac{1}{2}\left(X^2\frac{\partial^2}{\partial x_1^2} + 2XY\frac{\partial^2}{\partial x_1 \partial x_2} + Y^2\frac{\partial^2}{\partial x_2^2}\right)\frac{\partial \check{p}_{\mathfrak{S}0}(x_i)}{\partial x_3}\bigg|_{(x_1,x_2,x_3)=(x,y,z)} + \cdots.
\end{aligned} \tag{231b}$$

With the above preparation, we consider the momentum conservation equations in the (x_1,x_2,x_3) system. Let A be an equation, and be rewritten in the form $\overline{A} = 0$ with all the terms on the right-hand side shifted to the left. With this notation, first, take the x_2 and x_3 components of the momentum

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after some manipulation as

$$\partial\chi/\partial\chi_1 = 1 + O(k^2), \quad \partial y/\partial\chi_1 = o(k^{\alpha-1}).$$

conservation equations[83]

$$
\begin{aligned}
&\tfrac{1}{2}\overline{\text{Eq. (M-9.33)}_y}\cos\theta + \tfrac{1}{2}\overline{\text{Eq. (M-9.23)}_y}\,k\cos\theta + \overline{\text{Eq. (M-9.35)}}\,k\sin\theta \\
&\quad + \overline{\text{Eq. (M-9.36)}}\,k^2\cos\theta + \overline{\text{Eq. (M-9.35+)}}\,k^2\sin\theta = 0, \qquad (232\text{a}) \\
&\tfrac{1}{2}\overline{\text{Eq. (M-9.33)}_z} + \tfrac{1}{2}\overline{\text{Eq. (M-9.23)}_z}\,k + \overline{\text{Eq. (M-9.37)}}\,k^2 = 0. \qquad (232\text{b})
\end{aligned}
$$

Here, Eq. (M-9.33)$_y$ and Eq. (M-9.33)$_z$ are, respectively, the two equations of Eq. (M-9.33), i.e., $\partial\hat{p}_{\mathfrak{S}0}/\partial y = 0$ and $\partial\hat{p}_{\mathfrak{S}0}/\partial z = 0$; A similar convention applies to Eq. (M-9.23)$_y$ and Eq. (M-9.23)$_z$; Eq. (M-9.35+) is the equation corresponding to Eq. (M-9.35) to be derived in the next order in k, i.e., Eq. (M-9.22c) for $\psi = \zeta_x$. From Eqs. (232a) and (232b) at the level of the order of unity, noting Eqs. (228a) and (228b) with their note and the relations (231a) and (231b), we have Eqs. (M-9.33) with (x, y, z) identified with (x_1, x_2, x_3), i.e.,

$$
\frac{\partial\check{p}_{\mathfrak{S}0}(x_i)}{\partial x_2} = \frac{\partial\check{p}_{\mathfrak{S}0}(x_i)}{\partial x_3} = 0. \qquad (233)
$$

Owing to Eqs. (233), (231a), and (231b), the residues of $\partial\hat{p}_{\mathfrak{S}0}/\partial y$ and $\partial\hat{p}_{\mathfrak{S}0}/\partial z$ in Eqs. (228a) and (228b) are of the order of $o(k^2)$. A similar discussion applies to the second terms on the left-hand sides of Eqs. (232a) and (232b). The third term in Eq. (232a) is of the order of $o(k^{1+\alpha})$ because $\sin\theta = o(k^\alpha)$ [Eq. (224)], and the last term is of higher order than the third. Therefore, Eqs. (M-9.36) and (M-9.37) where (x, y, z) are identified with (x_1, x_2, x_3) in the arguments and derivatives are derived from Eqs. (232a) and (232b) at the order of k^2. Next, take the x_1 component of the momentum conservation equations

$$
-\tfrac{1}{2}\overline{\text{Eq. (M-9.33)}_y}\sin\theta - \tfrac{1}{2}\overline{\text{Eq. (M-9.23)}_y}\,k\sin\theta + \overline{\text{Eq. (M-9.35)}}\,k\cos\theta = 0. \qquad (234)
$$

The first and second terms on the left-hand side of Eq. (234) are of higher order than the third owing to the factor $\sin\theta$. Thus, Eq. (M-9.35) with (x, y, z) identified with (x_1, x_2, x_3) in the arguments and derivatives is derived from Eqs. (234) at the order of k. To summarize, Eqs. (M-9.33)–(M-9.39b) are the equations in the rectangular coordinate system (x, y, z) that determine the rectangular velocity $(\hat{v}_{x\mathfrak{S}0}, 0, 0)$ in the limit $k \to 0$ together with $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$, whether $\hat{v}_{y\mathfrak{S}1}$ and $\hat{v}_{z\mathfrak{S}1}$ are rectangular components or not.

[83]i) Note that Eqs. (M-9.33)–(M-9.37) [and Eqs. (M-9.23) and (M-9.35+)] are derived from the solvability conditions (M-9.22a)–(M-9.22c). The solvability conditions are the expansion form in k of the conservation equations (M-1.57)–(M-1.59) arranged for the nearly parallel flow considered in Sections M-9.1 and M-9.2. The equations corresponding to $\psi = \zeta_x$, ζ_y, and ζ_z are, respectively, the x, y, and z components of the momentum conservation equations. Their x_1, x_2, and x_3 components are derived from them with the aid of Eqs. (227a) and (227b). In this process, the summation of terms of different orders of k has to be considered because Eqs. (M-9.33), (M-9.35)–(M-9.37) [and Eqs. (M-9.23) and (M-9.35+)] come from equations at different orders of k.

ii) It should be noted that Eq. (M-9.33)$_{y\text{ or }z}$ and Eq. (M-9.23)$_{y\text{ or }z}$ are, respectively, the doubles of Eqs. (M-9.22a) and (M-9.22b) for $\psi = \zeta_y$ or ζ_z. In fact, the left-hand sides of Eqs. (M-9.22a) and (M-9.22b) for $\psi = \zeta_y$ or ζ_z are $(1/2)\partial\hat{p}_{\mathfrak{S}0}/\partial y$ or ∂z and $(1/2)\partial\hat{p}_{\mathfrak{S}1}/\partial y$ or ∂z. Thus, the factor 1/2 is put in front of $\overline{\text{Eq. (M-9.33)}}$ and $\overline{\text{Eq. (M-9.23)}}$ in Eqs. (232a), (232b), and (234).

7.2.3 Discussion

According to Eqs. (227a) and (227b), the x_1, x_2, and x_3 components of the flow velocity, i.e., $\hat{v}_1$, $\hat{v}_2$, and $\hat{v}_3$ are expressed as[84]

$$\hat{v}_1 = \hat{v}_{x\mathfrak{S}0} \cos\theta + \cdots , \tag{235a}$$

$$\hat{v}_2 = (\hat{v}_{y\mathfrak{S}1} \cos\theta)k + \hat{v}_{x\mathfrak{S}0} \sin\theta + \cdots , \tag{235b}$$

$$\hat{v}_3 = \hat{v}_{z\mathfrak{S}1} k + \cdots . \tag{235c}$$

Noting that $\cos\theta = 1 - o(k^{2\alpha})$ and $\sin\theta = o(k^{\alpha})$ [Eq. (224)], we have

$$\hat{v}_{10} = \hat{v}_{x\mathfrak{S}0}, \quad \hat{v}_{20} = 0, \quad \hat{v}_{30} = 0, \tag{236a}$$

$$\hat{v}_{21} = \hat{v}_{y\mathfrak{S}1}, \quad \hat{v}_{31} = \hat{v}_{z\mathfrak{S}1}. \tag{236b}$$

where $\hat{v}_1 = \hat{v}_{10} + \cdots$, $\hat{v}_2 = \hat{v}_{20} + \hat{v}_{21}k + \cdots$, and $\hat{v}_3 = \hat{v}_{30} + \hat{v}_{31}k + \cdots$. If we take Eqs. (M-9.33)–(M-9.39b) with (x, y, z) identified with (x_1, x_2, x_3) as the equations in the rectangular system from the above discussion, one easily raise a question where the term $\rho_{\mathfrak{S}0}\hat{v}_{x\mathfrak{S}0}^2/c^2$ on the left-hand side in Eq. (M-9.36) comes from.[85] The conservation equations (M-1.57)–(M-1.59) in a rectangular system have no such term in the convection term. To understand this, we have to examine the second term $\hat{v}_{x\mathfrak{S}0} \sin\theta$ on the right-hand side of Eq. (235b), which comes from the infinitesimal curvature of the flow $(\hat{v}_{x\mathfrak{S}0}, 0, 0)$, more carefully.

Owing to Eqs. (219) and (221), the leading-order term for small k of $\hat{v}_{x\mathfrak{S}0} \sin\theta$ in Eq. (235b) is expressed in the form

$$\hat{v}_{x\mathfrak{S}0} \sin\theta = -\hat{v}_{x\mathfrak{S}0} x/\hat{L} = -k\chi\hat{v}_{x\mathfrak{S}0}/c^2, \tag{237}$$

where the variable χ is used because it is a natural variable, instead of x_1, in the analysis of Eqs. (M-9.33)–(M-9.39b).[86] Then, from Eqs. (235a)–(235c),[87]

$$(\hat{v}_{10}, \hat{v}_{20}, \hat{v}_{30}) = (\hat{v}_{x\mathfrak{S}0}, 0, 0), \tag{238a}$$

$$\hat{v}_{21} = \hat{v}_{y\mathfrak{S}1} - \frac{\chi}{c^2}\hat{v}_{x\mathfrak{S}0}, \tag{238b}$$

$$\hat{v}_{31} = \hat{v}_{z\mathfrak{S}1}, \tag{238c}$$

In the range (224) of θ of our interest, the second term on the right-hand side of Eq. (238b), which comes from the infinitesimal curvature of the flow $(\hat{v}_{x\mathfrak{S}0}, 0, 0)$, is negligibly small, i.e.,

$$\frac{\chi}{c^2}\hat{v}_{x\mathfrak{S}0} = o(k^{\alpha-1}),$$

[84]Here, the arguments x, y, and z are identified with the rectangular components x_1, x_2, and x_3, as noted in the preceding paragraph.

[85]There is a similar term proportional to v_θ^2/r in the convection term of the r component of the momentum conservation equation in the cylindrical coordinate system [see, e.g., Eq. (M-9.73b)]. This is due to the curvature of the coordinate line $r =$ const, but not to the curvature of a flow. The term is not zero even for a straight flow. There is a term proportional to $v_r v_\theta/r$ in the convection term of the θ component [see, e.g., Eq. (M-9.73c)]. When a flow is along a coordinate line $r =$ const, the term $v_r v_\theta/r$ vanishes because $v_r = 0$.

[86]The length scale of variation of the variables $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, etc. is of the order of unity in the variable χ but of the order of $1/k$ in x or x_1.

[87]Note that $\cos\theta = 1 - k^2\chi^2/2c^4 + \cdots$.

because $\chi = kx = -k\hat{L}\theta = o(k^{\alpha-1})$. However, its derivative with respect to χ is of the order of unity, i.e.,

$$\frac{\partial}{\partial\chi}\frac{\chi\hat{v}_{x\mathfrak{S}0}}{c^2} = \frac{\hat{v}_{x\mathfrak{S}0}}{c^2} + \frac{\chi}{c^2}\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial\chi},$$

where the first term on the right-hand side is of the order of unity and the second is infinitesimal $[o(k^{\alpha-1})]$. If we express Eq. (M-9.36) in the variables $\hat{v}_{10}$, $\hat{v}_{21}$, and $\hat{v}_{31}$ in place of $\hat{v}_{x\mathfrak{S}0}$, $\hat{v}_{y\mathfrak{S}1}$, and $\hat{v}_{z\mathfrak{S}1}$ with the aid of Eqs. (238a)–(238c), the term $\rho_{\mathfrak{S}0}\hat{v}^2_{x\mathfrak{S}0}/c^2$ in Eq. (M-9.36) disappears in the equations in the new variables $\hat{v}_{10}$, $\hat{v}_{21}$, and $\hat{v}_{31}$, and its convection term (or its left-hand side) reduces to one of the momentum conservation equations [Eq. (M-1.58)] in the rectangular coordinate system.

The above somewhat strange relation between a functional value and its derivative is due to the present situation where an infinitesimal range $\chi = o(k^{\alpha-1})$ is interested in, though it is a straight channel or pipe with infinite length (in x_1). In this range of χ, the coordinate system (x, y, z) can be identified with the rectangular coordinate system (x_1, x_2, x_3). Equations (M-9.33)–(M-9.39b), without (x, y, z) identified with (x_1, x_2, x_3), are valid for any range of χ, and their process of solution for time-dependent problems is explained in Section 7.1.1. The corresponding process of solution in the infinitesimal χ range or at the given point $\chi = 0$ is obtained by paraphrasing the process in Section 7.1.1 in the following way.

Let a set consisting of a and its derivatives $\partial^n a/\partial\chi^n (n = 1, 2, 3, \cdots)$ on the cross section $(0, y, z)$ be indicated by $\{a\}$, where a is a quantity or an equation or equations. Prepare the sets of the equations: {Eq. (199)}–{Eq. (203)} and the initial data of $\{\hat{\rho}_{\mathfrak{S}0}\}$, $\{\hat{v}_{x\mathfrak{S}0}\}$, $\{\hat{v}_{y\mathfrak{S}1}\}$, $\{\hat{v}_{z\mathfrak{S}1}\}$, and $\{\hat{p}_{\mathfrak{S}0}\}$. The time derivatives $\{\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}\}$, $\{\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}\}$, $\{\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}\}$, $\{\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}\}$, and $\{\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}\}$ are expressed with $\{\hat{\rho}_{\mathfrak{S}0}\}$, $\{\hat{v}_{x\mathfrak{S}0}\}$, $\{\hat{v}_{y\mathfrak{S}1}\}$, $\{\hat{v}_{z\mathfrak{S}1}\}$, $\{\hat{p}_{\mathfrak{S}0}\}$, $\{\hat{p}_{\mathfrak{S}2}\}$ and their derivatives with respect to y and z by the sets of equations {Eq. (199)}–{Eq. (203)} with the aid of the supplementary conditions {Eq. (204a)} and {Eq. (204b)}. The sets of equations {Eq. (198)}[88] for all $\hat{t}$ can be replaced by the conditions {Eq. (198)} and {Eq. (208)} for the initial data and the set of equations {Eq. (212)} of $\{\hat{p}_{\mathfrak{S}2}\}$ for all $\hat{t}$, whose coefficients and inhomogeneous terms are expressed by $\{\hat{\rho}_{\mathfrak{S}0}\}$, $\{\hat{v}_{x\mathfrak{S}0}\}$, $\{\hat{v}_{y\mathfrak{S}1}\}$, $\{\hat{v}_{z\mathfrak{S}1}\}$, and $\{\hat{p}_{\mathfrak{S}0}\}$ and their derivatives with respect to y and z.[89] The set $\{\hat{p}_{\mathfrak{S}2}\}$ is determined except the set of additive functions $\{\psi\}$ of $\hat{t}$.[90] This $\{\hat{p}_{\mathfrak{S}2}\}$ being substituted into {Eq. (199)}–{Eq. (203)}, the time derivatives $\{\partial\hat{\rho}_{\mathfrak{S}0}/\partial\hat{t}\}$, $\{\partial\hat{v}_{x\mathfrak{S}0}/\partial\hat{t}\}$, $\{\partial\hat{v}_{y\mathfrak{S}1}/\partial\hat{t}\}$, $\{\partial\hat{v}_{z\mathfrak{S}1}/\partial\hat{t}\}$, and $\{\partial\hat{p}_{\mathfrak{S}0}/\partial\hat{t}\}$ are expressed with $\{\hat{\rho}_{\mathfrak{S}0}\}$, $\{\hat{v}_{x\mathfrak{S}0}\}$, $\{\hat{v}_{y\mathfrak{S}1}\}$, $\{\hat{v}_{z\mathfrak{S}1}\}$, $\{\hat{p}_{\mathfrak{S}0}\}$, and their derivatives with respect to y and z.[91] Then, the time evolution of $\{\hat{\rho}_{\mathfrak{S}0}\}$, $\{\hat{v}_{x\mathfrak{S}0}\}$, $\{\hat{v}_{y\mathfrak{S}1}\}$, $\{\hat{v}_{z\mathfrak{S}1}\}$,

[88] {Eq. (198)} are the two sets $\{\partial\hat{p}_{\mathfrak{S}0}/\partial y = 0\}$ and $\{\partial\hat{p}_{\mathfrak{S}0}/\partial z = 0\}$.

[89] See the discussion from Eq. (205) to Eq. (212) in Section 7.1.1.

[90] (i) Its boundary condition is of the form $\{n_y\partial\hat{p}_{\mathfrak{S}2}/\partial y + n_z\partial\hat{p}_{\mathfrak{S}2}/\partial z\} = \{\text{known data}\}$ [see the paragraph following that with Eq. (212) in Section 7.1.1].

(ii) Equation (212) and the boundary condition for $\hat{p}^c_{\mathfrak{S}2}$ can be transformed into those for $\hat{p}_{\mathfrak{S}2}$ keeping the property required in the discussion. Thus, they can be used interchangeably.

[91] In this process, $\{\psi\}$ does not contribute to {Eq. (199)}–{Eq. (203)}.

and $\{\hat{p}_{\mathfrak{S}0}\}$ is determined, satisfying the conditions {Eq. (198)} and {Eq. (208)} throughout.[92]

The above process of solution is formally consistent. However, we have to deal with an infinite series of equations. Generally, the series does not end at a finite order.[93] Exceptionally, the solution that is independent of χ is easily seen to be possible. Further, the series of equations cannot be solved successively from the lowest order with respect to differentiation $\partial^n/\partial\chi^n$.[94] Thus, the infinite series of equations has to be handled simultaneously. The velocity $\hat{v}_{x\mathfrak{S}0}$ at (χ, y, z) in the limit $k \to 0$ is expressed as

$$\hat{v}_{x\mathfrak{S}0} = (\hat{v}_{x\mathfrak{S}0})_{\chi=0} + \chi\left(\frac{\partial\hat{v}_{x\mathfrak{S}0}}{\partial\chi}\right)_{\chi=0} + \frac{1}{2}\chi^2\left(\frac{\partial^2\hat{v}_{x\mathfrak{S}0}}{\partial\chi^2}\right)_{\chi=0} + \cdots, \tag{239}$$

where the solution applies to finite χ. In the present case, where χ is negligibly small, the velocity field is expressed as

$$\hat{v}_{x\mathfrak{S}0}(x, y, z, \hat{t}) = \hat{v}_{x\mathfrak{S}0}(\chi; y, z, \hat{t})\big|_{\chi=0}, \tag{240}$$

where $\hat{v}_{x\mathfrak{S}0}$ expressed in the shrunk variable χ is indicated as $\hat{v}_{x\mathfrak{S}0}(\chi; y, z, \hat{t})$ with the semicolon after χ in order to avoid confusion with $\hat{v}_{x\mathfrak{S}0}(x, y, z, \hat{t})$ expressed in x.[95] The solution is uniform with respect to x irrespective of the initial data, but its variation with time depends on them.

Examples showing the effect of infinitesimal curvature are found in Sone & Doi [2005, 2007], where the instabilities of the plane Couette and Poiseuille flows are studied on the basis of Eqs. (M-9.49a)–(M-9.50e) with the time-derivative terms [or Eqs. (213a)–(214e)][96], in addition to the example in Section M-9.4 of the bifurcation of the time-independent plane Couette flow with infinitesimal curvature. In the papers, the solution that is independent of $\tilde{\chi}$, corresponding

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[92]The set $\{\psi\}$ in $\{\hat{p}_{\mathfrak{S}2}\}$ is undetermined in this process, but it does not influence $\{\hat{\rho}_{\mathfrak{S}0}\}$, $\{\hat{v}_{x\mathfrak{S}0}\}$, $\{\hat{v}_{y\mathfrak{S}1}\}$, $\{\hat{v}_{z\mathfrak{S}1}\}$, and $\{\hat{p}_{\mathfrak{S}0}\}$. In the higher-order analysis in k, which is unnecessary for the present purpose, equations for $\{\partial\hat{\rho}_{\mathfrak{S}2}/\partial\hat{t}\}$, $\{\partial\hat{v}_{x\mathfrak{S}2}/\partial\hat{t}\}$, $\{\partial\hat{v}_{y\mathfrak{S}3}/\partial\hat{t}\}$, $\{\partial\hat{v}_{z\mathfrak{S}3}/\partial\hat{t}\}$, and $\{\partial\hat{p}_{\mathfrak{S}2}/\partial\hat{t}\}$ are derived, where partially determined $\{\hat{p}_{\mathfrak{S}2}\}$ is in the same situation as $\{\hat{p}_{\mathfrak{S}0}\}$ partially determined by {Eq. (198)}.

[93]For example, if $\partial^s\hat{v}_{y\mathfrak{S}1}/\partial\chi^s$ $(s = 0, 1, \cdots, n)$ is nonzero and nonuniform in a region of the cross section, ∂^{n+1}[Eq. (201)]$/\partial\chi^{n+1}$ [or the equation for $\partial(\partial^{n+1}\hat{v}_{y\mathfrak{S}1}/\partial\chi^{n+1})/\partial\hat{t}$] contains a nonzero term $(\partial\hat{v}_{y\mathfrak{S}1}/\partial\chi)(\partial^{n+1}\hat{v}_{y\mathfrak{S}1}/\partial\chi^n\partial y)$. Similarly, the equation for $\partial(\partial^{n+1}(\partial\hat{v}_{y\mathfrak{S}1}/\partial y)\partial\chi^{n+1})/\partial\hat{t}$ contains a nonzero term $(\partial^2\hat{v}_{y\mathfrak{S}1}/\partial y\partial\chi)(\partial^{n+1}\hat{v}_{y\mathfrak{S}1}/\partial y\partial\chi^n)$. Therefore, $\partial^{n+1}\hat{v}_{y\mathfrak{S}1}/\partial\chi^{n+1}$ is nonzero and nonuniform.

[94]For example, ∂^n[Eq. (200)]$/\partial\chi^n$ [or the equation for $\partial(\partial^n\hat{v}_{x\mathfrak{S}0}/\partial\chi^n)/\partial\hat{t}$] contains $\hat{v}_{x\mathfrak{S}0}\partial^{n+1}\hat{v}_{x\mathfrak{S}0}/\partial\chi^{n+1}$.

[95]To present the result of analysis, the variables (x, y, z) are natural for the present problem. For analysis, the variables (χ, y, z) are convenient.

[96]i) Equations (M-9.49a)–(M-9.50e) are the simplified version for small but finite Mach numbers and temperature variations of Eqs. (M-9.33)–(M-9.39b). They are derived from Eqs. (M-9.33)–(M-9.39b) (see Section M-9.3).

ii) The time-derivative terms are given at the end of page M-465.

iii) The discussion in the preceding two paragraphs can be carried out in a similar way for these equations. Note the difference of notation owing to the difference of situations in Sections 7.1.1 and 7.1.2.

to χ in Section 7.1.1, is considered,[97] and is found to have the critical point of stability. Naturally, one can analyze the problems in a rectangular coordinate system without infinitesimal curvature term $[-\rho_{\mathfrak{S}0}\hat{v}^2_{x\mathfrak{S}0}/c^2$ in Eq. (M-9.36) or $-u_x^2/C^2$ in Eq. (M-9.50c)]. In this case, one has to take into account of the dependence on $\tilde{\chi}$ of the initial and boundary conditions modified according to the relation

$$u_2 = u_y - \frac{\tilde{\chi}}{C^2}u_x, \tag{241}$$

corresponding to Eq. (238b).

(Section 7.2: Version 10-00)

7.3 Ghost effect of infinitesimal curvature on the Poiseuille flow through a pipe

The fluid-dynamics-type equations with the ghost effect of infinitesimal curvature described in Sections M-9.2 and M-9.3 apply not only to flows through a straight channel between two parallel walls but also to flows through a straight pipe of uniform cross section. For flows through a channel, the bifurcation of the time-independent plane Couette flow (Section M-9.4) and the linear stability of the plane Couette and Poiseuille flows (Sone & Doi [2005, 2007]) are studied. In this section, we examine the effect of the infinitesimal curvature of the pipe axis on the Poiseuille flow through a circular pipe.

Here, we take the situation discussed in Section M-9.3, where the Mach number and the temperature variation are small but finite, and discuss the Poiseuille flow through a circular pipe. The fluid-dynamics-type equations for the time-independent case are given by Eqs. (M-9.49a)–(M-9.50e), i.e.,

$$\frac{\partial P_{01}}{\partial \tilde{\chi}} = \frac{\partial P_{01}}{\partial y} = \frac{\partial P_{01}}{\partial z} = 0, \quad P_{01} = \omega + \tau, \tag{242a}$$

$$\frac{\partial P_{02}}{\partial y} = \frac{\partial P_{02}}{\partial z} = 0, \tag{242b}$$

[97] For Eqs. (213a)–(214e), the solution in which the variables except P_{02} are all independent of $\tilde{\chi}$ but $\partial P/\partial\tilde{\chi}$ is a constant, including $\partial P/\partial\tilde{\chi} = 0$, is consistent with the equations. The Poiseuille flow is the case where $\partial P/\partial\tilde{\chi}$ is a nonzero constant.

$$\frac{\partial u_x}{\partial \tilde{\chi}} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \tag{243a}$$

$$u_x \frac{\partial u_x}{\partial \tilde{\chi}} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{2}\frac{\partial P_{02}}{\partial \tilde{\chi}} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right), \tag{243b}$$

$$u_x \frac{\partial u_y}{\partial \tilde{\chi}} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} - \frac{u_x^2}{C^2} = -\frac{1}{2}\frac{\partial P_{20}}{\partial y} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right), \tag{243c}$$

$$u_x \frac{\partial u_z}{\partial \tilde{\chi}} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{2}\frac{\partial P_{20}}{\partial z} + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right), \tag{243d}$$

$$u_x \frac{\partial \tau}{\partial \tilde{\chi}} + u_y \frac{\partial \tau}{\partial y} + u_z \frac{\partial \tau}{\partial z} = \frac{\gamma_2}{2}\left(\frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2}\right). \tag{243e}$$

The boundary condition for these equations is the nonslip condition. The velocity vanishes and the temperature is uniform on the surface ($y^2 + z^2 = 1$) of the cylinder, i.e.,

$$u_x = 0, \quad u_y = 0, \quad u_z = 0, \quad \tau = 0 \quad \text{at } y^2 + z^2 = 1. \tag{244}$$

Further, a constant pressure gradient is applied along the axis of the cylinder. Then, in view of Eq. (242b), $\partial P_{02}/\partial \tilde{\chi}$ is constant, i.e.,

$$\frac{\partial P_{02}}{\partial \tilde{\chi}} = \left(\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0. \tag{245}$$

Obviously, $\tau = 0$ is a solution independently of the velocity. From now on, we are interested only in the velocity field. First, consider the case where the infinitesimal curvature term u_x^2/C^2 is absent in Eq. (243c), and look for the solution with $u_y = u_z = 0$.[98] We easily find the solution as

$$u_x = \frac{-1}{4\gamma_1}\left(\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0 \left[1 - (y^2 + z^2)\right], \tag{246}$$

and P_{20} is uniform over the cross section. This is the Poiseuille flow with parabolic profile in the classical fluid dynamics. What we are interested in here is the infinitesimal curvature effect on the Poiseuille flow. In the case of flows through the channel, there are flows that have the same velocity profiles as those without the infinitesimal curvature term (the Couette and Poiseuille flows), for which the infinitesimal-curvature affects only P_{20} (see Section M-9.4.1 and Sone & Doi [2007]). We examine whether this is the case for the Poiseuille flow through the circular cylinder.

The solution where the variables $(u_x, u_y, u_z, \partial P_{02}/\partial \tilde{\chi}, P_{20}, \tau)$ are independent of $\tilde{\chi}$ (see Footnote 98) is consistent with the equations (242b)–(243e) and

[98]For the pipe of infinite length in the scale of x, the corresponding range of $\tilde{\chi}$ is infinitesimal $\tilde{\chi}$ (see Section 7.2.1), and the solution of the system (242a)–(243e) is interested in this infinitesimal $\tilde{\chi}$ range or at the point $\tilde{\chi} = 0$. The way to handle the system at the point $\tilde{\chi} = 0$ is discussed in Section 7.2.3. The condition $u_y = u_z = 0$ is taken to be so in a finite range of $\tilde{\chi}$ or to be $\partial^n u_y/\partial \tilde{\chi}^n = \partial u_z^n/\partial \tilde{\chi}^n = 0$ ($n = 1, 2, \cdots$) at the point $\tilde{\chi} = 0$ as well as $u_y = u_z = 0$ there.

boundary condition (244). We discuss this class of solutions. We examine whether the solution with $u_y = u_z = 0$ is consistent as in the Couette and Poiseuille flows through a channel. Obviously, Eq. (243a) is consistent. From Eq. (243b), we have

$$0 = -\frac{1}{2}\left(\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0 + \frac{\gamma_1}{2}\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right),$$

from which we obtain Eq. (246). From Eq. (243d), P_{20} is seen to be independent of z. Equation (243c) reduces to

$$\frac{u_x^2}{C^2} = \frac{1}{2}\frac{\partial P_{20}}{\partial y},$$

from which we obtain, with the aid of Eq. (246),

$$P_{20} = \frac{1}{8\gamma_1^2 C^2}\left(\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0^2 \left[b_0 + y(1-z^2)^2 - \frac{2}{3}y^3(1-z^2) + \frac{1}{5}y^5\right],$$

where b_0 is a constant. This result contradicts with the result from Eq. (243d) that P_{20} is independent of z. Thus, the solution with $u_y = u_z = 0$ does not exist. Thus, in the Poiseuille flow through a circular pipe, the flow $(u_x, 0, 0)$ with parabolic profile (246) is subject to change due to u_y and u_z induced by the infinitesimal curvature of the axis of the cylinder. Generally, in flows through pipes with various cross section, their velocity profiles without u_y and u_z depend on z as well as on y in contrast to the flows in the channel. So does the infinitesimal curvature term u_x^2/C^2 in Eq. (243c). This gives the dependence of P_{20} on z. On the other hand, in the momentum conservation equation (243d) in the z direction, there is no term of the curvature effect owing to the present infinitesimal curvature of the pipe. Thus, P_{20} is uniform with respect to z. Owing to this contradiction, u_y and u_z cannot be zero in a flow through a pipe. The infinitesimal flow (u_y, u_z) disturbs the main flow u_x.

Here, we rewrite Eqs. (242b)–(243d) for the class of solutions for which the variables $(u_x, u_y, u_z, \partial P_{02}/\partial \tilde{\chi}, P_{20})$ are independent of $\tilde{\chi}$ (see Footnote 98). From Eq. (243a) with $\partial u_x/\partial \tilde{\chi} = 0$, we can introduce the stream function Ψ such that

$$u_y = \frac{\partial \Psi}{\partial z}, \quad u_z = -\frac{\partial \Psi}{\partial y}. \tag{247}$$

This replaces Eq. (243a). From Eqs. (243c) and (243d), we can eliminate P_{20} by the operation ∂[Eq. (243d)]$/\partial y - \partial$[Eq. (243c)]$/\partial z$. Then, from Eqs. (243b)–(243d) and (247), we have

$$\frac{\gamma_1}{2}\triangle u_x - \mathfrak{D}u_x = \frac{1}{2}\left(\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0, \tag{248a}$$

$$\frac{\gamma_1}{2}\triangle \omega_x - \mathfrak{D}\omega_x = \partial_z \frac{u_x^2}{C^2}, \tag{248b}$$

$$\triangle \Psi = -\omega_x, \tag{248c}$$

$$u_y = \frac{\partial \Psi}{\partial z}, \quad u_z = -\frac{\partial \Psi}{\partial y}, \tag{248d}$$

where

$$\triangle = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{249a}$$

$$\mathfrak{D} = u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}, \tag{249b}$$

$$\partial_z = \frac{\partial}{\partial z}, \tag{249c}$$

$$\omega_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}. \tag{249d}$$

Here, ω_x is the axial component of the vorticity. The boundary condition is that the velocity (u_x, u_y, u_z) vanishes on the boundary $(y^2 + z^2 = 1)$.

In order to arrange the parameters scattered in Eqs. (248a)–(248d), we introduce the following variables:

$$U_x = u_x \left[\frac{1}{\gamma_1}\left(\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)\right]^{-1}, \quad U_y = \frac{2u_y}{\gamma_1}, \quad U_z = \frac{2u_z}{\gamma_1}, \tag{250a}$$

$$\bar{\Psi} = \frac{2\Psi}{\gamma_1}, \quad \Omega_x = \frac{2\omega_x}{\gamma_1}. \tag{250b}$$

Then, Eqs. (248a)–(248d) are rewritten as

$$\triangle U_x - \bar{\mathfrak{D}}U_x = 1, \tag{251a}$$

$$\triangle \Omega_x - \bar{\mathfrak{D}}\Omega_x = \left(\frac{2}{\gamma_1^2 C}\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0^2 \partial_z U_x^2, \tag{251b}$$

$$\triangle \bar{\Psi} = -\Omega_x, \tag{251c}$$

$$U_y = \frac{\partial \bar{\Psi}}{\partial z}, \quad U_z = -\frac{\partial \bar{\Psi}}{\partial y}, \tag{251d}$$

where

$$\bar{\mathfrak{D}} = U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z}, \tag{252a}$$

$$\Omega_x = \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}. \tag{252b}$$

The boundary condition on the surface of the pipe is given as

$$U_x = 0, \quad U_y = 0, \quad U_z = 0. \tag{253}$$

The system (251a)–(251d) contains only one parameter, i.e.,

$$\left(\frac{2}{\gamma_1^2 C}\frac{\partial P_{02}}{\partial \tilde{\chi}}\right)_0^2. \tag{254}$$

The variables U_y, U_z, and Ω_x are explicitly expressed with $\bar{\Psi}$ [Eqs. (251c) and (251d)]. Thus, it is a system for U_x and $\bar{\Psi}$. One of the boundary conditions for U_y and U_z can be replaced by

$$\bar{\Psi} = 0 \quad \text{on the boundary.} \tag{255}$$

Incidentally, this system applies to the corresponding problem for a pipe with an arbitrary cross section.

For the cylindrical pipe problem, the cylindrical coordinate system (x, r, θ) is convenient, which is defined by

$$x = x, \quad y = r\cos\theta, \quad z = r\sin\theta, \tag{256a}$$

$$U_y = U_r\cos\theta - U_\theta\sin\theta, \quad U_z = U_r\sin\theta + U_\theta\cos\theta. \tag{256b}$$

Then,

$$U_r = \frac{1}{r}\frac{\partial\bar{\Psi}}{\partial\theta}, \quad U_\theta = -\frac{\partial\bar{\Psi}}{\partial r}, \tag{257a}$$

$$\Omega_x = \frac{1}{r}\frac{\partial rU_\theta}{\partial r} - \frac{1}{r}\frac{\partial U_r}{\partial\theta}. \tag{257b}$$

The operators $\triangle$, $\bar{\mathfrak{D}}$, and ∂_z are expressed in the variables (x, r, θ) and (U_r, U_θ), instead of (x, y, z) and (U_y, U_z), as

$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{258a}$$

$$\bar{\mathfrak{D}} = U_r\frac{\partial}{\partial r} + \frac{U_\theta}{r}\frac{\partial}{\partial\theta}, \tag{258b}$$

$$\partial_z = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}. \tag{258c}$$

Substituting the expressions (258a)–(258c) into Eqs. (251a)–(251c), and replacing Eq. (251d) by Eq. (257a), we obtain the equations in the cylindrical system. The boundary condition at $r = 1$ is given by

$$\bar{\Psi} = 0, \quad \frac{\partial\bar{\Psi}}{\partial r} = 0. \tag{259a}$$

To find the disturbed Poiseuille flow, it is practical to solve the system (251a)–(251c) and (257a) with Eqs. (258a)–(258c) numerically. It is not a so

hard problem. Numerical computation of a structurally similar but more complicated system was carried out in the study of a ghost effect in the Bénard problem (see Section M-8.2). Following the process of solution there, we outline the method of numerical computation of the present system. The solution is obtained by iteration. First, choose initial data $U_r^{(0)}$ and $U_\theta^{(0)}$ of U_r and U_θ, which vanish on the boundary, and compute the initial data $\Omega^{(0)}$ of the boundary value of Ω by Eq. (257b).[99] With these initial data, we start iteration from $n = 1$ in the superscript (n) in the following expressions. One iteration consists of solving three partial differential equations successively.
(i) The first step is to find $U_x^{(n)}$ by solving the following boundary-value problem of a linear elliptic partial differential equation: The equation for $U_x^{(n)}$ is

$$\triangle U_x^{(n)} - \bar{\mathfrak{D}}^{(n-1)} U_x^{(n)} = 1, \tag{260}$$

where

$$\bar{\mathfrak{D}}^{(n)} = U_r^{(n)} \frac{\partial}{\partial r} + \frac{U_\theta^{(n)}}{r} \frac{\partial}{\partial \theta},$$

and its boundary condition at $r = 1$ is

$$U_x^{(n)} = 0. \tag{261}$$

(ii) The second step is to find $\Omega_x^{(n)}$ by solving the following boundary-value problem of a linear elliptic partial differential equation: The equation for $\Omega_x^{(n)}$ is

$$\triangle \Omega_x^{(n)} - \bar{\mathfrak{D}}^{(n-1)} \Omega_x^{(n)} = \left(\frac{2}{\gamma_1^2 C} \frac{\partial P_{02}}{\partial \tilde{\chi}} \right)_0^2 \partial_z (U_x^{(n)})^2, \tag{262}$$

and its boundary condition at $r = 1$ is

$$\Omega_x^{(n)} = \Omega_x^{(n-1)} - \vartheta U_\theta^{(n-1)}, \tag{263}$$

where ϑ is a constant to be chosen for the iteration to converge. This requires some explanation, which will be given after the main explanation of the process is finished.
(iii) The third step is to find $\bar{\Psi}^{(n)}$ by solving the following boundary-value problem of a linear elliptic partial differential equation: The equation for $\bar{\Psi}^{(n)}$ is

$$\triangle \bar{\Psi}^{(n)} = -\Omega_x^{(n)}, \tag{264}$$

and its boundary condition at $r = 1$ is

$$\bar{\Psi}^{(n)} = 0. \tag{265}$$

[99]In view of the boundary condition for U_r and U_θ, on the boundary ($r = 1$), Eq. (257b) reduces to

$$\Omega_x = \frac{\partial U_\theta}{\partial r}.$$

From $\bar{\Psi}^{(n)}$, compute $U_r^{(n)}$ and $U_\theta^{(n)}$ by Eq. (266):

$$U_r^{(n)} = \frac{1}{r}\frac{\partial \bar{\Psi}^{(n)}}{\partial \theta}, \quad U_\theta^{(n)} = -\frac{\partial \bar{\Psi}^{(n)}}{\partial r}. \tag{266}$$

These $U_r^{(n)}$ and $U_\theta^{(n)}$, together with the boundary value of $\Omega_x^{(n)}$ in the step (ii), serve as the initial data of the next iteration.
(iv) Now, we can go to the next iteration ($n \to n+1$) with the above mentioned initial data. Start again from the step (i), and continue the iteration until the solution is considered to have converged enough.

In the present problem, U_x, U_r, and U_θ, or U_x, $\bar{\Psi}$, and $\partial\bar{\Psi}/\partial r$, on the boundary are specified, but Ω_x on the boundary is not known until the final solution is obtained. Thus, it is not obvious what condition is to be chosen as the boundary condition for Eq. (262). In the process of iteration, the conditions $U_x^{(n)} = 0$ and $\bar{\Psi}^{(n)} = 0$ (or $U_r^{(n)} = 0$) are given as the boundary conditions for Eqs. (260) and (264) respectively. Thus, the information $\partial\bar{\Psi}/\partial r = 0$ (or $U_\theta = 0$) has to be taken in to the boundary condition for Eq. (262). In the iteration process, the condition $\partial\bar{\Psi}^{(n)}/\partial r = 0$ (or $U_\theta^{(n)} = 0$) can be replaced by the weaker condition $\partial\bar{\Psi}^{(n)}/\partial r \to 0$ (or $U_\theta^{(n)} \to 0$) as $n \to \infty$. When the solution of iteration converges, $\Omega_x^{(n+1)} - \Omega_x^{(n)} \to 0$ as $n \to 0$. Thus, we put

$$\Omega_x^{(n)} = \Omega_x^{(n-1)} - \vartheta U_\theta^{(n-1)},$$

where ϑ is some constant to be chosen for the iteration process to converge. Then, $U_\theta^{(n)}$ converges to zero as the solution converges in the limit of the iteration process. If a vorticity of positive value is put in a flow over the boundary wall, a flow with positive U_θ is induced on the wall. Thus, the constant ϑ should be positive. If it is positive but too large, the correction is in the correct direction but in excess, and the iteration may diverge. Proper size of ϑ should be chosen by examination in practical applications.

Finally, the effect of infinitesimal curvature is discussed for the Navier–Stokes equations with the nonslip condition of an incompressible fluid in Section M-9.5. The equations for the velocity field derived from them are of the same form as Eq. (242b)–(243d) with the nonslip condition. Thus, the results obtained in this section as well as those in Section M-9.4 and Sone & Doi [2005, 2007] apply to the Navier–Stokes system.

(Section 7.3: Version 11-00)

8 Appendix M-A

8.1 Note on the loss term of the collision integral [From Eq. (M-A.18) to Eq. (M-A.21)]

Consider the following collision term of the Boltzmann equation (M-A.18):[100]

$$\frac{d_m^2}{2m}\int_{\text{all } e,\ \text{all } \boldsymbol{\xi}_*} |(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot \boldsymbol{e}|[f(\boldsymbol{\xi}')f(\boldsymbol{\xi}_*') - f(\boldsymbol{\xi})f(\boldsymbol{\xi}_*)]\mathrm{d}\Omega(\boldsymbol{e})\mathbf{d}\boldsymbol{\xi}_*, \tag{267}$$

where

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + [\boldsymbol{\alpha}\cdot(\boldsymbol{\xi}_* - \boldsymbol{\xi})]\boldsymbol{\alpha}, \quad \boldsymbol{\xi}_*' = \boldsymbol{\xi}_* - [\boldsymbol{\alpha}\cdot(\boldsymbol{\xi}_* - \boldsymbol{\xi})]\boldsymbol{\alpha}. \tag{268}$$

The change (M-A.20) of the variable of integration from $\boldsymbol{e}$ to $\boldsymbol{\alpha}$, i.e.,

$$|(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot \boldsymbol{e}|\mathrm{d}\Omega(\boldsymbol{e}) = \frac{2}{d_m^2}B\mathrm{d}\Omega(\boldsymbol{\alpha}), \tag{269}$$

is introduced instead of expressing $\boldsymbol{\alpha}$ in Eq. (268) in terms of $\boldsymbol{e}$. The part of the integral of Eq. (267)

$$\frac{d_m^2}{2m}\int_{\text{all } e,\ \text{all } \boldsymbol{\xi}_*} |(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot \boldsymbol{e}|f(\boldsymbol{\xi})f(\boldsymbol{\xi}_*)\mathrm{d}\Omega(\boldsymbol{e})\mathbf{d}\boldsymbol{\xi}_*,$$

which comes from I_- in Eq. (M-A.8) and corresponds to the loss term (see Section M-1.2) of the collision integral of the Boltzmann equation (M-1.5) or (M-A.21), does not contain $\boldsymbol{\alpha}$, and the change (269) of the variable of integration is not required.[101] Thus, the result is determined uniquely irrespective of the relation between $\boldsymbol{\alpha}$ and $\boldsymbol{e}$, that is, the loss term of the collision integral is independent of the intermolecular potential when d_m is of a finite value. That is, the loss term of the collision integral is determined only by $d_m^2/2m$ and $f(\boldsymbol{\xi})$, and is the same as that for the hard-sphere molecule with the same d_m.

(Section 8.1: Version 6-00)

8.2 Note on the loss term of the kernel representation of the linearized collision integral [Section M-A.2.10]

In Section M-A.2.10, we discussed the kernel representation of the linearized collision integral $\mathcal{L}(\phi)$ introduced in Section M-1.10, and gave its explicit form for a hard-sphere molecule. From the discussion in Section 8.1, the kernel representation of the loss term of the linearized collision integral for a hard-sphere molecule applies to any intermolecular potential with a finite d_m.

[100] The factor $d_m^2/2m$ can be rewritten as $nd_m^2/2\rho$, where n is the number of molecules in unit volume. The numerator nd_m^2 is of the order of the inverse of the mean free path (Section M-1.5). Note Footnote M-4 in Section M-A.1.

[101] Transformation (M-A.20) or (269) is carried out to make the variable of integration to be the same. Thus, it is simply one of the changes of variable $\boldsymbol{e}$ of integration to some variable.

In Section M-A.2.10, the linearized collision integral $\mathcal{L}(\phi)$ is expressed by Eqs. (M-137a)–(M-A.139c) as

$$\begin{aligned}\mathcal{L}(\phi) &= \int E_*(\phi' + \phi'_* - \phi - \phi_*)\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_* \\ &= \mathcal{L}^G(\phi) - \mathcal{L}^{L2}(\phi) - \nu_L(\boldsymbol{\zeta})\phi, \end{aligned}\tag{270}$$

where

$$\mathcal{L}^G(\phi) = \int E_*(\phi' + \phi'_*)\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_*, \tag{271a}$$

$$\begin{aligned}\mathcal{L}^{L2}(\phi) &= \int E_*\phi_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_* \\ &= \int K_2(\boldsymbol{\zeta},\boldsymbol{\zeta}_*)\phi(\boldsymbol{\zeta}_*)\mathbf{d}\boldsymbol{\zeta}_*, \end{aligned}\tag{271b}$$

$$\nu_L(\boldsymbol{\zeta}) = \int E_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\zeta}_*. \tag{271c}$$

The loss term is the sum of Eqs. (271b) and (271c) multiplied by ϕ, i.e., $\mathcal{L}^{L2}(\phi)+\nu_L(\boldsymbol{\zeta})\phi$.[102] The kernel $K_2(\boldsymbol{\zeta},\boldsymbol{\zeta}_*)$ and the function $\nu_L(\boldsymbol{\zeta})$ for a hard-sphere molecule are given by Eqs. (M-A.149b) and (M-A.149c) as

$$K_2(\boldsymbol{\zeta},\boldsymbol{\zeta}_*) = \frac{|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|}{2\sqrt{2}\,\pi}\exp\left(-\zeta_*^2\right), \tag{272a}$$

$$\nu_L(\boldsymbol{\zeta}) = \frac{1}{2\sqrt{2}}\left[\exp(-\zeta^2) + \left(2\zeta + \frac{1}{\zeta}\right)\int_0^{\zeta}\exp(-\zeta_*^2)\mathrm{d}\zeta_*\right], \tag{272b}$$

where

$$\zeta = |\boldsymbol{\zeta}|.$$

These formulas apply to any potential with a finite d_m as well as to a hard-sphere molecule.

(Section 8.2: Version 6-00)

8.3 Parity of the collision integral: Supplement to Section M-A.2.7

In Section M-A.2.7, we discussed the parity of the linearized collision integral. It may be better to explain a similar property of the collision integral defined

[102]Only the term $\nu_L(\boldsymbol{\zeta})\phi$ is often called the loss term, and the rest, i.e., $\mathcal{L}^G(\phi) - \mathcal{L}^{L2}(\phi)$, is called the gain term by misunderstanding. This is probably because the loss term of the original collision integral (267) is often written in the form $\nu_c f$, where ν_c is the collision frequency defined by Eq. (M-1.18) as

$$\nu_c = m^{-1}\int_{\text{all }\boldsymbol{\alpha},\text{ all }\boldsymbol{\xi}_*} f(\boldsymbol{\xi}_*)B\mathrm{d}\Omega(\boldsymbol{\alpha})\mathbf{d}\boldsymbol{\xi}_* = (d_m^2/2m)\int_{\text{all }\boldsymbol{e},\text{ all }\boldsymbol{\xi}_*}|(\boldsymbol{\xi}_* - \boldsymbol{\xi})\cdot\boldsymbol{e}|f(\boldsymbol{\xi}_*)\mathrm{d}\Omega(\boldsymbol{e})\mathbf{d}\boldsymbol{\xi}_*.$$

Not to mention, $\mathcal{L}^{L2}(\phi)$ is derived from $\nu_c f$.

by Eq. (M-1.9), i.e.,

$$\hat{J}(\hat{f},\hat{g}) = \frac{1}{2}\int (\hat{f}'\hat{g}'_* + \hat{f}'_*\hat{g}' - \hat{f}\hat{g}_* - \hat{f}_*\hat{g})\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{\zeta}_*, \tag{273}$$
$$\widehat{B} = \widehat{B}(|\boldsymbol{\alpha}\cdot\boldsymbol{V}|/|\boldsymbol{V}|, |\boldsymbol{V}|),$$
$$\hat{f} = \hat{f}(\zeta_i),\ \hat{f}_* = \hat{f}(\zeta_{i*}),\ \hat{f}' = \hat{f}(\zeta'_i),\ \hat{f}'_* = \hat{f}(\zeta'_{i*}),$$
and a similar notation for $\hat{g},\ \hat{g}_*,\ \hat{g}',$ and $\hat{g}'_*$,
$$\zeta'_i = \zeta_i + \alpha_j V_j \alpha_i,\ \zeta'_{i*} = \zeta_{i*} - \alpha_j V_j \alpha_i,\ \zeta_{i*} = V_i + \zeta_i.$$

Here, we discuss the relation of the parity of $\hat{J}(\hat{f},\hat{g})$ with respect to a component (ζ_1, ζ_2, or ζ_3) of the variable $\boldsymbol{\zeta}$ to that of $\hat{f}$ and $\hat{g}$. Put the integral (273) in the sum

$$\hat{J}(\hat{f},\hat{g}) = \frac{1}{2}\left(\mathit{IV} + \mathit{III} - \mathit{II} - \mathit{I}\right), \tag{274}$$

where

$$\mathit{I} = \int \hat{f}_*\hat{g}\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{275a}$$

$$\mathit{II} = \int \hat{f}\hat{g}_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{275b}$$

$$\mathit{III} = \int \hat{f}'_*\hat{g}'\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{275c}$$

$$\mathit{IV} = \int \hat{f}'\hat{g}'_*\widehat{B}\,\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}, \tag{275d}$$

and discuss each term separately.[103] In Eqs. (275a)–(275d), the variable of integration is changed from $\boldsymbol{\zeta}_*$ to $\boldsymbol{V}$ $(=\boldsymbol{\zeta}_* - \boldsymbol{\zeta})$. The following change of the variables

$$\widetilde{V}_1 = -V_1,\quad \widetilde{V}_s = V_s,\quad \widetilde{\alpha}_1 = -\alpha_1,\quad \widetilde{\alpha}_s = \alpha_s \quad (s = 2,3) \tag{276}$$

is performed in the integrals I, II, III, and IV. Noting that

$$\zeta_{i*} = V_i + \zeta_i,\qquad |\widetilde{V}_i| = |V_i|,\qquad \widetilde{\alpha}_i\widetilde{V}_i = \alpha_i V_i, \tag{277}$$

we can transform the integrals I, II, III, and IV in the following way, where the subscript s indicates $s = 2$ and 3:

$$\begin{aligned}\mathit{I}(\zeta_1,\zeta_s) &= \int \hat{f}(V_1+\zeta_1, V_s+\zeta_s)\hat{g}(\zeta_1,\zeta_s)\widehat{B}\,(|\alpha_i V_i|/|V_i|, |V_i|)\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V}\\ &= \int \hat{f}(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\hat{g}(\zeta_1,\zeta_s)\,\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}};\end{aligned} \tag{278a}$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in I, we have

$$\mathit{II}(\zeta_1,\zeta_s) = \int \hat{f}(\zeta_1,\zeta_s)\hat{g}(-\widetilde{V}_1+\zeta_1, \widetilde{V}_s+\zeta_s)\,\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}; \tag{278b}$$

[103]The separation is made only for convenience of explanation.

$$
\begin{aligned}
\mathit{III}(\zeta_1,\zeta_s) &= \int \hat{f}(V_i+\zeta_i-\alpha_j V_j\alpha_i)\hat{g}(\zeta_i+\alpha_j V_j\alpha_i)\widehat{B}\,(|\alpha_i V_i|/|V_i|,|V_i|)\mathrm{d}\Omega(\boldsymbol{\alpha})\,\mathbf{d}\boldsymbol{V} \\
&= \int \hat{f}(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad\times \hat{g}(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}};
\end{aligned}
\tag{278c}
$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in III, we have

$$
\begin{aligned}
\mathit{IV}(\zeta_1,\zeta_s) = \int & \hat{f}(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\times \hat{g}(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\times \widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}.
\end{aligned}
\tag{278d}
$$

Now we examine the parity of the integrals I, II, III, and IV with respect to ζ_1 on the basis of Eqs. (278a)–(278d). Here, we introduce the notation: (i) the parity of $\hat{f}$ (or $\hat{g}$) is indicated by the subscript attached to it, i.e., the subscript E is attached when it is even and the subscript O when it is odd; (ii) the first subscript of I, II, III, and IV indicates the parity of $\hat{f}$ in them and the second indicates the parity of $\hat{g}$. First, when $\hat{f}$ and $\hat{g}$ are even functions of ζ_1.

$$
\begin{aligned}
I_{EE}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_E(\zeta_1,\zeta_s) \\
&\qquad\times \widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= \int \hat{f}_E(\widetilde{V}_1-\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_E(-\zeta_1,\zeta_s) \\
&\qquad\times \widehat{B}\,(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= I_{EE}(-\zeta_1,\zeta_s),
\end{aligned}
\tag{279a}
$$

where the last relation holds owing to the first relation of Eq. (278a); Interchanging the arguments of $\hat{f}_E$ and $\hat{g}_E$ in I_{EE}, we have

$$
\mathit{II}_{EE}(\zeta_1,\zeta_s) = \mathit{II}_{EE}(-\zeta_1,\zeta_s);
\tag{279b}
$$

$$
\begin{aligned}
\mathit{III}_{EE}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad\times \hat{g}_E(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= \int \hat{f}_E(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
&\qquad\times \hat{g}_E(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
&= \mathit{III}_{EE}(-\zeta_1,\zeta_s);
\end{aligned}
\tag{279c}
$$

Interchanging the arguments of $\hat{f}_E$ and $\hat{g}_E$ in III_{EE}, we have

$$IV_{EE}(\zeta_1,\zeta_s) = IV_{EE}(-\zeta_1,\zeta_s). \tag{279d}$$

When both $\hat{f}$ and $\hat{g}$ are odd with respect to ζ_1,

$$\begin{aligned}
I_{OO}(\zeta_1,\zeta_s) &= \int \hat{f}_O(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_O(\zeta_1,\zeta_s)\,\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= \int \hat{f}_O(\widetilde{V}_1-\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_O(-\zeta_1,\zeta_s)\,\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= I_{OO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{280a}$$

Interchanging the arguments of $\hat{f}_O$ and $\hat{g}_O$ in II_{OO}, we have

$$II_{OO}(\zeta_1,\zeta_s) = II_{OO}(-\zeta_1,\zeta_s); \tag{280b}$$

$$\begin{aligned}
III_{OO}(\zeta_1,\zeta_s) &= \int \hat{f}_O(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\qquad\times \hat{g}_O(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\qquad\times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= \int \hat{f}_O(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\\
&\qquad\times \hat{g}_O(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1,\zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= III_{OO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{280c}$$

Interchanging the arguments of $\hat{f}$ and $\hat{g}$ in III_{OO}, we have

$$IV_{OO}(\zeta_1,\zeta_s) = IV_{OO}(-\zeta_1,\zeta_s). \tag{280d}$$

When $\hat{f}$ is even and $\hat{g}$ is odd with respect to ζ_1,

$$\begin{aligned}
I_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_O(\zeta_1,\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= -\int \hat{f}_E(\widetilde{V}_1-\zeta_1,\widetilde{V}_s+\zeta_s)\hat{g}_O(-\zeta_1,\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= -I_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{281a}$$

$$\begin{aligned}
II_{EO}(\zeta_1,\zeta_s) &= \int \hat{f}_E(\zeta_1,\zeta_s)\hat{g}_O(-\widetilde{V}_1+\zeta_1,\widetilde{V}_s+\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= -\int \hat{f}_E(-\zeta_1,\zeta_s)\hat{g}_O(\widetilde{V}_1-\zeta_1,\widetilde{V}_s+\zeta_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|,|\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}}\\
&= -II_{EO}(-\zeta_1,\zeta_s);
\end{aligned} \tag{281b}$$

$$
\begin{aligned}
\mathit{III}_{EO}(\zeta_1,\zeta_s) = \int & \hat{f}_E(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
& \times \hat{g}_O(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
= -\int & \hat{f}_E(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
& \times \hat{g}_O(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s)\widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\mathbf{d}\widetilde{\boldsymbol{V}} \\
= -\mathit{III}_{EO}&(-\zeta_1,\zeta_s); \qquad (281c)
\end{aligned}
$$

$$
\begin{aligned}
\mathit{IV}_{EO}(\zeta_1,\zeta_s) = \int & \hat{f}_E(\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
& \times \hat{g}_O(-\widetilde{V}_1+\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
& \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
= -\int & \hat{f}_E(-\zeta_1+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \zeta_s+\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
& \times \hat{g}_O(\widetilde{V}_1-\zeta_1-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_1, \widetilde{V}_s+\zeta_s-\widetilde{\alpha}_j\widetilde{V}_j\widetilde{\alpha}_s) \\
& \times \widehat{B}(|\widetilde{\alpha}_i\widetilde{V}_i|/|\widetilde{V}_i|, |\widetilde{V}_i|)\mathrm{d}\Omega(\widetilde{\boldsymbol{\alpha}})\,\mathbf{d}\widetilde{\boldsymbol{V}} \\
= -\mathit{IV}_{EO}&(-\zeta_1,\zeta_s). \qquad (281d)
\end{aligned}
$$

For I_{OE}, II_{OE}, III_{OE}, and IV_{OE}, interchanging the role of $\hat{f}$ and $\hat{g}$, respectively, in II_{EO}, I_{EO}, IV_{EO}, and III_{EO}, we have

$$I_{OE}(\zeta_1,\zeta_s) = -I_{OE}(\zeta_1,\zeta_s), \qquad (282a)$$
$$\mathit{II}_{OE}(\zeta_1,\zeta_s) = -\mathit{II}_{OE}(\zeta_1,\zeta_s), \qquad (282b)$$
$$\mathit{III}_{OE}(\zeta_1,\zeta_s) = -\mathit{III}_{OE}(\zeta_1,\zeta_s), \qquad (282c)$$
$$\mathit{IV}_{OE}(\zeta_1,\zeta_s) = -\mathit{IV}_{OE}(\zeta_1,\zeta_s). \qquad (282d)$$

The parity is common to I, II, III, and IV. Therefore, the parity of $\hat{J}(\hat{f},\hat{g})$ is the same as I, i.e.,

$$\hat{J}(\hat{f}_E,\hat{g}_E)(\zeta_1,\zeta_s) = \hat{J}(\hat{f}_E,\hat{g}_E)(-\zeta_1,\zeta_s), \qquad (283a)$$
$$\hat{J}(\hat{f}_O,\hat{g}_O)(\zeta_1,\zeta_s) = \hat{J}(\hat{f}_O,\hat{g}_O)(-\zeta_1,\zeta_s), \qquad (283b)$$
$$\hat{J}(\hat{f}_E,\hat{g}_O)(\zeta_1,\zeta_s) = -\hat{J}(\hat{f}_E,\hat{g}_O)(-\zeta_1,\zeta_s), \qquad (283c)$$
$$\hat{J}(\hat{f}_O,\hat{g}_E)(\zeta_1,\zeta_s) = -\hat{J}(\hat{f}_O,\hat{g}_E)(-\zeta_1,\zeta_s). \qquad (283d)$$

Obviously, the same parity holds for the other components, i.e., ζ_2, ζ_3, of $\boldsymbol{\zeta}$.
(Section 8.3: Version 4-00)

8.4 Supplement to Section M-A.10

8.4.1 On the equality condition of Eq. (M-A.266)

Here we will discuss the equality condition in the Darrozes–Guiraud inequality in Section M-A.10 in more detail. The equality in the Jensen inequality (M-A.265) is proved to hold when and only when ϕ is independent of $\boldsymbol{\xi}$ (see, e.g., Reference M-129). It should be noted that the uniqueness condition of the equality applies only to the region of $\boldsymbol{\xi}$ where $\psi > 0$ and that no condition is required of ϕ where $\psi = 0$. Choose a $\boldsymbol{\xi}$ in $(\xi_i - v_{wi})n_i > 0$, and consider the condition for equality in Eq. (M-A.266). According to the above note, the equality holds only when $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is a constant (say, c_1) in the region D_1 of $\boldsymbol{\xi}_*$, joint or disjoint, where $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$. If we choose another $\boldsymbol{\xi}$, $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ in a different range D_2 of $\boldsymbol{\xi}_*$, and $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*) = c_2$ (c_2 : const) is required in D_2. The constants c_1 and c_2 may be different if D_1 and D_2 are disjoint. The two constants are required to be the same ($c_1 = c_2$), if D_1 and D_2 overlap for some range of $\boldsymbol{\xi}_*$ (their intersection is neither empty nor measure zero).[104] From the condition (M-1.27b), there is a region of $\boldsymbol{\xi}$ where $K_B > 0$ for any $\boldsymbol{\xi}_*$ in $(\xi_{i*} - v_{wi})n_i < 0$. Thus, the collection of the regions of $\boldsymbol{\xi}_*$ where $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ with respect to all $\boldsymbol{\xi}$ in $(\xi_i - v_{wi})n_i > 0$ covers $(\xi_{i*} - v_{wi})n_i < 0$. If K_B is such a kernel that the series of the ranges $\boldsymbol{\xi}_*$ of different $\boldsymbol{\xi}$ constituting the above collection overlap with nonzero measure at the intersecting points, the constant is unique over $(\xi_{i*} - v_{wi})n_i < 0$, i.e., $f(\boldsymbol{\xi}_*) = c_0 f_0(\boldsymbol{\xi}_*)$ (c_0 : a constant) in $(\xi_{i*} - v_{wi})n_i < 0$ (see Fig. 1).[105] Then, from the condition (M-1.27c),

$$f(\boldsymbol{\xi}) = c_0 f_0(\boldsymbol{\xi}) \quad \text{for all } \boldsymbol{\xi}. \tag{284}$$

Incidentally, the kernel K_B that is positive almost everywhere (Footnote M-5 in Section M-1.2) is classified as positive, and Eq. (284) holds almost everywhere of $\boldsymbol{\xi}$. When the overlap-covering condition is not satisfied, the above Maxwellian is not necessarily required for the equality.[106]

The equality condition of Eq. (M-A.267) is seen to be the same as that of Eq. (M-A.266) in the following way. Obviously, $B = A \Leftrightarrow \int_V a(\boldsymbol{\xi})[B(\boldsymbol{\xi}) - A(\boldsymbol{\xi})]\mathbf{d}\boldsymbol{\xi} = 0$ if $A(\boldsymbol{\xi}) \leq B(\boldsymbol{\xi})$ and $a(\boldsymbol{\xi}) > 0$. Taking

$$A(\boldsymbol{\xi}) = F\left(\frac{f(\boldsymbol{\xi})}{f_0(\boldsymbol{\xi})}\right), \qquad B(\boldsymbol{\xi}) = \int_{(\xi_{i*} - v_{wi})n_i < 0} \frac{K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f_0(\boldsymbol{\xi}_*)}{f_0(\boldsymbol{\xi})} F\left(\frac{f(\boldsymbol{\xi}_*)}{f_0(\boldsymbol{\xi}_*)}\right) \mathbf{d}\boldsymbol{\xi}_*,$$

[104](i) In the common region, $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ cannot take two values. On a set with measure zero, whether $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is determined or not can be ignored. (See Footnote M-5 in Section M-1.2 for the set with measure zero.)

(ii) If the intersection is empty or measure zero, the integrations with respect to $\boldsymbol{\xi}_*$ at different $\boldsymbol{\xi}$'s, are not influenced by the $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ determined by the other $\boldsymbol{\xi}$.

(iii) The equality only on a set of $\boldsymbol{\xi}$ with measure zero is ignored. Thus, the above set of $\boldsymbol{\xi}_*$ where $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is constant is required to have some extent with measure nonzero with respect to $\boldsymbol{\xi}$ including the intersections.

[105]The collection has to have some extent mentioned in Footnote 104 (iii).

[106]In fact, Takata (private communication) constructed a kernel K_B, which is zero in $[(\xi_i - v_{wi})n_i - C][(\xi_{i*} - v_{wi})n_i + C_*] > 0$ (C and C_*: some positive constants) and satisfies the conditions (M-1.27a)–(M-1.27c), for which the equality holds for another function.

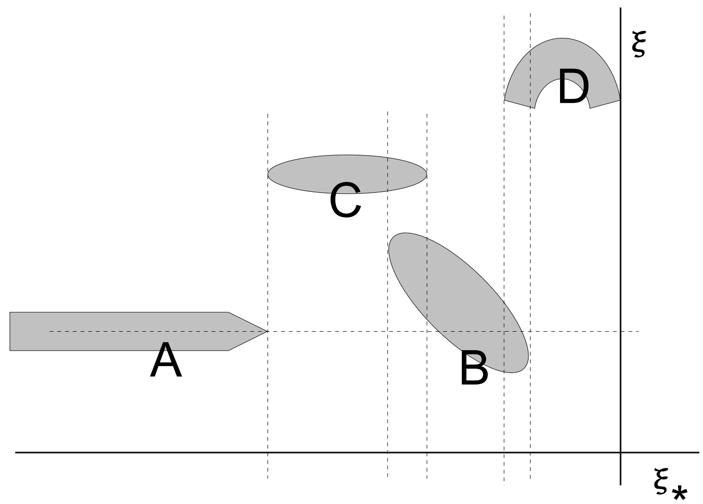

Figure 1: $K_B(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ that requires $f(\boldsymbol{\xi}) = c_0 f_0(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$. The quarter in the figure is the range $(\xi_{i*} - v_{wi})n_i < 0$ and $(\xi_i - v_{wi})n_i > 0$ in the space $(\boldsymbol{\xi_*}, \boldsymbol{\xi})$. Let $K_B > 0$ in the regions A, B, C, and D at least, and their ranges of $\boldsymbol{\xi}_*$ cover $(\xi_{i*} - v_{wi})n_i < 0$. Then, $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*)$ is constant in each of A, B, C, and D (say, a in A, b in B, c in C, and d in D). Some ranges in A and B being on a common $\boldsymbol{\xi}$ having some extent, $a = b$. In view of the intersection of the ranges of $\boldsymbol{\xi}_*$ of B and C and that of B and D, $c = b\,(= a)$, and $d = b\,(= a)$. Thus, $f(\boldsymbol{\xi}_*)/f_0(\boldsymbol{\xi}_*) = a$ in $(\xi_{i*} - v_{wi})n_i < 0$. It may be noted that the regions of $\boldsymbol{\xi}_*$ of A and C are required to be only in contact with each other because the intersection of the ranges of $\boldsymbol{\xi}_*$ of C and B is not measure zero.

and $(\xi_i - v_{wi})n_i > 0$ as the domain V of integration, and comparing Eq. (M-A.266) and its next equation without number, we find the equivalence of the equality conditions of Eqs. (M-A.266) and (M-A.267). The above discussion being common for a strictly convex function F, the equality condition applies to the Darrozes–Guiraud inequality (M-A.262) and Eq. (M-A.268).

(Section 8.4.1: Version 5-00)

8.4.2 Extension of the Darrozes–Guiraud inequality to an interface

Darrozes–Guiraud inequality (M-A.262) or (M-A.267) is proved for a function f satisfying the boundary condition (M-1.26) on a simple boundary (M-Darrozes & Guiraud [1966]). Here, we discuss its extension to f that satisfies the boundary condition (M-1.30) on an interface of a gas and its condensed phase.

The boundary condition on the interface is given as[107]

$$f(\boldsymbol{\xi}) = g_I(\boldsymbol{\xi}) + \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0], \tag{285}$$

where g_I and K_I are independent of f. Further, g_I and K_I satisfy the following conditions [see Eqs. (M-1.31a)–(M-1.31c)]:

(i) Nonnegativity of g_I

$$g_I(\boldsymbol{\xi}) \geq 0 \qquad [(\xi_i - v_{wi})n_i > 0]. \tag{286a}$$

(ii) Nonnegativity of K_I

$$K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*) \geq 0 \qquad [(\xi_i - v_{wi})n_i > 0, \quad (\xi_{i*} - v_{wi})n_i < 0]. \tag{286b}$$

(iii) Condition of establishment of the equilibrium state

$$f_w(\boldsymbol{\xi}) = g_I(\boldsymbol{\xi}) + \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0], \tag{286c}$$

where f_w is the Maxwellian determined by the temperature T_w and velocity v_{wi} of the interface and the saturation gas density ρ_w at temperature T_w i.e.,

$$f_w(\boldsymbol{\xi}) = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_{wi})^2}{2RT_w}\right). \tag{287}$$

It is also required here that if $f(\boldsymbol{\xi}_*)$ for $(\xi_{i*} - v_{wi})n_i < 0$ is the corresponding part of another Maxwellian [say, $f_e(\boldsymbol{\xi})$], it does not give $f_e(\boldsymbol{\xi})$ for $(\xi_i - v_{wi})n_i > 0$, which will be called the uniqueness condition of Eq. (286c) for shortness.

In the following discussion, we impose another condition in addition to Eqs. (286a)–(286c), i.e., putting

$$\alpha(\boldsymbol{\xi}_*) = -\int_{(\xi_i-v_{wi})n_i>0} \frac{(\xi_i - v_{wi})n_i}{(\xi_{j*} - v_{wj})n_j} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi} \quad [(\xi_{j*} - v_{wj})n_j < 0], \tag{288}$$

[107] The variables $\boldsymbol{X}$ and t are not shown here because they are not important in the present discussion [see Footnote M-10 (ii) in Section M-1.5].

we assume[108] that

$$0 \leq \alpha(\boldsymbol{\xi}_*) \leq 1 \qquad [(\xi_{i*} - v_{wi})n_i < 0]. \tag{289}$$

Incidentally, from Eqs. (286a)–(286c),

$$f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) \geq 0. \tag{290}$$

We will show that the inequality (M-A.267) with f_0 being replaced by f_w, i.e.,

$$\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) F[f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]\mathbf{d}\boldsymbol{\xi} \leq 0, \tag{291}$$

holds when $F(x)$ is such a strictly convex function (see Footnote M-52 in Section M-A.10) that

$$F(x) \geq 0 \text{ and } F(1) = 0.$$

The equality of the relation (291) holds when $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$, and this relation is required except for some boundary conditions shown later. The inequality is proved with the aid of the Jensen inequality (see Eq. (M-A.265) or M-Jensen [1906], M-Lieb & Moss [2001], M-Parzen [1960], or M-Rudin [1976])

$$F\left(\int \phi\psi\mathbf{d}\boldsymbol{\xi} \bigg/ \int \psi\mathbf{d}\boldsymbol{\xi}\right) \leq \int \psi F(\phi)\mathbf{d}\boldsymbol{\xi} \bigg/ \int \psi\mathbf{d}\boldsymbol{\xi} \qquad (\psi \geq 0), \tag{292}$$

where $F(x)$ is a strictly convex function, and ϕ and ψ ($\psi \geq 0$) are arbitrary functions of $\boldsymbol{\xi}$. The equality sign holds when ϕ is independent of $\boldsymbol{\xi}$; it is also required where $\psi > 0$ for the equality.

[108](i) This condition corresponds to Eq. (M-1.27b) for a simple boundary. The simple boundary consists of molecules of different kinds from the gas molecules, and they stay there forever. The gas molecules impinging on the boundary are reflected without time delay (in the time scale of our interest), and there is no net mass flux to the boundary in this process. The condition (M-1.27b) is derived from this situation, as explained in Footnote M-13 in Section M-1.6.1. In the case of an interface, the condition (289) is derived similarly, if we consider that some of the molecules impinging on the interface do not reflect and stay there. However, the interface is the condensed phase of the gas and consists of the same kind of molecules as the gas. On the interface, molecules leave it depending on the condition of the interface even if there is no impinging molecules; this is the g_I part in Eq. (285). When a molecule impinges on the interface, it interacts with molecules of the interface, and some molecules leave the interface. Whether the impinging molecule is reflected or kicks out another molecule has no difference. Further, depending on the condition (e.g., speed or direction) of the impinging molecule and that of the interface, more than one molecule may be kicked out or no molecule may be kicked out or reflected. Thus, it is not clear that the condition (289) holds or not. However, it is sure that the size of the kernel K_I is limited owing to the conditions (286a)–(286c), e.g., $K_I = 0$ if $g_I = f_w$ (the complete condensation). See also Footnote 111 in Section 8.4.2.

(ii-a) The case $\alpha(\boldsymbol{\xi}_*) = 1$ for $(\xi_{j*} - v_{wj})n_j < 0$ is excluded by the uniqueness condition of Eq. (286c). In fact, multiplying Eq. (285) by $(\xi_j - v_{wj})n_j$ and integrating with respect to $\boldsymbol{\xi}$ over $(\xi_j - v_{wj})n_j > 0$, we obtain $g_I(\xi) = 0$. Thus, Cf_w (C : a constant) also satisfies Eq. (285).

(ii-b) When $\alpha(\boldsymbol{\xi}_*) = 0$ for $(\xi_{j*} - v_{wj})n_j < 0$, the kernel $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ degenerates, i.e., $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) = 0$ for $(\xi_j - v_{wj})n_j > 0$. This is the case of the complete condensation.

Let $F(x)$ be a nonnegative strictly convex function that takes value zero at $x=1$,[109] i.e.,

$$F(x) \geq 0, \quad F(1) = 0. \tag{293}$$

Consider the function $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$, where $f_w(\boldsymbol{\xi})$ is given by Eq. (287). The function $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$ for $(\xi_i - v_{wi})n_i > 0$ is bounded by an integral of $f(\boldsymbol{\xi})$ for $(\xi_i - v_{wi})n_i < 0$ with the aid of Eq. (285) in the following way:

$$\begin{aligned}
F\left(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right) &= F\left(\frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} + \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} f(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_*\right) \\
&= F\left[\frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} + \left(1 - \frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)\int_{<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\right] \\
&\leq \frac{g_I}{f_w}F(1) + \left(1 - \frac{g_I}{f_w}\right) F\left(\int_{<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\right) \\
&= \left(1 - \frac{g_I(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right) F\left(\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\mathbf{d}\boldsymbol{\xi}_*\right) \\
&\qquad\qquad [(\xi_i - v_{wi})n_i > 0].
\end{aligned} \tag{294}$$

Here, we, for a moment, consider the point of $\boldsymbol{\xi}$ $[(\xi_i - v_{wi})n_i > 0]$ where

$$f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) > 0,$$

for which

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})}\mathbf{d}\boldsymbol{\xi}_* = 1 \qquad [(\xi_i - v_{wi})n_i > 0],$$

because of Eq. (286c); in the second and third lines, the simple $<$ sign of the subscript of the integral sign $\int$ indicates $(\xi_{i*} - v_{wi})n_i < 0$; the convex property of $F(x)$ is used from the second line to the third, and $F(1) = 0$ is used from the third to the fourth.

Now, we apply the Jensen inequality (292) to the function F on the fourth line in Eq. (294). Here, we choose $\phi(\boldsymbol{\xi}_*)$ and $\psi(\boldsymbol{\xi}_*)$ as

$$\begin{aligned}
\phi(\boldsymbol{\xi}_*) &= \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}, \\
\psi(\boldsymbol{\xi}_*) &= \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \geq 0 \quad [(\xi_i - v_{wi})n_i > 0, \quad (\xi_{i*} - v_{wi})n_i < 0].
\end{aligned}$$

It should be noted that $\phi(\boldsymbol{\xi}_*)$ is defined for the whole range of $\boldsymbol{\xi}_*$ and that $\psi(\boldsymbol{\xi}_*)$ depends also on $\boldsymbol{\xi}$ and satisfies the relation, irrespective of $\boldsymbol{\xi}$,

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \psi(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* = 1 \qquad [(\xi_i - v_{wi})n_i > 0].$$

[109] Note that $x = 1$ is the unique zero point of $F(x)$.

Then, $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$ for $(\xi_i - v_{wi})n_i > 0$ is bounded as

$$F\bigg(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\bigg) \leq \bigg(1 - \frac{g_I}{f_w}\bigg) F\bigg(\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})]f_w(\boldsymbol{\xi})} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)} \mathbf{d}\boldsymbol{\xi}_*\bigg)$$
$$\leq \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} F\bigg(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\bigg) \mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0]. \tag{295}$$

Up to this point, we limited our discussion to the point of $\boldsymbol{\xi}$ $[(\xi_i - v_{wi})n_i > 0]$ where

$$f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) > 0.$$

If it vanishes at some $\boldsymbol{\xi}_A$ $[(\xi_{iA} - v_{wi})n_i > 0]$, i.e.,

$$f_w(\boldsymbol{\xi}_A) - g_I(\boldsymbol{\xi}_A) = 0, \tag{296}$$

the integral $\int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_*$ vanishes there, i.e.,

$$\int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* = 0,$$

because of the condition (286c). The function $f_w(\boldsymbol{\xi}_*)$ being positive for all $\boldsymbol{\xi}_*$, the kernel $K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*)$ must vanish for $(\xi_{i*} - v_{wi})n_i < 0$, i.e.,

$$K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*) = 0 \qquad [(\xi_{i*} - v_{wi})n_i < 0]. \tag{297}$$

Thus, from the boundary condition (285),

$$f(\boldsymbol{\xi}_A) = g_I(\boldsymbol{\xi}_A) = f_w(\boldsymbol{\xi}_A).$$

Therefore, the function $F(f(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A))$ vanishes, i.e.,

$$F(f(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A)) = F(1) = 0. \tag{298}$$

From Eqs. (297) and (298), the equality holds between the left-most side and the right-most of Eq. (295) at $\boldsymbol{\xi} = \boldsymbol{\xi}_A$. In conclusion, the inequality

$$F\bigg(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\bigg) \leq \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} F\bigg(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\bigg) \mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0], \tag{299}$$

holds without the assumption of $f_w(\boldsymbol{\xi}) - g_I(\boldsymbol{\xi}) > 0$.

When $f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}) = 1$ for all $\boldsymbol{\xi}$, $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}))$ vanishes in Eq. (299), and the equality holds there. We look for the other possibilities of the equality. The first inequality in Eq. (295) comes from that of Eq. (294), for which the equality holds at $\boldsymbol{\xi} = \boldsymbol{\xi}_A$ when (i) $g_I(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A) = 0$ or (ii) $g_I(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A) = 1$, or (iii) the arguments of two F's on the third line of Eq. (294) are equal, i.e.,

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi}_A,\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{[1-g_I(\boldsymbol{\xi}_A)/f_w(\boldsymbol{\xi}_A)]f_w(\boldsymbol{\xi}_A)} \frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)} \mathbf{d}\boldsymbol{\xi}_* = 1, \tag{300}$$

for some $f(\boldsymbol{\xi}_*)$. In the third case, the equality relation being imposed between the first and the second line on the right-hand side of Eq. (295) under the condition (300), we find that

$$f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*) \text{ in } \mathrm{B_A}(\boldsymbol{\xi}_*),$$

where $\mathrm{B}_A(\boldsymbol{\xi}_*)$ is the region of $\boldsymbol{\xi}_*$ in which $K_I(\boldsymbol{\xi}_A, \boldsymbol{\xi}_*) > 0$.

If $g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}) = 0$ for $(\xi_i - v_{wi})n_i > 0$, the boundary condition (285) reduces to

$$f(\boldsymbol{\xi}) = \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(\boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_*. \tag{301}$$

Then, the Maxwellian $a_0 f_w(\boldsymbol{\xi})$ (a_0 : a constant) also satisfies the boundary condition (285), which is not allowed by the uniqueness condition of Eq. (286c). Thus, this case is excluded. If $g_I(\boldsymbol{\xi})/f_w(\boldsymbol{\xi}) = 1$ for $(\xi_i - v_{wi})n_i > 0$, the kernel $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ vanishes for $(\xi_i - v_{wi})n_i > 0$ and $(\xi_{i*} - v_{wi})n_i < 0$ from the discussion in the preceding paragraph. That is, $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ in $(\xi_i - v_{wi})n_i > 0$ irrespective of $f(\boldsymbol{\xi})$ in $(\xi_i - v_{wi})n_i < 0$ (this is the case of the complete condensation condition). For this case the equality holds in Eq. (299). If the third condition holds for $(\xi_i - v_{wi})n_i > 0$, we have

$$f_w(\boldsymbol{\xi}) = g_I(\boldsymbol{\xi}) + \int_{(\xi_{i*}-v_{wi})n_i<0} K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(\boldsymbol{\xi}_*) \mathbf{d}\boldsymbol{\xi}_* \quad [(\xi_i - v_{wi})n_i > 0]. \tag{302}$$

From the discussion of the preceding paragraph,

$$f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*) \text{ in } \mathrm{B}(\boldsymbol{\xi}_*), \tag{303}$$

where $\mathrm{B}(\boldsymbol{\xi}_*)$ is the region of $\boldsymbol{\xi}_*$ in which $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ for some $\boldsymbol{\xi}$. This condition is paraphrased as

$$f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*) \text{ except in the region } \alpha(\boldsymbol{\xi}_*) = 0. \tag{304}$$

Whether $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ or $\alpha(\boldsymbol{\xi}_*) = 0$ in $(\xi_{i*} - v_{wi})n_i < 0$,

$$f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi}) \ \ [(\xi_i - v_{wi})n_i > 0].$$

Let us consider the case where the three situations (i), (ii), and (iii) listed just before Eq. (300) take place for different $\boldsymbol{\xi}$, say, (i) for $\boldsymbol{\xi}$ in A_1, (ii) for $\boldsymbol{\xi}$ in A_2, and (iii) for $\boldsymbol{\xi}$ in A_3. The A_2 part does not contribute to the restriction on $f(\boldsymbol{\xi}_*)$. When A_1 is empty, the condition is the same as for the case of Eq. (302), i.e., Eq. (303) or (304). When A_1 is not empty, from the discussion for $\boldsymbol{\xi}$ in A_3, $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ in the region of $\boldsymbol{\xi}_*$ where $K_I(\boldsymbol{\xi}, \boldsymbol{\xi}_*) > 0$ for some $\boldsymbol{\xi}$ in A_3 [say, $\mathrm{B}_3(\boldsymbol{\xi}_*)$], and the condition for the remaining $\boldsymbol{\xi}_*$ is determined only by the behavior of K_I for $\boldsymbol{\xi}$ in A_1, that is, the region $f(\boldsymbol{\xi}_*)/f_w(\boldsymbol{\xi}_*) = \text{const}$ [say, $\mathrm{B}_1(\boldsymbol{\xi}_*)$] is looked for in the range $(\xi_{i*} - v_{wi})n_i < 0$ in the same way as in Section 8.4.1 and if B_1 has a common region with B_3, $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ in B_1. In the region of the remaining $\boldsymbol{\xi}_*$ [say, $\mathrm{R}(\boldsymbol{\xi}_*)$], $f(\boldsymbol{\xi}_*)$ other than $f_w(\boldsymbol{\xi}_*)$ can exist. The region $\alpha(\boldsymbol{\xi}_*) = 1$ in $\mathrm{R}(\boldsymbol{\xi}_*)$ is denoted by $\mathrm{R}_{\alpha=1}$ for the convenience in the later citation.

When A_3 is empty, the boundary condition (285) is expressed as

$$f(\boldsymbol{\xi}) = \begin{pmatrix} 0 \\ f_w(\boldsymbol{\xi}) \end{pmatrix} + \int_{(\xi_{i*}-v_{wi})n_i<0} \begin{pmatrix} K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*) \\ 0 \end{pmatrix} f(\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}_* \quad \begin{matrix} [\boldsymbol{\xi} \text{ in } \mathrm{A}_1] \\ [\boldsymbol{\xi} \text{ in } \mathrm{A}_2] \end{matrix}, \tag{305}$$

where

$$\int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})}\mathbf{d}\boldsymbol{\xi}_* = 1 \qquad [(\xi_i - v_{wi})n_i > 0 \text{ and } \boldsymbol{\xi} \text{ in } \mathrm{A}_1].$$

The boundary condition (305) obviously satisfies the conditions (286a)–(286c).[110] In this case, the restriction on $f(\boldsymbol{\xi}_*)$ is determined by K_I in A_1. Substituting $f(\boldsymbol{\xi}_*) = C_D f_w(\boldsymbol{\xi}_*)$ $[(\xi_{i*} - v_{wi})n_i < 0,\ C_D$: independent of $\boldsymbol{\xi}_*]$, which is the strongest restriction on $f(\boldsymbol{\xi}_*)$, into Eq. (305), we have $f(\boldsymbol{\xi}) = C_D f_w(\boldsymbol{\xi})$ [in A_1] and $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ [in A_2] for $(\xi_i - v_{wi})n_i > 0$. For this $f(\boldsymbol{\xi})$, the equality holds in Eq. (299). Thus, for the boundary condition (305) as well as the complete condensation condition, the equality in Eq. (299) holds for $f(\boldsymbol{\xi})$ other than $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ [$f(\boldsymbol{\xi}_*) = C_D f_w(\boldsymbol{\xi}_*)$ for $(\xi_{i*} - v_{wi})n_i < 0$ for Eq. (305), and $f(\boldsymbol{\xi}_*)$ is arbitrary for $(\xi_{i*} - v_{wi})n_i < 0$ for the complete condensation]. This is an example of $f(\boldsymbol{\xi}_*)$ that satisfies the equality in Eq. (299).

With the aid of the inequality (299) and Eq. (288), we have

$$\begin{aligned}
&\int_{(\xi_i-v_{wi})n_i>0} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) F\left(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)\mathbf{d}\boldsymbol{\xi} \\
&\le \int_{(\xi_i-v_{wi})n_i>0} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) \int_{(\xi_{i*}-v_{wi})n_i<0} \frac{K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)f_w(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi})} F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right)\mathbf{d}\boldsymbol{\xi}_*\mathbf{d}\boldsymbol{\xi} \\
&= \int_{(\xi_{i*}-v_{wi})n_i<0} f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right) \int_{(\xi_i-v_{wi})n_i>0} (\xi_i - v_{wi})n_i K_I(\boldsymbol{\xi},\boldsymbol{\xi}_*)\mathbf{d}\boldsymbol{\xi}\mathbf{d}\boldsymbol{\xi}_* \\
&= -\int_{(\xi_{i*}-v_{wi})n_i<0} \alpha(\boldsymbol{\xi}_*)(\xi_{i*} - v_{wi})n_i f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right)\mathbf{d}\boldsymbol{\xi}_*,
\end{aligned} \tag{306}$$

where $0 \le \alpha(\boldsymbol{\xi}_*) \le 1$ [the assumption (289)]. Thus, we obtain the extension of Eq. (M-A.267) to the case of an interface as follows:

$$\begin{aligned}
&\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f_w(\boldsymbol{\xi}) F\left(\frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})}\right)\mathbf{d}\boldsymbol{\xi} \\
&\qquad \le \int_{(\xi_{i*}-v_{wi})n_i<0} [1-\alpha(\boldsymbol{\xi}_*)](\xi_{i*} - v_{wi})n_i f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)}\right)\mathbf{d}\boldsymbol{\xi}_* \le 0.
\end{aligned} \tag{307}$$

Obviously, the equal sign holds in the two inequalities of Eq. (307) when $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$. Conversely, it is required for the equal sign to hold in the inequalities

[110] To confirm the uniqueness condition of Eq. (286c) is simple. Note $f(\boldsymbol{\xi})$ $[(\xi_i - v_{wi})n_i > 0]$ for $\boldsymbol{\xi}$ in A_2.

that $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$ when $\mathrm{R}_{\alpha=1}$ is empty.[111] It should be noted that $F(x)$ is required to satisfy that $F(x) \geq 0$ and $F(1) = 0$ in addition to convexity. Here, we take

$$F(x) = x(\ln x - 1) + 1,$$

which is strictly convex, nonnegative, and zero at $x = 1$. Then,

$$\int_{\text{all }\boldsymbol{\xi}} (\xi_i - v_{wi})n_i \left[f(\boldsymbol{\xi}) \left(\ln \frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} - 1 \right) + f_w(\boldsymbol{\xi}) \right] \mathbf{d}\boldsymbol{\xi} \leq 0,$$

or

$$\int_{\text{all }\boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln \frac{f(\boldsymbol{\xi})}{f_w(\boldsymbol{\xi})} \mathbf{d}\boldsymbol{\xi} \leq \rho(v_i - v_{wi})n_i. \tag{308}$$

This is the extension of Eq. (M-A.262) for a simple boundary to an interface.

We try to express the inequality (308) in terms of macroscopic variables. It is simply transformed in the following form:

$$\begin{aligned}
&\int_{\text{all }\boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln \frac{f(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi} \\
&\leq \int_{\text{all }\boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln \frac{f_w(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi} + \rho(v_i - v_{wi})n_i \\
&= -\frac{1}{RT_w} \left[q_i n_i + (v_j - v_{wj})\tilde{p}_{ij} n_i + \rho(v_i - v_{wi})n_i \left(\frac{5}{2}RT + \frac{1}{2}(v_j - v_{wj})^2 \right) \right] \\
&\quad + \rho(v_i - v_{wi})n_i \left(\ln \frac{\rho_w}{(2\pi RT_w)^{3/2} c_0} + 1 \right),
\end{aligned}$$

where c_0 is a constant to make the argument of the logarithmic function dimensionless, and

$$\tilde{p}_{ij} = p_{ij} - p\delta_{ij}, \tag{309}$$

[111] (i) The integration of a nonnegative function multiplied by a positive function does not change the equality condition. Thus, the equality condition of the inequality of Eq. (306) is the same as that of Eq. (299) $[B = A \Leftrightarrow \int a(\boldsymbol{\xi})[B(\boldsymbol{\xi}) - A(\boldsymbol{\xi})]\mathbf{d}\boldsymbol{\xi} = 0$ if $A(\boldsymbol{\xi}) \leq B(\boldsymbol{\xi})$ and $a(\boldsymbol{\xi}) > 0]$. Thus, the range where $f(\boldsymbol{\xi}_*) = f_w(\boldsymbol{\xi}_*)$ is required is outside R. For the equality of the Darrozes–Guiraud inequality, we have to examine the equality of the second inequality in Eq. (307). The second equal sign holds only when $F(f(\boldsymbol{\xi}_*)/f_w(\boldsymbol{\xi}_*)) = 0$ in R outside $\mathrm{R}_{\alpha=1}$ because $f_w(\boldsymbol{\xi}_*) > 0$ and $1 - \alpha(\boldsymbol{\xi}_*) > 0$ there. Thus, $f(\boldsymbol{\xi}_*)/f_w(\boldsymbol{\xi}_*) = 1$ outside $\mathrm{R}_{\alpha=1}$ in $(\xi_{i*} - v_{wi})n_i < 0$ (see Footnote 109 in Section 8.4.2). When $\mathrm{R}_{\alpha=1}$ is empty, the integral $\int_{\text{all }\boldsymbol{\xi}}$ on the left-most side reduces to $\int_{(\xi_i - v_{wi})n_i > 0}$. This vanishes only when $F(f(\boldsymbol{\xi})/f_w(\boldsymbol{\xi})) = 0$, i.e., $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for $(\xi_i - v_{wi})n_i > 0$. Thus, $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$ when $\mathrm{R}_{\alpha=1}$ is empty. It may be noted that when A_3 is empty [or for the boundary condition (305)], $\mathrm{R}_{\alpha=1}$ is the range of $\boldsymbol{\xi}_*$ where $\alpha(\boldsymbol{\xi}_*) = 1$ in $(\xi_{i*} - v_{wi})n_i < 0$. Incidentally, $g_I(\boldsymbol{\xi})$ that is positive almost everywhere (Footnote M-5 in Section M-1.2) is classified positive, for which A_1 in the paragraph following to that of Eq. (304) is empty and Eq. (304) holds (that is, $\mathrm{R}_{\alpha=1}$ is empty), and therefore the equal signs hold in Eq. (307) only when $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$.

(ii) If $\alpha(\boldsymbol{\xi}_*)$ exceeds unity for some range of $\boldsymbol{\xi}_*$ in $(\xi_{i*} - v_{wi})n_i < 0$ and the assumption (289) is violated, but the integral

$$\int_{(\xi_{i*} - v_{wi})n_i < 0} [1 - \alpha(\boldsymbol{\xi}_*)](\xi_{i*} - v_{wi})n_i f_w(\boldsymbol{\xi}_*) F\left(\frac{f(\boldsymbol{\xi}_*)}{f_w(\boldsymbol{\xi}_*)} \right) \mathbf{d}\boldsymbol{\xi}_*$$

is nonpositive, the inequality holds.

The $\tilde{p}_{ij}$ is the part of stress tensor with the pressure contribution subtracted. Only the tangential component of the stress $\tilde{p}_{ij}n_i$ contributes to $(v_j - v_{wj})\tilde{p}_{ij}n_i$ when no flow to the boundary. Further, $\ln \rho_w/(2\pi RT_w)^{3/2}c_0$ is related to the H function H_w for $f(\boldsymbol{\xi}) = f_w(\boldsymbol{\xi})$ as

$$\frac{H_w}{\rho_w} = \ln \frac{\rho_w}{(2\pi RT_w)^{3/2}c_0} - \frac{3}{2}, \tag{310}$$

which is independent of v_{wi}. That is,

$$H_w = \int_{\text{all } \boldsymbol{\xi}} f_w(\boldsymbol{\xi}) \ln \frac{f_w(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi} = \int_{\text{all } \boldsymbol{\xi}} f_w^{(v)}(\boldsymbol{\xi}) \ln \frac{f_w^{(v)}(\boldsymbol{\xi})}{c_0} \mathbf{d}\boldsymbol{\xi},$$

where

$$f_w^{(v)}(\boldsymbol{\xi}) = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_i)^2}{2RT_w}\right).$$

On the other hand, by definition (see Section M-1.7),

$$\int_{\text{all } \boldsymbol{\xi}} (\xi_i - v_{wi})n_i f(\boldsymbol{\xi}) \ln[f(\boldsymbol{\xi})/c_0]\mathbf{d}\boldsymbol{\xi} = (H_i - Hv_{wi})n_i.$$

Therefore,

$$\begin{aligned}(H_i &- Hv_{wi})n_i \\ &\leq -\frac{1}{RT_w}\left[q_i n_i + (v_j - v_{wj})\tilde{p}_{ij}n_i\right] \\ &\quad + \rho(v_i - v_{wi})n_i \left[\frac{H_w}{\rho_w} - \frac{1}{RT_w}\left(\frac{5}{2}R(T - T_w) + \frac{1}{2}(v_j - v_{wj})^2\right)\right]. \end{aligned} \tag{311}$$

When $f = f_w$, both sides of the inequality vanish and the equal sign holds. Conversely, for the kernel K_I with $R_{\alpha=1}$ empty, e.g., g_I that is positive almost everywhere, the equal sign holds only when $f = f_w$.

Finally, we consider the variation of the integral $\overline{H}$ of H over the domain D. According to Eq. (M-1.36),

$$\frac{\mathrm{d}\overline{H}}{\mathrm{d}t} = \int_{\partial D} (H_i - Hv_{wi})n_i + \int_D G\boldsymbol{dX},$$

where

$$\overline{H} = \int_D H\boldsymbol{dX}.$$

With the aid of Eq. (311), the variation is bounded as

$$\begin{aligned}\frac{\mathrm{d}\overline{H}}{\mathrm{d}t} &\leq -\frac{1}{RT_w}\left[q_i n_i + (v_j - v_{wj})\tilde{p}_{ij}n_i\right] \\ &\quad + \rho(v_i - v_{wi})n_i \left[\frac{H_w}{\rho_w} - \frac{1}{RT_w}\left(\frac{5}{2}R(T - T_w) + \frac{1}{2}(v_j - v_{wj})^2\right)\right], \end{aligned} \tag{312}$$

because $\int_D G\boldsymbol{dX} \leq 0$ [see Eq. (M-1.34b)].

(Section 8.4.2: Version 5-00)

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