Supplement to Kinetic Theory and Fluid Dynamics

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(Birkhäuser, Boston, 2002)

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Bibliography

Update of bibliography

- [161] Sone, Y., M. Handa, and H. Sugimoto [2002]:
  (to be published) → 299–332
- p. 151, the eighth line from the bottom:
  Sone, Handa & Doi [unpublished] → Sone, Handa & Doi [2003]
  The following reference is to be added to Bibliography:
Sone, Y., M. Handa, and T. Doi (2003), Ghost effect and bifurcation in a gas between coaxial circular cylinder with different temperatures, *Phys. Fluids* **15**, 2903–2915.
- p. 159, the sixth line
  Sone & Doi [unpublished] → Sone & Doi [2003]
  The following reference is to be added to Bibliography:
Sone, Y. and T. Doi (2003), Bifurcation of and ghost effect on temperature field in the Bénard problem of a gas in the continuum limit, *Phys. Fluids* **15**, 1405–1423.

Errata

- p. 11, the 10th line

specular condition $\rightarrow$ specular reflection

- p. 33, the first line in Eq. (3.31c):

$$\frac{\partial S_{ijG0}}{\partial x_i} \quad \rightarrow \quad \frac{\partial S_{ijG0}}{\partial x_j}$$

- p. 33, the first line in Eq. (3.31d):

$$\frac{\partial S_{ijG1}}{\partial x_i} \quad \rightarrow \quad \frac{\partial S_{ijG1}}{\partial x_j}$$

- p. 36, Footnote 7:

Footnote 7 should be replaced by the following statement:

In some cases, the boundary data u_{wi}, τ_w, or ω_w (thus, P_w) are unknown beforehand and may depend on the Knudsen number, for example, the surface temperature of a particle set freely in a gas. To include the case, the data u_{wi}, τ_w, ω_w, and P_w also are expanded in power series of k in the following discussion. Even when they are independent of k, the ϕ_w depends on k because ϕ_w generally depends on ϕ ($\zeta_i n_i < 0$).

- p. 104, the second line in Footnote 18:

only Ω_8 and Θ_8 $\rightarrow$ (Ω_8, Θ_8), (Ω_9, Θ_9), and $(\Omega_{10}, \Theta_{10})$

- p. 146, the fifth line:

$\hat{Y}_1(\tilde{\eta}) = \hat{Y}_1(\tilde{\eta})$ $\rightarrow$ $\hat{Y}_1(\tilde{\eta}) = Y_1(\tilde{\eta})$

- p. 177, Eq. (6.53):

The following note should be made to Eq. (6.53):

Equation (6.54), which is derived from Eqs. (6.50), (6.52), and $\hat{v}_{wi} n_i = 0$, is used in the derivation of Eq. (6.53) from the solvability condition. See the second paragraph in page 130 of Y. Sone, Molecular Gas Dynamics (Birkhäuser, Boston, 2007).

To improve the indefinite expression, the third paragraph on p. 177, which starts from its 9th line from the bottom and ends at the second line on p. 178, should be replaced by the following paragraph:[1]

Now return to the discussion of $\hat{f}_{Vm}$. The first relation of the solvability condition (6.48) with $m = 1$, where

$$Ih_1 = \zeta_i n_i (\partial \hat{f}_{V0}/\partial y), \tag{6.51}$$

reduces to

$$\frac{\partial(\hat{\rho}_{V0} \hat{v}_{iV0} n_i)}{\partial y} = 0, \tag{6.52}$$

from which with the aid of Eqs. (6.50) and $\hat{v}_{wi} n_i = 0$, we have

$$\hat{v}_{iV0} n_i = 0. \tag{6.54}$$

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[1] In the new paragraph, the numbering (6.54) comes earlier than (6.53) to avoid the correction of the quotations in the rest of the book.

With the use of Eq. (6.54), the scalar product of the second relation of Eq. (6.48) and n_i reduces to

$$\partial \hat{p}_{V0}/\partial y = 0. \tag{6.53}$$

Thus, $\hat{p}_{V0}$ is a function of χ_1 and χ_2 only, i.e.,

$$\hat{p}_{V0} = \hat{p}_{V0}(\chi_1, \chi_2). \tag{6.55}$$

The remaining three relations [or the remaining two of the second relation of Eq. (6.48) and its third relation] reduce to identities with the aid of Eq. (6.54).

- p.215, the third line in Footnote 6:
$$M_n \quad \to \quad -M_n$$

- p. 220, the fifth line:
$$F_s \quad \to \quad -M_n F_s$$
$$F_b \quad \to \quad -M_n F_b$$

Supplementary Notes

In the present supplementary notes, the letter K is attached to the labels of equations and sections, etc. in the book *Kinetic Theory and Fluid Dynamics*, the letter M is attached to those in Y. Sone, *Molecular Gas Dynamics* (Birkhäuser, Boston, 2007), and the letter MS is attached to those in Y. Sone [2018] (*Supplement to Molecular Gas Dynamics*) to avoid confusion. The two books, *Kinetic Theory and Fluid Dynamics* and *Molecular Gas Dynamics*, and the supplement of the latter (Sone [2018]) are, respectively, referred to as KF, MGD, and MS.[2]

1 Chapter K-2

1.1 Background of the Boltzmann equation (Sections K-2.1 and K-2.2)

The situation of a monatomic gas the description of which is the purpose of the Boltzmann equation is explained in more detail in Section MS-1.53 (Version 13.00) [the last half of Section MS-1.5.2 (Version 11-02)]. This will serve as the supplement to Sections K-2.1 and K-2.2, though it is prepared for the discussion of the parameters in the nondimensional Boltzmann equation.

(Section 1.1: Version 5-00)

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[2]Labels in MS depend on its version.

1.2 Bulk viscosity

The assumptions (K-2.14) and (K-2.15) for the stress tensor and heat-flow vector in classical gas dynamics are what is to be studied by kinetic theory (see KF). For a monatomic gas, consisting of identical molecules whose intermolecular potential is spherically symmetric, which is discussed in KF, the bulk viscosity is easily seen to vanish. From Eqs. (K-2.2d) and (K-2.2f),[3]

$$p_{ii} = 3p. \tag{1}$$

On the other hand, the trace of the first relation of Eq. (K-2.15) is

$$p_{ii} = 3p - 3\mu_B \frac{\partial v_i}{\partial X_i}.$$

Thus, from the two relations, we have

$$\mu_B = 0. \tag{2}$$

(Section 1.2: Version 3-00)

1.3 Nondimensional form of the Boltzmann equation for an infinite-range potential (Sections K-2.9)

As mentioned in Footnote K-10 in Section K-2.9, the reference quantity B_0 of B in Eq. (K-2.41d) for or an intermolecular potential extending up to infinity diverges to infinity[4] because B_0 is derived from the loss term of collision term. Therefore, the nondimensional formulas for the collision term there are useless for an infinite-range potential and so is the corresponding Boltzmann equation (K-2.40a) in nondimensional form. The collision integral that is finite when the gain and loss terms evaluated together as a whole is considered here. The recipe to avoid the divergence in Footnote K-10 (or Section K-2.5) is conventional or not concrete. The natural and concrete process to derive the nondimensinal form

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[3]For molecules with internal degree of freedom (e.g., rotational and vibrational freedoms), this freedom contributes to the integrands of Eqs. (K-2.2c)–(K-2.2g). Thus, Eq. (1) does not generally hold. (More precisely, the velocity distribution function f depends also on the variables of the internal degree of freedom of a molecule. The integration with respect to these variables in Eqs. (K-2.2a)–(K-2.2g) has to be carried out. The angular momentum due to the rotation of molecules of infinitesimal size per unit mass is negligible even when the energy of rotation is not negligible.) The density ρ and the specific internal energy e can be clearly defined whether the gas is in an equilibrium state or not. The specific internal energy e/i_f per unit freedom of a molecule is taken as $RT/2$, i.e., $e = i_f RT/2$, where i_f is the degree of freedom of a molecule; thus, the relation between e and T is independent of the state of the gas (equilibrium or nonequilibrium). The pressure is defined by the equation of state, i.e., the perfect gas relation $p = \rho RT$; thus, except for a monatomic gas without internal degree of freedom, the pressure differs generally from the isotropic part of stress tensor in a nonequilibrium state.

[4]What is called the Boltzmann equation for an infinite-range potential [Footnote K-10 in Section K-2.9 or more concretely in the paragraph with Eq. (M-A.59)] is conventional one because some points have to be confirmed [see in (ii) of Footnote MS-19]. The process to eliminate the adjective "conventional" is discussed in Section MS-1.5.5 (Version 14.00).

of the Boltzamnn equation for infinite-range potential is discussed in Section MS-1.5 (Version 9-00 and later).

In this relation, it is better to make the dependence of U_0 (on the 2nd line of Footnote K-11 in Section K-2.9) on m explicit. That is,

$$U_0 \rightarrow m\bar{\mathcal{U}}_0,$$

where $\bar{\mathcal{U}}_0$ has the dimension of RT_0. Correspondingly, $U_0/k_B T_0$ on the 3rd line of Footnote K-11 in Section K-2.9 should be replaced by $\bar{\mathcal{U}}_0/RT_0$ [see item (iii) of Footnote MS-19 (Version 11-02 and later)]. Incidentally, some notes are added to Footnote K-11 in Section K-2.9 here: U is better to be replaced by $\hat{U}$, which is used in Section M-A.2.4; only the value 1 is allowed as m of a^m on the 2nd line from below for the function B [compare the 2nd and 3rd lines in Eq. (K-2.41d); see Section M-A.2.4 for $B(|\alpha \cdot (\boldsymbol{\xi}_* - \boldsymbol{\xi})/|\boldsymbol{\xi}_* - \boldsymbol{\xi}|, |\boldsymbol{\xi}_* - \boldsymbol{\xi}|)]$.

(Section 1.3: Version 5-00)

2 Chapter K-3

2.1 Flows induced by temperature fields and video files of their experiments

2.1.1 Thermal creep flow

The video file of the simple experiment (K-Sone [1991b]) demonstrating thermal creep flow in Section K-3.11.1 can be downloaded from Sone [1991] at Kyoto University Research Information Repository (https://hdl.handle.net/2433/120983).

(Section 2.1.1: Version 5-00)

The thermal creep flow (see the formula on page 61 of KF), i.e.,

$$v_1 = -K_1 \left(\frac{\pi R T_0}{2} \right)^{1/2} \frac{\ell_0}{T_0} \frac{\mathrm{d}T_w}{\mathrm{d}X_1}, \tag{3}$$

vanishes in the limit that the mean free path tends to zero ($\ell_0 \rightarrow 0$).[5] At the standard state, the mean free path is small but finite, and therefore, the thermal creep flow does not vanish. The mean free path is related to the thermal conductivity λ by Eq. (K-3.91), i.e.,

$$\ell_0 = \frac{4(2RT_0)^{1/2}}{5\sqrt{\pi}\gamma_2 R p_0} \lambda. \tag{4}$$

The above-mentioned formula of the thermal creep flow where the mean free path ℓ_0 replaced by the thermal conductivity λ with the above relation, i.e.,

$$v_1 = -\frac{4K_1\lambda}{5\gamma_2 p_0} \frac{\mathrm{d}T_w}{\mathrm{d}X_1}, \tag{5}$$

makes the thermal creep flow more accessible. This kind of replacement can be made between mean free path and viscosity [see Eq. (K-3.90)].

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[5]Except in the thin layer adjacent to the boundary, Y_1 is negligible.

2.1.2 Thermal edge flow

The video files of the simple experiment demonstrating thermal edge flow in Section K-3.11.4 (K-Sone & Yoshimoto [1997]) can be downloaded from Sone & Yoshimoto [1997] at Kyoto University Research Information Repository (https://hdl.handle.net/2433/122357).

<div align="right">(Section 2.1.2: Version 5-00)</div>

3 Chapter K-4

3.1 Notes on basic equations in classical fluid dynamics

3.1.1 Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

The mass, momentum, and energy-conservation equations of fluid flow are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0, \tag{6}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial X_j}(\rho v_i v_j + p_{ij}) = 0, \tag{7}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}v_i^2\right)\right] + \frac{\partial}{\partial X_j}\left[\rho v_j\left(e + \frac{1}{2}v_i^2\right) + v_i p_{ij} + q_j\right] = 0, \tag{8}$$

where ρ is the density, v_i is the flow velocity, e is the internal energy per unit mass, p_{ij}, which is symmetric with respect to i and j, is the stress tenor, and q_i is the heat-flow vector. The pressure p and the internal energy e are given by the equations of state as functions of T and ρ, i.e.,[6]

$$p = p(T, \rho), \quad e = e(T, \rho). \tag{9}$$

Especially, for a perfect gas (see Section 1.6 in Liepmann & Roshko [1957]),

$$p = R\rho T, \quad e = e(T). \tag{10}$$

Equations (7) and (8) are rewritten with the aid of Eq. (6) in the form

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial X_j} + \frac{\partial p_{ij}}{\partial X_j} = 0, \tag{11}$$

$$\rho\frac{\partial}{\partial t}\left(e + \frac{1}{2}v_i^2\right) + \rho v_j\frac{\partial}{\partial X_j}\left(e + \frac{1}{2}v_i^2\right) + \frac{\partial}{\partial X_j}(v_i p_{ij} + q_j) = 0. \tag{12}$$

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[6]The case where the first equation $p = p(T, \rho)$ cannot be solved with respect to ρ, or ρ is independent of p and T is called *incompressible*. The incompressible condition is a special case of the equation of state. In a system consisting of a single incompressible substance, the density is a constant.

The operator $\partial/\partial t + v_j \partial/\partial x_j$, which expresses the time variation along the fluid particle, is denoted by $\mathrm{D}/\mathrm{D}t$, i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}.$$

Multiplying Eq. (11) by v_i we obtain the equation for the variation of kinetic energy as

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v_i^2\right) = -v_i \frac{\partial p_{ij}}{\partial X_j}. \tag{13}$$

Another form of Eq. (8), where Eq. (13) is subtracted from Eq. (12), is given as

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p_{ij}\frac{\partial v_i}{\partial X_j} - \frac{\partial q_j}{\partial X_j}. \tag{14}$$

Noting the thermodynamic relation

$$\frac{\mathrm{D}e}{\mathrm{D}t} = T\frac{\mathrm{D}s}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{15}$$

where s is the entropy per unit mass, and Eq. (6), Eq. (14) is rewritten as

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = -\frac{1}{T}\left[(p_{ij} - p\delta_{ij})\frac{\partial v_i}{\partial X_j} + \frac{\partial q_j}{\partial X_j}\right]. \tag{16}$$

Equation (16) expresses the variation of the entropy of a fluid particle.

Equations (6)–(9) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor p_{ij} and the heat-flow vector q_i are assumed in some ways. The *Navier–Stokes set of equations* (or the *Navier–Stokes equations*) is Eqs. (6)–(9) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3}\frac{\partial v_k}{\partial X_k}\delta_{ij}\right) - \mu_B\frac{\partial v_k}{\partial X_k}\delta_{ij}, \tag{17}$$

$$q_i = -\lambda\frac{\partial T}{\partial X_i}, \tag{18}$$

where μ, μ_B, and λ are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of T and ρ. The *Euler set of equations* (or the *Euler equations*) is Eqs. (6)–(9) where p_{ij} and q_i are given by

$$p_{ij} = p\delta_{ij}, \quad q_i = 0, \tag{19}$$

or the Navier–Stokes equations with $\mu = \mu_B = \lambda = 0$.

For the Navier–Stokes equations, in view of the relations (17) and (18), the

entropy variation is expressed in the form[7]

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T} \left[\frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \mu_B \left(\frac{\partial v_k}{\partial X_k} \right)^2 + \frac{\partial}{\partial X_i} \left(\lambda \frac{\partial T}{\partial X_i} \right) \right].$$
(20)

For the Euler equations, for which p_{ij} and q_i are given by Eq.(19), the entropy of a fluid particle is invariant, i.e.,

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = 0.$$
(21)

The incompressible condition in Footnote 6 can be expressed as[8]

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \ \text{ or } \ \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0.$$
(22)

Thus, from Eqs. (6) and (22),

$$\frac{\partial v_i}{\partial X_i} = 0.$$
(23)

Equation (17) for the Navier–Stokes-stress tensor reduces to

$$p_{ij} = p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right).$$
(24)

The first term on the right-hand side of Eq. (14) reduces to

$$-p_{ij} \frac{\partial v_i}{\partial X_j} = - \left[p\delta_{ij} - \mu \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right] \frac{\partial v_i}{\partial X_j}$$

$$= \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2.$$

Thus, Eq. (14) reduces to

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\mu}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left(\lambda \frac{\partial T}{\partial X_j} \right).$$
(25)

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[7]Note the following transformation:

$$\frac{\partial v_i}{\partial X_j} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)$$

$$= \frac{1}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} + \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)$$

$$= \frac{1}{2} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \frac{1}{3} \frac{\partial v_l}{\partial X_l} \delta_{ij} \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right).$$

The second term in the last expression is easily seen to vanish.

[8](i) The density is invariant along fluid-particle paths. If ρ is of uniform value ρ_0 initially, it is a constant, i.e.,

$$\rho = \rho_0.$$

In a time-independent (or steady) problem, the density is constant along streamlines.

(ii) This condition is useful in the system consisting of non-mixing different incompressible substances (e.g., oil and water). Needless to say, e, μ, and λ depend on substances.

To summarize, the Navier–Stokes equations for incompressible fluid are

$$\frac{\partial v_i}{\partial X_i} = 0, \tag{26a}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j}\left[\mu\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)\right], \tag{26b}$$

$$\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \frac{\mu}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)^2 + \frac{\partial}{\partial X_j}\left(\lambda \frac{\partial T}{\partial X_j}\right), \tag{26c}$$

with the incompressible condition (22) being supplemented, i.e.,[9]

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. \tag{27}$$

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3.1.2 Boundary condition for the Euler set

In Chapter K-7, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentum-conservation

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[9]The energy equation (26c) for incompressible fluid contains the heat-production term due to the work done by viscous stress [the first term on its right-hand side] if we literally put Eq. (17) in the energy equation (14). The incompressible Navier--Stokes set of equations is generally used for the case where the flow velocity is not so large. That is, it is used for the case where the dynamic energy $v_i^2/2$ per unit mass is negligibly small compared with the internal energy e per unit mass ($v_i^2 \ll e$). The Mach number is a common reference scale of flow speed, but it is not useful for incompressible fluid because the speed of sound is not well defined. Let e_0 be e of the reference state. We take the speed $(2e_0)^{1/2}$ as the reference speed, which is comparable to the sound speed for perfect gas. Let the typical flow speed of the flow under consideration be U_0. With the nondimensional small parameter $U_0/(2e_0)^{1/2}$, we compare the size of the terms in the energy equation (26c). The first term on the right-hand side of Eq. (26c) is seen to be of higher-order of smallness compared with the convection and conduction terms when the Reynolds number based on U_0 and the Prandtl number are of the order of unity. Thus, the neglected form is often given in literature, with the condition being implicitly assumed.

equations of the Euler set are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{28}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{29}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{30}$$

where ρ is the density, which is uniform, (u, v) is the flow velocity, and p is the pressure. Owing to Eq. (28), the stream function Ψ can be introduced as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{31}$$

Eliminating p from Eqs. (29) and (30), we have[10]

$$u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} = 0, \tag{32}$$

where Ω is the vorticity, i.e.,

$$\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}. \tag{33}$$

From Eqs. (31) and (32),

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Omega}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Omega}{\partial y} = 0. \tag{34}$$

This equation shows that Ω is a function of Ψ,[11] i.e.,

$$\Omega = F(\Psi). \tag{35}$$

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[10]The following equation is formed from them:

$$\partial \text{Eq. (29)}/\partial y - \partial \text{Eq. (30)}/\partial x = 0.$$

[11]This can be seen with the aid of theorems on implicit functions (see Bronshtein & Semendyayev [1997], Buck [1965], Takagi [1961]):

$$\Omega = \Omega(x, y), \quad \Psi = \Psi(x, y). \tag{$*$}$$

Solving the second equation with respect to x, we have

$$x = \hat{x}(\Psi, y). \tag{$**$}$$

With this relation into Eq. ($*$),

$$\Omega = \Omega(\hat{x}(\Psi, y), y) = \hat{\Omega}(\Psi, y), \tag{$\sharp$a}$$

$$\Psi = \Psi(\hat{x}(\Psi, y), y) = \hat{\Psi}(\Psi, y). \tag{$\sharp$b}$$

This functional relation between Ω and Ψ is a local relation, and therefore F may be a multivalued function of Ψ. From Eqs. (33) and (35),

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = F(\Psi). \tag{36}$$

Consider a boundary-value problem in a simply-connected bounded domain, where Ψ is given on the boundary ($\Psi = \Psi_B$). Introduce a coordinate s ($0 \leq s < S$) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where $\partial \Psi_B / \partial s < 0$, and the fluid flows out from the domain on the boundary where $\partial \Psi_B / \partial s > 0$. When F is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose F on the part where $\partial \Psi_B / \partial s < 0$ or $\partial \Psi_B / \partial s > 0$. For example, take the case where $\partial \Psi_B / \partial s < 0$ for $0 < s < S_m$ and $\partial \Psi_B / \partial s > 0$ for $S_m < s < S$, and choose the distribution $\Omega_B(s)$ of Ω along the boundary for the part $0 < s < S_m$. By the choice of Ω_B, the function $F(\Psi)$ is determined in the following way. Inverting the relation $\Psi = \Psi_B(s)$ between Ψ and s on the part $0 < s < S_m$, i.e., $s(\Psi)$, and noting the relation (35), we find that F is given by

$$F(\Psi) = \Omega_B(s(\Psi)). \tag{37}$$

Then, the boundary-value problem is fixed. That is, Eq. (36) is fixed as[12]

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \Omega_B(s(\Psi)), \tag{38}$$

and the boundary condition is given as $\Psi = \Psi_B(s)$. This system is a standard from the point of counting of the number of boundary conditions. Obviously,

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That is, Ω is expressed as a function of Ψ and y. From Eqs. ($\sharp$a) and ($\sharp$b),

$$\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = \frac{\partial \Omega(\hat{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Omega(x, y)}{\partial x} \frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Omega(x, y)}{\partial y}, \tag{$\sharp\sharp$a}$$

$$\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = 0. \tag{$\sharp\sharp$b}$$

On the other hand,

$$\frac{\partial \hat{\Psi}(\Psi, y)}{\partial y} = \frac{\partial \Psi(\hat{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y}.$$

Thus,

$$\frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \hat{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y} = 0. \tag{$\ddagger$}$$

From Eqs. (34), ($\sharp\sharp$a) and ($\ddagger$), we have

$$\frac{\partial \hat{\Omega}(\Psi, y)}{\partial y} = 0, \quad \text{or} \quad \Omega = \hat{\Omega}(\Psi).$$

[12]There is still some ambiguity. The case where there is a region with closed stream lines $\Psi(x, y) = \text{const}$ inside the domain is not excluded.

from Eq. (33), the solution of the above system automatically satisfies condition $\Omega = \Omega_B(s)$ along the boundary for $0 < s < S_m$. We cannot choose the distribution of Ω on the boundary for $S_m < s < S$.

The energy-conservation equation of the incompressible Euler set is given by Eq. (25) with $\mu = \lambda = 0$, i.e.,

$$u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} = 0, \text{ or } \frac{\partial \Psi}{\partial y}\frac{\partial e}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial e}{\partial y} = 0, \tag{39}$$

where e is the internal energy. Thus, e is a function of Ψ, i.e.,

$$e = F_1(\Psi). \tag{40}$$

In the above boundary-value problem, therefore, e can be specified on the the part $(0 < s < S_m)$ of the boundary, but no condition can be specified on other part $(S_m < s < S)$ and vice versa.[13]

To summarize, we can specify three conditions for Ψ, Ω, and e on the part $\partial \Psi_B/\partial s < 0$ $(\partial \Psi_B/\partial s > 0)$ of boundary but one condition for Ψ on the other part $\partial \Psi_B/\partial s > 0$ $(\partial \Psi_B/\partial s < 0)$. The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

3.1.3 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set although the difference is obvious from the set of equations derived, especially from the equation of state and the energy equation.[14] The difference is explained in the two books KF and MGD in connection with the set of fluid-dynamic type equations derived by the S expansion from the Boltzmann equation in Sections K-4.3, M-3.2, and M-3.7. To make difference clearer and to eliminate the misunderstanding, we will show the process of analysis from the compressible Navier–Stokes set with the boundary condition taken into account. Thus, it is made clear how the solution is constructed and how the behavior of the solution in the two systems is different in the initial and boundary-value problem. In the time-dependent case, the energy equation contains another time-derivative term, in addition to the difference in the time-independent case. An ambiguity of solution in the incompressible system in a bounded domain with the simple boundary is eliminated in the compressible system.

Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the

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[13]From the second relation on e of Eq. (9) and the uniform-density condition, the condition on e can be replaced by the condition on the temperature T.

[14]In a time-dependent case, we here consider the case where the variables vary in the diffusion time scale. In a shorter time scale, the sound wave propagates.

nondimensional variables introduced by Eq. (K-2.57) in Section K-2.10 as follows:

$$\text{Sh}\frac{\partial \omega}{\partial \hat{t}} + \frac{\partial (1+\omega)u_i}{\partial x_i} = 0, \tag{41}$$

$$\text{Sh}\frac{\partial (1+\omega)u_i}{\partial \hat{t}} + \frac{\partial}{\partial x_j}\left((1+\omega)u_i u_j + \frac{1}{2}P_{ij}\right) = 0, \tag{42}$$

$$\text{Sh}\frac{\partial}{\partial \hat{t}}\left[(1+\omega)\left(\frac{3}{2}(1+\tau) + u_i^2\right)\right]$$
$$+ \frac{\partial}{\partial x_j}\left[(1+\omega)u_j\left(\frac{3}{2}(1+\tau) + u_i^2\right) + u_i(\delta_{ij} + P_{ij}) + Q_j\right] = 0. \tag{43}$$

The nondimensional stress tensor P_{ij}, and heat-flow vector Q_i are expressed as[15]

$$P_{ij} = P\delta_{ij} - \frac{\mu_0(2RT_0)^{1/2}}{p_0 L}(1+\bar{\mu})\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \tag{44a}$$

$$Q_i = -\frac{\lambda_0 T_0}{Lp_0(2RT_0)^{1/2}}(1+\bar{\lambda})\frac{\partial \tau}{\partial x_i}. \tag{44b}$$

Here, $\bar{\mu}$ and $\bar{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

$$\mu = \mu_0(1+\bar{\mu}), \quad \lambda = \lambda_0(1+\bar{\lambda}),$$

where μ_0 and λ_0 are, respectively, the values of the viscosity μ and the thermal conductivity λ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of τ and ω. The first relation of the equation of state [Eq. (10)] is expressed as

$$P = \omega + \tau + \omega\tau. \tag{45}$$

Take a small parameter ε, and consider the case where

$$u_i = O(\varepsilon), \quad \omega = O(\varepsilon), \quad \tau = O(\varepsilon), \quad \text{Sh} = O(\varepsilon), \tag{46a}$$

$$\frac{\mu_0(2RT_0)^{1/2}}{p_0 L} = \gamma_1\varepsilon, \quad \frac{\lambda_0 T_0}{Lp_0(2RT_0)^{1/2}} = \frac{5}{4}\gamma_2\varepsilon, \tag{46b}$$

where γ_1 and γ_2 are constants of the order of unity. Thus,

$$P = O(\varepsilon), \quad \bar{\mu} = O(\varepsilon), \quad \bar{\lambda} = O(\varepsilon).$$

Some notes on the conditions (46a) and (46b) may be in order.
(i-a) The first three relations in Eq. (46a) are the condition on the size of perturbation of gas dynamic variables from the background state ($v_i = 0$, $\rho = \rho_0$ and $T = T_0$) with respect to their reference values [$v_i = (2RT_0)^{1/2}$, $\rho = \rho_0$,

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[15]For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B = 0$ (see Section 1.2).

and $T = T_0$]. The last relation is the condition of the time t_0 of appreciable variation of the perturbed quantities. This condition shows that the time-scale t_0 of variation of the variables is chosen as

$$t_0 = \frac{L}{(2RT_0)^{1/2}\varepsilon},\tag{47}$$

which is the time that the typical gas flow proceeds over the distance L. In other words, we are interested in the behavior of the gas, the perturbed quantities of which vary appreciably in time t_0. Naturally, the variation of boundary data is to be consistent with the above time scale.

(i-b) The discussion in this section (Section 3.1.3) is based on the Navier–Stokes equation, which is assumed to be valid without the restriction on the size of the parameters. We are going to derive the set of equations for small ε under the conditions (46a) and (46b) for the two kinds of fluid (perfect gas and incompressible fluid), and compare the difference between their results. Here we are interested in the leading nontrivial set. The equations at the higher-orders are obviously within the framework of the Navier–Stokes equations. This applies to the boundary condition. Nonslip or non-jump condition is used on the simple boundary irrespective of the size of ε. The boundary condition on the interface is borrowed from the leading-order results of the asymptotic analysis for small Knudsen numbers of the Boltzmann system for the case in (iii).

(ii) In the Navier–Stokes system, the viscosity μ and the thermal conductivity λ characterize the property of the fluid. The relation (46b) is the condition between these transport coefficients and the typical size ε of the perturbations. Form these relations, we have

$$\frac{\mu_0/\rho_0}{\varepsilon(2RT_0)^{1/2}L} = \frac{\gamma_1}{2}, \qquad \frac{2\lambda_0}{5R\mu_0} = \frac{\gamma_2}{\gamma_1}.\tag{48}$$

The quantities on the left-hand sides of the two relations consist of the parameters[16] of the flow under consideration. Thus, we are considering the case where the two combinations of the parameters are finite because γ_1 and γ_2 are constants.

(iii-a) In the S expansion in Chapter K-4 or in Section M-3.2, the parameter k characterizes the degree of rarefaction of the gas under consideration. The analysis there is carried out under the condition that

$$\varepsilon = k,\tag{49}$$

where ε is the size of perturbation ϕ of the velocity distribution function from the background equilibrium state f_0 at rest with the density ρ_0 and the temperature T_0 corresponding to Eq. (46a), with the time-independent condition imposed. The time-dependent case is discussed for the two cases, $t_0 = O[L/(2RT_0)^{1/2}]$

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[16]Note that (a) $\varepsilon(2RT_0)^{1/2}$ is the typical flow speed, and $\varepsilon(2RT_0)^{1/2}L/(\mu_0/\rho_0)$ is the Reynolds number based on that flow speed and that (b) $3R/2$ is the heat capacity at constant volume because the internal energy per unit mass is $3RT_0/2$ for the gas under consideration (perfect gas).

and $t_0 = O[L/(2RT_0)^{1/2}\varepsilon]$, in Section K-4.9 and K-4.10.3 and Sections M-3.7.1 and M-3.7.2.

(iii-b) This is one of the several cases where the behavior of a gas for small Knudsen numbers (or $k \ll 1$) is discussed on the basis of the Boltzmann system (Boltzmann equation and its boundary condition), and the fluid-dynamic type equation and its boundary condition are derived for various limiting processes (see the two books KF and MGD). The nontrivial leading-order results of the case (49) corresponds to the set of equations of the case with Eqs. (46a) and (46b) of the Navier–Stokes equation for perfect gas. We will use the boundary condition on the interface obtained by the analysis of the Boltzmann system to the discussion in this section (Section 3.1.3). The higher-order results include higher-order effects of the Knudsen number, which is not covered by the Navier–Stokes system. The higher-order boundary condition does not contribute to the comparison. Thus, the leading-order boundary condition is used with adjustments as the higher-order one if necessary.

(iv) The expansion parameter ε can be chosen rather freely by a finite factor. So are the reference quantities. If we use a different reference velocity, e.g., $(2e_0)^{1/2}$, instead of $(2RT_0)^{1/2}$, with the same ε, the coefficients of the resulting equations are obviously different.

According to the definition of u_i in Eq. (K-2.57), ε is of the order of the Mach number of typical flow speed U_0 of the flow field. Here, we take ε as $\varepsilon = U_0/(2RT_0)^{1/2}$. The relation (46b) between ε and viscosity μ_0 or thermal conductivity λ_0 is taken from the result of analysis in Chapter K-4, where the situation that the Knudsen number and the perturbation scale is of the same of order of smallness, i.e., $k = \varepsilon$, is discussed. According to Eq. (K-2.41a), the condition $\mathrm{Sh} = O(\varepsilon)$ in Eq. (46a) means that the time scale t_0 of the variation of variables is of the order of $L/(2RT_0)^{1/2}\varepsilon$, which is of the order of time scale of viscous or thermal diffusion owing to the relation (46b). We can

take $t_0 = L/(2RT_0)^{1/2}\varepsilon$ without loss of generality.[17] Then,

$$\mathrm{Sh} = \varepsilon. \tag{50}$$

The new symbol $\tilde{t}$, instead of $\hat{t}$ is introduced to make this time scale explicit:

$$\tilde{t} = \hat{t}. \tag{51}$$

Corresponding to the above situation, u_i, ω, P, and τ are expanded in power series of ε, i.e.,

$$u_i = u_{i1}\varepsilon + u_{i2}\varepsilon^2 + \cdots , \tag{52a}$$

$$\omega = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots , \tag{52b}$$

$$P = P_1\varepsilon + P_2\varepsilon^2 + \cdots , \tag{52c}$$

$$\tau = \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots , \tag{52d}$$

$$\bar{\mu} = \bar{\mu}_1\varepsilon + \bar{\mu}_2\varepsilon^2 + \cdots , \tag{52e}$$

$$\bar{\lambda} = \bar{\lambda}_1\varepsilon + \bar{\lambda}_2\varepsilon^2 + \cdots , \tag{52f}$$

$$P_{ij} = P_1\delta_{ij}\varepsilon + P_{ij2}\varepsilon^2 + \cdots , \tag{52g}$$

$$Q_i = Q_{i2}\varepsilon^2 + \cdots . \tag{52h}$$

Substituting Eqs. (52a)–(52h) with Eqs. (46b) and (50) into Eqs. (41)–(43) with Eqs. (44a) and (44b), and arranging the same order terms of ε, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \qquad \frac{\partial P_1}{\partial x_i} = 0, \qquad \frac{\partial u_{i1}}{\partial x_i} = 0,$$

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[17]This time scale, $t_0 = L/(2RT_0)^{1/2}\varepsilon$, is called diffusion time scale in view of the following situation. The viscous diffusion of the parallel flow in X_1 direction with nonuniform velocity v_1 is given by

$$\frac{\partial v_1}{\partial t} = \frac{\mu_0}{\rho_0}\frac{\partial^2 v_1}{\partial X_2^2},$$

where the variation with respect to x_3 is assumed to be absent. Let the length scale and the time scale of variation of v_1 be L and t_0 respectively. Putting $t = \tilde{t}\, t_0$ and $X_2 = x_2 L$, we have

$$\frac{\partial v_1}{\partial \tilde{t}} = \frac{\mu_0 t_0}{\rho_0 L^2}\frac{\partial^2 v_1}{\partial x_2^2}$$

$$= \frac{\gamma_1 t_0 (2RT_0)^{1/2}\varepsilon}{2L}\frac{\partial^2 v_1}{\partial x_2^2},$$

where the first relation of Eq. (46b) is used in the final step. When $t_0 = L/(2RT_0)^{1/2}\varepsilon$, the coefficient is of the order of unity because $\gamma_2/2$ is so. Then, the time derivative $\partial v_1/\partial \tilde{t}$ and the space derivative $\partial^2 v_1/\partial x_2^2$ are of the same order $O(v_1)$. Thus, in the diffusion process, the time scale of variation is $t_0 = L/(2RT_0)^{1/2}\varepsilon$ when the length scale of variation is L.

The diffusion of heat can be discussed in the same way, with ρ_0 being replaced by $\alpha\rho_0 RT_0$ ($\alpha = 3/2$ or $5/2$ depending on the diffusion under constant volume or constant pressure), μ_0 by λ_0, and the velocity v_1 by temperature T. With the aid of the second relation of Eq. (46b),

$$\frac{\partial T}{\partial \tilde{t}} = \frac{5\gamma_2 t_0 (2RT_0)^{1/2}\varepsilon}{4\alpha L}\frac{\partial^2 T}{\partial x_2^2}.$$

The remaining discussion is the same because γ_2 is a constant of the order of unity.

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$$\frac{\partial \omega_1}{\partial \tilde{t}} + \frac{\partial \omega_1 u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} = 0,$$

$$\frac{\partial u_{i1}}{\partial \tilde{t}} + \frac{\partial u_{i1} u_{j1}}{\partial x_j} + \frac{1}{2}\frac{\partial P_2}{\partial x_i} - \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right) = 0,$$

$$\frac{3}{2}\frac{\partial P_1}{\partial \tilde{t}} + \frac{\partial}{\partial x_j}\left(\frac{5}{2}u_{j2} + \frac{5}{2}P_1 u_{j1} - \frac{5}{4}\gamma_2\frac{\partial \tau_1}{\partial x_j}\right) = 0,$$

and so on. At the leading order, the equations derived from Eqs. (41) and (43) degenerate into the same equation $\partial u_{i1}/\partial x_i = 0$. Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{53}$$

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{54a}$$

$$\frac{\partial u_{i1}}{\partial \tilde{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{54b}$$

$$\frac{5}{2}\frac{\partial \tau_1}{\partial \tilde{t}} - \frac{\partial P_1}{\partial \tilde{t}} + \frac{5}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{54c}$$

$$\frac{\partial u_{i2}}{\partial x_i} = -\frac{\partial \omega_1}{\partial \tilde{t}} - \frac{\partial \omega_1 u_{i1}}{\partial x_i}, \tag{55a}$$

$$\frac{\partial u_{i2}}{\partial \tilde{t}} + u_{j1}\frac{\partial u_{i2}}{\partial x_j} + u_{j2}\frac{\partial u_{i1}}{\partial x_j}$$
$$= -\frac{1}{2}\left(\frac{\partial P_3}{\partial x_i} - \omega_1\frac{\partial P_2}{\partial x_i}\right) + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3}\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right)$$
$$- \frac{\gamma_1\omega_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2}\frac{\partial}{\partial x_j}\left[\bar{\mu}_1\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)\right], \tag{55b}$$

$$\frac{3}{2}\frac{\partial P_2}{\partial \tilde{t}} + \frac{3}{2}u_{j1}\frac{\partial P_2}{\partial x_j} + \frac{5}{2}\left(P_1\frac{\partial u_{j2}}{\partial x_j} - \frac{\partial \omega_2}{\partial \tilde{t}} - \frac{\partial(\omega_1 u_{j2} + \omega_2 u_{j1})}{\partial x_j}\right)$$
$$= \frac{5\gamma_2}{4}\frac{\partial}{\partial x_i}\left(\frac{\partial \tau_2}{\partial x_i} + \bar{\lambda}_1\frac{\partial \tau_1}{\partial x_i}\right) + \frac{\gamma_1}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right)^2, \tag{55c}$$

where

$$P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1\tau_1. \tag{56}$$

These equations are very similar to the corresponding equations obtained by the S expansion of the Boltzmann equation in Section K-4.9 or Section M-3.7.2.[18]

[18] (i) Equations (53)–(54c) and (56) are of the same form as Eqs (K-4.86a)–(K-4.86d) and (K-4.8d) or Eqs. (M-3.265)–(M-3.266c) and (M-3.268). Thus, the following discussion in this section (Section 3.1.3) applies to the two systems. Naturally, it applies to the S solution in Section K-4.10.3.

(ii) It may be noted that the second-order velocity u_{i2} is not solenoidal from Eq. (55a).

In order to compare Eqs. (53)–(54c) and the first relation of Eq. (56) with the incompressible Navier–Stokes equations (26a)–(27), we will rewrite the latter equations for the situation [Eqs. (46a) and (46b)] where the former equations are derived. The starting equations are Eqs. (41)–(44b)[19] and the nondimensional form of Eq. (22), i.e.,

$$\varepsilon \frac{\partial \omega}{\partial \tilde{t}} + u_i \frac{\partial \omega}{\partial x_i} = 0, \tag{57}$$

instead of Eq. (45). The analysis is carried out in a similar way[20] and the equations corresponding to Eqs. (53)–(54c) are[21]

$$\frac{\partial P_1}{\partial x_i} = 0, \tag{58}$$

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[19]As the internal energy e, $3RT/2 \; [= 3RT_0(1 + \tau)/2]$ is chosen for consistency.

[20](i) For incompressible fluid, the sound speed is not well defined. Thus, the speed $(2e_0)^{1/2}$ defined by the internal energy e_0 per unit mass is taken as the reference speed, which is comparable to the sound speed for perfect gas. Thus, the condition that the Mach number is small can be replaced by the condition that the flow speed $|v_i|/(2e_0)^{1/2}$ divided by that reference speed is small. This is the case where the work done by the viscous force is negligible in Eq. (26c) (see Footnote 9).

(ii) Different equation of state corresponds to different substance. Thus, various parameters differ in different systems. However, we are interested in the difference of solutions due to the change of the equation of state among the pressure, temperature and density (perfect gas or incompressible). Thus, we take a state at rest ($v_i = 0$) with pressure p_0 and temperature T_0. Thus, the density $\rho_0 = p_0/RT_0$ for perfect gas. We imagine the incompressible fluid at rest with density ρ_0 and the other properties (internal energy e_0, viscosity μ_0, thermal conductivity λ_0) of the two kinds of fluid are taken to be the same. (This is not so real because the density differs much for gas and liquid, the latter of which is much closer to incompressible fluid. Here, we put aside the reality for the present purpose avoiding the similarity discussion. Incompressible fluid is not gas according to the classification of gas and liquid.) Taking the state at rest with pressure p_0 and temperature T_0 as the background state (thus, the density $\rho_0 = p_0/RT_0$; this relation holds only at the background state; the relation without subscript 0 applies only to perfect gas but not to incompressible fluid), we discuss the behavior of two kinds of fluid for the case where (a) the typical flow velocity, say U_0, is small compared with the reference speed $(2e_0)^{1/2}$, i.e., $U_0/(2e_0)^{1/2} \ll 1$, and (b) the relative variations P and τ of pressure and temperature is of the order of $U_0/(2e_0)^{1/2}$. In the main text, $(2RT_0)^{1/2}$ is used instead of $(2e_0)^{1/2}[= (3RT_0)^{1/2}]$ to define the expansion [note the definition $u_i = v_i/(2RT_0)^{1/2}$ and the discussion in the paragraph with Eq. (46a)].

(iii) In the preceding analysis, the equation of state, Eq. (45), is partially used in the middle of analysis. It is already used in Eq. (54c). Thus, the results, e.g., Eq. (54c), cannot directly be transferred to the case of incompressible fluid. The expansion, with P not related to ω and τ, should be done independently and apply the incompressible condition when necessary. It is much simpler to start with $\omega_n = 0$ for all n.

[21](i) From Eqs. (41) and (57), we have $\partial u_i/\partial x_i = 0$. Obviously, one of Eqs. (41) and (57) can be replaced by $\partial u_i/\partial x_i = 0$, but both cannot be replaced by it. Some confusion about the incompressibility is due to the misunderstanding of the statement.

(ii) It should be noted that Eqs. (54a) and (59a) are derived from Eq. (41) under the condition (46a) without the help of the equation of state. Incompressibility cannot be judged by Eq. (54a) or (59a).

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{59a}$$

$$\frac{\partial u_{i1}}{\partial \tilde{t}} + u_{j1}\frac{\partial u_{i1}}{\partial x_j} = -\frac{1}{2}\frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2}\frac{\partial^2 u_{i1}}{\partial x_j^2}, \tag{59b}$$

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \tilde{t}} + \frac{3}{2}u_{j1}\frac{\partial \tau_1}{\partial x_j} = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}, \tag{59c}$$

and the equation corresponding to the first relation of Eq. (56) is obtained from the incompressible condition (57) as[22]

$$\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i} = 0. \tag{60}$$

Now the basic equations, the behavior of solutions of which we are going to compare, are prepared [Eqs. (53)–(54c), and (56) for perfect gas and Eqs. (58)–(60) for incompressible fluid]. For the comparison, the initial and boundary conditions have to be chosen commonly. The diffusion time scale being natural time scale of the solution of the incompressible Navier–Stokes equation, this scale solution is the subject of comparison.[23] Incidentally, the boundary data must be consistent with the diffusion time scale. The form of the two sets of equations requires two conditions on the initial data. That is, the initial values of u_{i1} and P_1 must be solenoidal and uniform respectively in the domain under consideration, i.e., Eqs. (54a) and (53) for perfect gas and Eqs. (59a) and (58) for incompressible fluid). The initial condition common to perfect gas and incompressible fluid is determined in the following way: From the request of incompressible fluid, the density is uniform, i.e., $\omega_1 = 0$; from common request of uniformity of pressure, i.e., $P_1 = 0$; on the plane $\omega_1 = 0$, the temperature $\tau_1 = 0$ from Eq. (56) for perfect gas.[24] (see also Footnote 20). Only the uniform state with respect to density, pressure and temperature can be the common initial condition to perfect gas and incompressible fluid. The velocity can be chosen freely under the solenoidal condition. When comparing solutions in the two kinds of fluid, we have to choose their initial condition that satisfies the above condition. In this case, the time-variation of the boundary data make the difference clear. An example of comparison of this kind is shown in K-4.10.3, where nontrivial difference of their temperature fields are shown.[25] Next, we

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[22]From the choice of the background state in Footnote 20, $\omega = 0$ and $\partial u_i/\partial x_i = 0$ or $\omega_n = 0$ and $\partial u_{in}/\partial x_i = 0$ for any n.

[23]In perfect gas, there are solutions with another time scale $[t_0 = L/(2RT_0)^{1/2}]$ of variation, expressing the propagation of sound waves.

[24](i) On the surface $\omega_1 = 0$, the pressure P_1 and temperature τ_1 can be chosen freely for incompressible fluid.

(ii) We have chosen zero for the constant values for ω_1, P_1, and τ_1 without loss of generality. It is a problem of choosing the background state.

[25]The example in Section K-4.10.3 is a simple problem with the simple boundary and the interface. In the example, the S solution of the Boltzmann equation (note Footnote 18) and the corresponding one of the incompressible fluid are shown. In the former, the density varies with time, and further, the temperature field is quite different from that of the latter owing to the time-dependent boundary condition on P_{S1}, corresponding to P_1 here.

examine the two sets of equations (perfect gas and incompressible fluid) and make clear the effect or mechanism that makes the time development of the two sets different. Then, we discuss the process of solution (or how the solution is constructed) for each set with its boundary condition taken into account and show an important difference between them.

Equations (58), (59a), and (59b) are, respectively, of the same form as Eqs. (53), (54a), and (54b). Equation (54c) is rewritten with the aid of Eqs. (53) and (56) as

$$\frac{3}{2}\frac{\partial \tau_1}{\partial \tilde{t}} + \frac{3}{2}u_{i1}\frac{\partial \tau_1}{\partial x_i} - \left(\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right) = \frac{5}{4}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{61}$$

The difference of Eq. (54c) or (61) from Eq. (59c) is

$$\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}, \tag{62}$$

which vanishes for an incompressible fluid. The work W done per unit time on unit volume of fluid by pressure, given by $-p_0(2RT_0)^{1/2}L^{-1}\partial(1+P)u_i/\partial x_i$, is transformed with the aid of Eqs. (53), (54a), and (55a) in the following way:

$$\begin{aligned}
\frac{W}{p_0(2RT_0)^{1/2}L^{-1}} &= -\frac{\partial(1+P)u_i}{\partial x_i} \\
&= -\frac{\partial u_{i1}}{\partial x_i}\varepsilon - \left(P_1\frac{\partial u_{i1}}{\partial x_i} + u_{i1}\frac{\partial P_1}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i}\right)\varepsilon^2 + \cdots \\
&= -\frac{\partial u_{i2}}{\partial x_i}\varepsilon^2 + \cdots \\
&= \left(\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i}\right)\varepsilon^2 + \cdots . \tag{63}
\end{aligned}$$

The work vanishes up to the order considered here for incompressible fluid, because $\partial u_i/\partial x_i = 0$ and $\partial P_1/\partial x_i = 0$ [see Footnotes 21 and Eq. (58)]. That is, Eq. (54c) differs from Eq. (59c) by the amount of the work done by pressure. Thus, naturally, the temperature τ_1 fields in the two cases are different owing to this difference.

The variation (62) of the density ω_1 along a fluid-particle path is expressed with u_{i1}, τ_1, and P_1. That is, ω_1 in Eq. (62) is replaced by $P_1 - \tau_1$ with the aide of Eq. (56), and Eqs. (53) and (54c) are applied to the result successively. Then, we have

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i} &= \frac{\partial P_1}{\partial \tilde{t}} - \frac{\partial \tau_1}{\partial \tilde{t}} - u_{i1}\frac{\partial \tau_1}{\partial x_i} \\
&= \frac{3}{5}\frac{\partial P_1}{\partial \tilde{t}} - \frac{1}{2}\gamma_2\frac{\partial^2 \tau_1}{\partial x_j^2}. \tag{64}
\end{aligned}$$

Therefore, the density ω_1 generally varies along a fluid-particle path.

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Here, we will explain the process of solution (how the solution is determined from the basic equations and boundary condition) for the two systems (perfect gas and incompressible fluid). In the two systems, the difference is the energy equation among the conservation equations [Eqs. (53)–(54c) and Eqs. (58)–(59c)] and the equation of state [Eqs. (56) and (60)].[26] When the pressure P_1 is independent of $\tilde{t}$, the difference of the energy equations (54c) and (59c) in the two systems is the ratio of the convection term to the heat-conduction term. That is, the heat-conduction term is of the same form, but the convection term differs only by numerical factor (5/2 and 3/2). Thus, we can say the difference is the convection term by the ratio (5 : 3) or the thermal conductivity by the ratio (3 : 5). The source of this difference is due to the work done by pressure in perfect gas. The pressure P_1 being constant, the variation of the flow along a fluid particle path takes place under constant pressure during its motion, and the work done by pressure can be incorporated into the variation of enthalpy defined by $e + p/\rho$, which is equal to $5RT/2$ for perfect gas.[27] Thus, the change during the motion is expressed by the variation of the enthalpy. That is, the energy equation expresses that the enthalpy variation along a fluid particle path is equal to the energy supplied by heat conduction. On the other hands, when P_1 depends on $\tilde{t}$, the variation along a fluid particle path is neither under constant pressure nor under constant volume, and thus, the extra term $\partial P_1/\partial \tilde{t}$ enters Eq. (54c) in addition to the above difference of the coefficients. To find the solution, the boundary condition is required, which depends on the kind of the boundary. The present discussion is for the Navier–Stokes system apart from the kinetic theory except that sizes of viscosity and thermal conductivity compared with the parameter ε are borrowed from its result. The boundary condition is the one used in the Navier–Stokes system. That is, we take the non-slip condition [Eq. (K-4.61a) and (K-4.61b) or Eq. (M-3.113a) and (M-3.113b)] on the simple boundary and the leading-order result of the kinetic theory [Eq. (K-4.68a) and Eq. (K-4.68c) or Eq. (M-3.119a) and Eq. (M-3.119c)] on the interface.[28] The same boundary condition is used for incompressible fluid. The process of solution of a similar system, the S solution in Chapter K-4 or Section M-3.2, is given in Section MS-3.1.1 for the simple boundary. This discussion applies to the present case.

In an unbounded domain where the pressure at infinity is imposed, $P_1(\tilde{t})$ is

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[26]The difference of the equation of state is often treated carelessly. Equation (54a) or (59a) is mistaken to be incompressible condition with discarding Eqs. (56) and (60).

[27](i) The factor $5R/2$ is the heat capacity at constant pressure for perfect gas.

(ii) In incompressible fluid, the pressure produces no work as noted just after Eq. (63).

[28](i) The formulas for the interface is the leading-order result of S expansion of kinetic theory analysis. The non-slip condition is also confirmed by it.

(ii) The formulas quoted above are derived for time-independent problems. The results are shown to be applicable to the time-dependent problem with the time scale under discussion in Section M-3.7.3.

(iii) In the formulas in the two books, the subscript S is to be neglected. The subscript 1 showing the order is extended to $2, \cdots$. The formulas with subscript K is to be discarded.

(iv) Keeping the fundamental form of the condition on the interface, we generalize the formula formulas allowing the coefficients, C_4^* and d_4^*, to be functions of position and time, and the discussion is made under the generalized boundary condition.

determined, but in a bounded domain with the simple boundary, it is undetermined. According to the process in Section MS-3.1.1, the velocity field u_{i1} is determined, which is independent of P_1, for the simple boundary. On the other hand, when the boundary is the interface or a part of it is the interface, P_1 is determined by the boundary condition, as shown in Footnote 29 below, owing to the presence of P_1 in the boundary condition on $u_{i1}n_i$.[29] From P_1 determined, the boundary value of $u_{i1}n_i$ on the interface is determined by one of the relations in Eq. (K-4.68c) or Eq. (M-3.119c). From $u_{i1}n_i$ just determined, the boundary value of τ_1 on the interface is determined by another relation in Eq. (K-4.68c) or Eq. (M-3.119c).[30] Thus, all kinds of boundary data are prepared. The velocity field u_{i1} is determined under the boundary data just obtained as in Section MS-3.1.1 with the aid of Footnote 31 below.[31] Then, we can determine the

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[29](i) For the case of the interface, $P(\tilde{t})$ enters the boundary condition on the velocity normal to the boundary. On the other hand, the integral of $u_{i1}n_i$ over the boundary vanishes owing to Eq. (54a), i.e.,

$$\int_S u_{i1}n_i \mathrm{d}S = -\int_V \frac{\partial u_{i1}}{\partial x_i}\mathbf{d}\boldsymbol{x} = 0.$$

This determines the relation of P_1 and the integral of the boundary data P_{w1}, i.e.

$$P_1(\tilde{t})\int_S (1/C_4^*)\mathrm{d}S - \int_S (P_{w1}/C_4^*)\mathrm{d}S = \int_S u_{i1}n_i\mathrm{d}S, = 0.$$

Thus,

$$P_1(\tilde{t}) = \frac{\int_S (P_{w1}/C_4^*)\mathrm{d}S}{\int_S (1/C_4^*)\mathrm{d}S},$$

where S is the surface of the boundary [see Eq. (K-4.68c) or Eq. (M-3.119c)]. It is noted that C_4^* is a constant in the formula by the kinetic theory where the complete condensation boundary is considered. Here, C_4^* is allowed to be a given function of x_i and t to express more general condition of the interface ($0 < -1/C_4^* \le c_{M1}$; c_{M1} : const). Thus, P_1 is determined. With this $P_1(\tilde{t})$, the boundary data of $u_{i1}n_i$ is specified by Eq. (K-4.68c) or Eq. (M-3.119c). Thus, the boundary value of u_{i1} is given by this $u_{i1}n_i$ and Eq. (K-4.68a) or (M-3.119a).

(ii) When the boundary consists of the two kinds of boundaries, By the same reason, the integral $u_{i1}n_i$ over the boundary vanishes, which is divided into the contributions of the two kinds of boundaries. That is,

$$\int_{S_S} u_{i1}n_i\mathrm{d}S + \int_{S_I} u_{i1}n_i\mathrm{d}S = 0,$$

where S_S and S_I indicate, respectively, the simple boundary and the interface. The first integral vanishes because $u_{i1}n_i = 0$ on the simple boundary. Thus, the second integral also vanishes. Then, similarly to note (i), we have

$$P_1(\tilde{t}) = \frac{\int_{S_I} (P_{w1}/C_4^*)\mathrm{d}S}{\int_{S_I} (1/C_4^*)\mathrm{d}S}.$$

Thus, $P_1(\tilde{t})$ is determined, from which $u_{i1}n_i$ on the interface is given by Eq. (K-4.68c) or Eq. (M-3.119c). With this data, the boundary value of u_{i1} is specified on the interface. Together with the condition on the simple boundary, the boundary value of u_{i1} is determined.

[30]The constant d_4^* in the formula is allowed to be a given function of x_i and t as C_4^* is allowed to be so in Footnote 29 ($0 < -d_4^* \le c_{M2}$; c_{M2} : const).

[31]In the process solving the velocity field u_{iS1} in the simple boundary problem in Section MS-3.1.1, P_{S2} is required to satisfy the relation

$$\frac{\partial^2 P_{S2}}{\partial x_i^2} = -2\frac{\partial u_{jS1}}{\partial x_i}\frac{\partial u_{iS1}}{\partial x_j},$$

temperature τ_1 from Eq. (54c) except for the bounded domain with the simple boundary. With the determined P_1 and τ_1, the density ω_1 is determined as

$$\omega_1 = P_1 - \tau_1, \tag{65}$$

by the first relation of Eq. (56).

In the exceptional case, Eq. (54c) contains two unknown functions τ_1 and P_1. We need another condition to determine τ_1 and P_1. When the temperature of the boundary is time-dependent and nonuniform, so is the solution τ_1 in the domain, irrespective of $\partial P_1 / \partial \tilde{t}$. The density ω_1 is given by $P_1 - \tau_1$, which is time-dependent and nonuniform and includes undetermined P_1. In a bounded domain of the simple boundary, the mass of the fluid in the domain is invariant. This has to be confirmed.[32] The condition that the mass in the domain V is invariant is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_V \omega_1 \mathbf{dx} = \frac{\mathrm{d}P_1(\tilde{t})}{\mathrm{d}\tilde{t}} V - \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_V \tau_1 \mathbf{dx} = 0, \tag{66}$$

where V also indicates its volume. On the other hand, the integral form of Eq. (54c) is

$$\frac{\mathrm{d}P_1(\tilde{t})}{\mathrm{d}\tilde{t}} V - \frac{5}{2} \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_V \tau_1 \mathbf{dx} = \int_V \left(\frac{5}{2} u_{i1} \frac{\partial \tau_1}{\partial x_i} - \frac{5}{4} \gamma_2 \frac{\partial^2 \tau_1}{\partial x_j^2} \right) \mathbf{dx}. \tag{67}$$

in order to ensure the subsequent (or future) solenoidal condition of u_{iS1}. According to Section MS-3.1.1, to determine the boundary value of $n_i \partial P_{S2} / \partial x_i$, the time-derivative of the boundary data of $u_{iS1} n_i$ is required (u_{iS1} and P_{S2} corresponds to u_{i1} and P_2 here). It vanishes because $u_{iS1} n_i = 0$ on the simple boundary. In the interface problem, the boundary value of $u_{i1} n_i$ is given by the condition (K-4.68c) or (M-3.119c) and expressed with known data $P_1(\tilde{t})$ and P_{w1} [see Footnote 29 (i)]. Thus, $\partial u_{i1} n_i / \partial \tilde{t}$ on the interface is known. This is used in constructing the boundary data $n_i \partial P_2 / \partial x_i$ as explained in Section MS-3.1.1. Thus, P_2 is determined with an arbitrary additive function $f_2(\tilde{t})$ of $\tilde{t}$. For the combined boundary in (ii) of Footnote 29, we can obtain P_2 by the combination of $n_i \partial P_2 / \partial x_i$ of the two kinds of boundaries, with an arbitrary additive function of $\tilde{t}$. From P_2 thus obtained, we can obtain the solution u_{i1} of Eq (54b) with the boundary condition on u_{i1} for the cases (i) and (ii) of Footnote 29 in the same way as Section MS-3.1.1. This u_{i1} satisfies the solenoidal condition in subsequent $\tilde{t}$. The additive function in P_2 does not influence the solution u_{i1}.

[32](i) It has been made clear by the analysis up to now that the condition $u_{i1} n_i = 0$ on the boundary does not guarantee that mass flow in the diffusion time scale $\tilde{t} = O(1)$ or $t = O[L/(2RT_0)^{1/2}\varepsilon]$ is negligible compared with the quantity of $O(\varepsilon)$ under concern.

(ii) Here, the order of variation of density by inflow to or outflow from a volume $O(L^3)$ by the higher-order velocity $(2RT_0)^{1/2}\varepsilon^2$ in time $L/(2RT_0)^{1/2}\varepsilon$ is estimated. The inflow or outflow of the mass of fluid over the surface $O(L^2)$ of the volume during the time is of the order of $\rho_0 \times (2RT_0)^{1/2}\varepsilon^2 \times L/(2RT_0)^{1/2}\varepsilon \times L^2$, which is $\rho_0 \varepsilon L^3$. Thus, the density varies by the order of $\rho_0 \varepsilon$, which is of the same order as the term ω_1 of the expansion of ρ in ε.

(iii) The flow u_{i2} in the preceding note (ii) is not obtained at the present step of analysis, but the mass conservation condition $\mathrm{d} \int_V \omega_1 \mathbf{dx} / \mathrm{d}\tilde{t} = 0$ in volume V, which is weaker than the simple boundary condition $u_i n_i = 0$, determines $\mathrm{d}P_1 / \mathrm{d}\tilde{t}$ or future P_1 as will be shown in the main text. Then, from the integral of Eq. (55a) over V, we obtain

$$\int_S u_{i2} n_i \mathrm{d}S = 0,$$

with the aid of $u_{i1} n_i = 0$ on S, $\partial u_{i1} / \partial x_i = 0$, and the mass conservation condition, This condition is required to obtain the solution of Eqs. (55a)–(56) satisfying the condition $u_{i2} n_i = 0$ on the simple boundary. A similar structure is repeated in higher-order analysis.

From these two equations, the equation for P_1 is obtained as

$$\frac{\mathrm{d}P_1(\tilde{t})}{\mathrm{d}\tilde{t}} = -\frac{5}{3V} \int_V \left(u_{i1} \frac{\partial \tau_1}{\partial x_i} - \frac{1}{2}\gamma_2 \frac{\partial^2 \tau_1}{\partial x_j^2} \right) \mathbf{dx}. \tag{68}$$

From two equations (54c) and (68), The temperature $\tau(x_i, \tilde{t})$ and the pressure $P_1(\tilde{t})$ are determined. Then, ω_1 is determined by Eq. (65). Thus, we have obtained the required information for perfect gas, up to the order of ε under concern.

At the final stage of the discussion of the process of solution, we briefly describe the process for incompressible fluid, where $\omega = 0$. The process up to the step to obtain the velocity u_{i1} is the same as for perfect gas. That is, the velocity field u_{i1} is determined independently of undetermined $P_1(\tilde{t})$ for a bounded domain with the simple boundary. In the other cases, the velocity field u_{i1} is determined together with $P_1(\tilde{t})$. The energy equation (59c) contains only τ_1 and determined u_{i1} without $P_1(\tilde{t})$ term. It is similar to the equation for perfect gas with difference of numerical coefficient and the absence of $P_1(\tilde{t})$ term. Thus, the solution τ_1 is determined. The pressure $P_1(\tilde{t})$, however, is left undetermined for a bounded domain with the simple boundary. This situation corresponds to the situation described in Section MS-3.2.3. Obviously, the mass is invariant in a bounded domain because the density is invariant, which does not work to determine $P_1(\tilde{t})$. This ambiguity is due to the combination of the two limiting characters of the incompressible fluid and the simple boundary.[33]

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[33]Incompressible fluid is the extreme (or limiting) case of fluid very hard to compress. Suppose that the equation of state is given by

$$\omega = \left(\frac{P - \tau}{1 + \tau} \right)^{2m+1} \quad (m = 0, 1, 2, \cdots), \tag{†1}$$

where $m = 0$ corresponds to perfect gas. Then,

$$\omega_n = 0 \quad \text{for } n \leq 2m, \tag{†2a}$$

$$\omega_{2m+1} = (P_1 - \tau_1)^{2m+1}. \tag{†2b}$$

The equation of state (†1) is reduced to incompressible fluid in the limit $m \to \infty$ because $\omega = 0$ in the limit irrespective of the perturbations P and τ of the order of ε. For $m \geq 1$ (fluid hard to compress; harder for larger m), the conservation equations are the same as those for incompressible fluid up to $n = 2m$, That is, Eqs. (58)–(59c), where the contribution of work done by pressure is absent, apply to fluid hard to compress commonly. In a bounded domain with the simple boundary, the velocity u_{i1} and temperature τ_1 are determined irrespective of P_1, but P_1 cannot be specified within the above set of equations, and the mass in V is invariant up to the level ω_{2m}. The undetermined pressure P_1 is determined by Eq. (†2b) and the mass conservation at the level ω_{2m+1} in the volume V, i.e., $\mathrm{d}\int_V \omega_{2m;+1}\mathbf{dx}/\mathrm{d}\tilde{t} = 0$. From them, we obtain the condition

$$\int_V (P_1 - \tau_1)^{2m} \left(\frac{\mathrm{d}P_1}{\mathrm{d}\tilde{t}} - \frac{\partial \tau_1}{\partial \tilde{t}} \right) \mathbf{dx} = 0.$$

With the aid of Eq. (59c),

$$\frac{\mathrm{d}P_1}{\mathrm{d}\tilde{t}} = \int_V (P_1 - \tau_1)^{2m} \left(-u_{i1} \frac{\partial \tau_1}{\partial x_i} + \frac{5}{6}\gamma_2 \frac{\partial^2 \tau_1}{\partial x_i^2} \right) \mathbf{dx} \bigg/ \int_V (P_1 - \tau_1)^{2m}\mathbf{dx}, \tag{†3}$$

To summarize, the mass and momentum equations (54a) and (54b) of the set derived from the compressible Navier–Stokes set [Eqs. (41)–(44b) and (45)] under the situation given by Eqs. (46a) and (46b) with small ε are of the same form as the corresponding equations (59a) and (59b) of the set derived from the incompressible Navier–Stokes set [Eqs. (41)–(44b) and (57)], but the energy equations (54c) and (59c) of the two sets differ by the work done by pressure. The density ω_1 obtained from u_{i1}, τ_1, and P_1 by the first relation of Eq. (56) does not generally satisfy the incompressible condition (60). Both the density and temperature fields (ω_1, τ_1) are different in the two sets. The variation of the density ω_1 along a fluid-particle path is given by Eq. (64). Even if the temperature τ_1 varies according to Eq. (59c), the density ω_1 determined by Eq. (64) does not generally satisfy the incompressible condition. Further, the two systems have a decisive difference in bounded domain problems with the simple boundary. That is, the pressure P_1 is undetermined in the incompressible fluid system, but no such ambiguity exists in the perfect gas system, where the pressure P_1 is determined.

Finally, it may be repeated that under the situation (46a), the solenoidal condition for u_{i1}, i.e., Eq. (54a) or (59a), is derived only from the mass conservation equation (41) without the help of the equation of state [perfect gas condition (45) or the incompressible condition (57)]. That is, the mass conserva-

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where $\mathrm{d}P_1/\mathrm{d}\tilde{t}$ is expressed with the data of the present state (u_{i1}, τ_1, P_1). Thus, the future P_1, thus (u_{i1}, τ_1, P_1), is determined. In incompressible fluid, the limit $m \to \infty$ is taken first. After the limit $m \to \infty$, the number n showing the level of expansion of the solution in ε is smaller than $2m + 1$ ($=\infty$), and therefore $\omega_n = 0$ for any n, and P_1 remains undetermined. This is the ambiguity mentioned in Section MS-3.2.3.

To summarize, for any finite positive m (≥ 1), the solution (u_{i1}, τ_1, P_1) in a bounded domain with the simple boundary is determined by the conservation equations (58)–(59c) with the mass conservation condition (†3) in V, and the velocity u_{i1} and temperature τ_1 are the same as those of incompressible fluid. This solution is distinct from the solution for perfect gas because the work done by pressure is absent in the energy equation (59c). The solution for incompressible fluid is different from both the solutions, perfect gas ($m = 0$) or fluid hard to compress ($m \geq 1$), on the point of the ambiguity of P_1. The solution for perfect gas is totally different from the solution for incompressible fluid. The solution for fluid hard to compress partially agrees with the solution for incompressible fluid. In this sense, the solution hard to compress is on the side of incompressible fluid. The incompressible condition is too strongly simplified to approximate the solution for fluid hard to compress (note that any fluid is not exactly incompressible). It sacrifices the determinacy of P_1, which is the qualitative difference from perfect gas and fluid hard to coompress.

This kind of situation is often seen when some small parameter is approximated by its extreme value. Here, we have two parameters taken to their limits first (incompressible fluid and the simple boundary, the latter of which can be taken, for example, as the limiting case with the coefficient $1/C_4^*$ in the interface condition being taken to zero). Another well-known example is the approximation of fluid with small viscosity (Navier–Stokes equation with large Reynolds number) by ideal fluid (Euler equation). The ghost effect of infinitesimal curvature in References [8]–[10] and Section MS-7.3 (see also Chapter M-9 and Section MS-7.2) is another aspect of this kind of behavior, where we see that a plane wall or straight pipe is too strong simplification (any plane wall or straight pipe to be discussed is not exactly so). The ghost effect of infinitesimal curvature is shown by analysis with the limiting processes as above taken into consideration.

From the above discussion, it is clear that the two systems under consideration (perfect gas and incompressible fluid) are distinct.

tion equation at the level of $O(\varepsilon)$ are common to perfect gas and incompressible fluid, i.e.,

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \tag{69}$$

but the equation of state is different for the two kinds of fluid, i.e.,

$$\omega_1 = P_1 - \tau_1 \quad \text{(perfect gas)}, \tag{70a}$$
$$\omega_1 = 0 \quad \text{(incompressible fluid)}. \tag{70b}$$

The solenoidal condition (54a) or (59a), i.e., $\partial u_{i1}/\partial x_i = 0$, does not guarantee the invariance of the density ω_1 in the diffusion time scale $\tilde{t} = O(1)$ (see Footnote 32).[34]

3.1.4 Equations derived from the compressible Euler set when the Mach number and the temperature variation are small

Take the Euler set of equations for a monatomic gas, which is derived from the Boltzmann equation in the limit $k \to 0$ (see Section M-3.71). Its nondimensional form with the notation (K-2.36) is given as follows [see Eqs. (M-3.250a)–(M-3.250c) and the equation of state]:

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial \hat{\rho} \hat{v}_j}{\partial x_j} = 0, \tag{71a}$$

$$\frac{\partial \hat{\rho} \hat{v}_i}{\partial \hat{t}} + \frac{\partial \hat{\rho} \hat{v}_j \hat{v}_i}{\partial x_j} + \frac{1}{2}\frac{\partial \hat{p}}{\partial x_i} = 0, \tag{71b}$$

$$\frac{\partial}{\partial \hat{t}}\left[\hat{\rho}\left(\hat{v}_i^2 + \frac{3}{2}\hat{T}\right)\right] + \frac{\partial}{\partial x_j}\left[\hat{\rho}\hat{v}_j\left(\hat{v}_i^2 + \frac{5}{2}\hat{T}\right)\right] = 0, \tag{71c}$$

$$\hat{p} = \hat{\rho}\hat{T}, \tag{71d}$$

where Strouhal number Sh is taken to be unity without loss of generality. We consider the situation where the state of the gas deviates slightly from a uniform equilibrium state at rest. That is,

$$\hat{\rho} = 1 + \hat{\omega}, \ \hat{p} = 1 + \hat{P}, \ T = 1 + \hat{\tau}, \ \hat{v}_i = \hat{u}_i, \tag{72}$$

where the perturbed quantities $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ are small, say of the order of ε. They are expanded as

$$\hat{h} = \hat{h}_1\varepsilon + \hat{h}_2\varepsilon^2 + \cdots, \tag{73}$$

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[34]Note the size ε of the variation of quantities, the diffusion time-scale $[L/(2RT_0)^{1/2}\varepsilon]$ under consideration, and the nonlinearity in the mass conservation equation (41). Owing to these situations, the density variation, i.e,

$$\frac{\partial \omega_1}{\partial \tilde{t}} + u_{i1}\frac{\partial \omega_1}{\partial x_i},$$

along the fluid particle path is obtained from u_i, τ_1, and P_1, and is generally finite ($\neq 0$), affecting $\partial u_{i2}/\partial x_i$ in Eq. (55a).

where $\hat{h} = \hat{\omega}$, $\hat{P}$, $\hat{\tau}$, or $\hat{u}_i$.

We discuss the two cases with different time scale. The first case is

$$\frac{\partial \hat{h}}{\partial \hat{t}} = O(\hat{h}). \tag{74}$$

Substituting the expansions (73) of the variables $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ into the Euler equations (71a)–(71d) and arranging the same-order terms with Eq. (74) in mind, we find that the leading-order variables are governed by the following set of equations:

$$\frac{\partial \hat{\omega}_1}{\partial \hat{t}} + \frac{\partial \hat{u}_{j1}}{\partial x_j} = 0, \tag{75a}$$

$$\frac{\partial \hat{u}_{i1}}{\partial \hat{t}} + \frac{1}{2}\frac{\partial \hat{P}_1}{\partial x_i} = 0, \tag{75b}$$

$$\frac{\partial \hat{P}_1}{\partial \hat{t}} + \frac{5}{3}\frac{\partial \hat{u}_j}{\partial x_j} = 0, \tag{75c}$$

$$\hat{P}_1 = \hat{\omega}_1 + \hat{\tau}_1. \tag{75d}$$

This set is the well-known acoustic equations, which are explained in a standard textbook of gas dynamics, e.g., K-Liepmann & Roshko [1957].

The second case is the case where the variables are slowly varying or the time scale of variation of the variables is long and of the order $1/\varepsilon$:

$$\frac{\partial \hat{h}}{\partial \hat{t}} = \varepsilon O(\hat{h}). \tag{76}$$

Here, we introduce the shrunk time $\hat{t}_\varepsilon$:

$$\hat{t}_\varepsilon = \varepsilon \hat{t}. \tag{77}$$

Then,

$$\frac{\partial \hat{h}}{\partial \hat{t}_\varepsilon} = O(\hat{h}). \tag{78}$$

Substituting the expansion (73) of the variables $\hat{\omega}$, $\hat{P}$, $\hat{\tau}$, and $\hat{u}_i$ into the Euler equations (71a)–(71d) and arranging the same-order terms with Eq. (78) in

mind, we obtain the equations that determine the leading-order variables as[35]

$$\frac{\partial \hat{P}_1}{\partial x_i} = 0, \tag{79a}$$

$$\frac{\partial \hat{u}_{j1}}{\partial x_j} = 0, \tag{79b}$$

$$\frac{\partial \hat{u}_{i1}}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{u}_{i1}}{\partial x_j} + \frac{1}{2}\frac{\partial \hat{P}_2}{\partial x_i} = 0, \tag{79c}$$

$$\frac{5}{2}\frac{\partial \hat{\tau}_1}{\partial \hat{t}_\varepsilon} - \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \frac{5}{2}\hat{u}_{j1}\frac{\partial \hat{\tau}_1}{\partial x_j} = 0, \tag{79d}$$

$$\hat{P}_1 = \hat{\omega}_1 + \hat{\tau}_1. \tag{79e}$$

From Eq. (79a), $\hat{P}_1$ is a function of $\hat{t}_\varepsilon$ only, and thus is determined by the boundary condition.[36] The relation

$$\frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} = \frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{P}_1}{\partial x_j}, \tag{80}$$

obvious from Eq. (79a), is conveniently used in the following discussion. The energy equation (79d) is transformed as

$$\frac{3}{2}\left(\frac{\partial \hat{\tau}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{\tau}_1}{\partial x_j}\right) - \frac{\partial \hat{\omega}_1}{\partial \hat{t}_\varepsilon} - \hat{u}_{j1}\frac{\partial \hat{\omega}_1}{\partial x_j} = 0, \tag{81}$$

by using Eqs. (79e) and (80) for $\partial \hat{P}_1/\partial \hat{t}_\varepsilon$. From Eqs. (81) and (79e), the variation of $\hat{\omega}_1$ along the fluid-particle path is expressed as follows:

$$\frac{\partial \hat{\omega}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{\omega}_1}{\partial x_j} = \frac{3}{5}\left(\frac{\partial \hat{P}_1}{\partial \hat{t}_\varepsilon} + \hat{u}_{j1}\frac{\partial \hat{P}_1}{\partial x_j}\right) = \frac{3}{5}\frac{d\hat{P}_1}{d\hat{t}_\varepsilon}. \tag{82}$$

Equations (81) and (82) are the linearized forms of the isentropic variations of $\hat{\omega}_1$ versus $\hat{\tau}_1$ and $\hat{P}_1$ along the fluid-particle path. The energy equation (81) is conveniently compared with the energy equation of incompressible fluid. For the latter, the last two terms are absent and the temperature is invariant along the fluid-particle path. The difference is the work done by pressure, which can be shown as is done in Section 3.1.3.

The behavior of the gas governed by Eqs. (79a)–(79e) is summarized as follows:

(1): Equations (79b) and (79c) for the velocity field are of the same form as those of incompressible fluid.

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[35] Under the assumptions (72) and (76) or (78), the solenoidal condition (79b) for $\hat{u}_{i1}$ is derived solely from the mass conservation equation (71a). It should not be confused with the incompressible condition.

[36] For example, the pressure is specified at infinity in an unbounded problem.

(2): Depending on the condition of the boundary, $\hat{P}_1$ can be time dependent or independent. (i) If $\hat{P}_1$ is time dependent, the density $\hat{\omega}_1$ varies along the fluid-particle path owing to Eq. (82). (ii) If $\hat{P}_1$ is time independent, the temperature $\hat{\tau}_1$ and the density $\hat{\omega}_1$ are invariant along the fluid-particle path owing to Eqs. (79d) and (82).

(Section 3.1.4: Version 4-00)

4 Appendix K-A

4.1 Boundary condition for Euler equations

In Appendix K-A.10, we discussed the boundary condition for the linearized Euler equations for simple examples. Related discussion is given in Section 3.1.2 in this notes.

5 Appendix K-C

5.1 Numerical procedure for discrete ξ_i and η (Supplement to Sections K-C.2 and K-C.3)

In Appendix K-C, a numerical method of solution of conservation equations, e.g., fluid-dynamic equations, by a kinetic equation is discussed. As noted there, the range of the variables ξ_i and η may be a set of discrete points. Obviously, the integrals with respect to ξ_i and η in the discussion in Section K-C.2 are taken to be the summation over the discrete points because the velocity distribution function f is the collection of delta functions of ξ_i and η with their center at the discrete points. To solve the conservation equations by kinetic-equation approach, a finite number of discrete ξ_i's and η's are sufficient. The number depends on the size of the conservation equations. Here, we will describe the procedure of solution of conservation equations by a kinetic equation with the formulas expressed in discrete ξ_i and η.

The Navier–Stokes equations (K-C.34a)–(K-C.34c) are taken as an example. In this case, $r = 0, 1, \ldots, 4$, i or $j = 1, 2, 3$, and γ_r's are chosen as

$$\gamma_0 = 1, \ \ \gamma_i = \xi_i, \ \ \gamma_4 = \xi_i^2 + (\alpha - 3)\eta^2, \tag{83}$$

where $\alpha \geq 3$ is a constant.[37] From the Navier–Stokes equations (K-C.34a)–(K-C.34c), we can identify ρ_r and $\hat{H}_i^r$ as follows:

$$\rho_0 = \rho, \tag{84a}$$

$$\rho_i = \rho u_i, \tag{84b}$$

$$\rho_4 = \rho(u_j^2 + 2e), \tag{84c}$$

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[37]The constant α is the freedom of a molecule of the gas under consideration. For a monatomic gas without internal degree of freedom, $\alpha = 3$.

$$\hat{H}_i^0 = \rho u_i, \tag{85a}$$

$$\hat{H}_i^j = \rho u_i u_j + p_{ij}, \tag{85b}$$

$$\hat{H}_i^4 = \rho u_i(u_j^2 + 2e) + 2u_j p_{ij} + 2q_i, \tag{85c}$$

where

$$e = \frac{\alpha RT}{2}, \tag{86a}$$

$$p = \rho RT, \tag{86b}$$

$$p_{ij} = p\delta_{ij} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - \mu_B\frac{\partial u_k}{\partial x_k}\delta_{ij}, \tag{86c}$$

$$q_i = -\lambda\frac{\partial T}{\partial x_i}. \tag{86d}$$

The velocity distribution function $f^{(m,n)}$ of Chapman–Enskog type that gives a given set $(\rho_r, \hat{H}_i^r)$ is obtained by the following simultaneous linear algebraic equations, which correspond to Eqs. (K-C.1) and (K–C.5):[38]

$$\sum_{m,n} f^{(m,n)} = \rho_0, \tag{87a}$$

$$\sum_{m,n} \xi_i^{(m)} f^{(m,n)} = \rho_i, \tag{87b}$$

$$\sum_{m,n} [(\xi_i^{(m)})^2 + (\alpha - 3)(\eta^{(n)})^2]f^{(m,n)} = \rho_4, \tag{87c}$$

$$\sum_{m,n} \xi_i^{(m)}\xi_j^{(m)} f^{(m,n)} = \hat{H}_i^j \quad (j \le i), \tag{87d}$$

$$\sum_{m,n} \xi_i^{(m)}[(\xi_j^{(m)})^2 + (\alpha - 3)(\eta^{(n)})^2]f^{(m,n)} = \hat{H}_i^4. \tag{87e}$$

The set of points $(\xi_i^{(m)}, \eta^{(n)})$'s has to be chosen in the way that the determinant formed with the coefficients of $f^{(m,n)}$ does not vanish and that the number of the points is 14 [5 for ρ_r, 6 for $\hat{H}_i^j$ ($j \le i$), and 3 for $\hat{H}_i^4$] in the present case.[39] If one chooses more points for one's convenience, one should impose some extra conditions to make $f^{(m,n)}$ unique.[40]

Once the way to determine the velocity distribution function $f^{(m,n)}$ of the Chapman–Enskog type is fixed, we can proceed to the next procedure described

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[38]Note that the relation between ρ_i and $f^{(m,n)}$ and that between $\hat{H}_i^0$ and $f^{(m,n)}$ are the same. That is, these relations are commonly expressed by Eq. (87b).

[39]i) When the determinant vanishes, we cannot obtain the solution for arbitrary data on the right-hand side.

ii) In the case of $\alpha = 3$ (monatomic gas), the left-hand side of Eq. (87c) is derived from Eq. (87d) with $i = j$. The right-hand side of Eq. (87c) is obtained in the same way because the bulk viscosity vanishes ($\mu_B = 0$) for a monatomic gas. Thus, Eq. (87c) is unnecessary in the analysis.

[40]Sometimes, it is convenient to choose more $(\xi^{(m)}, \eta^{(n)})$ than required to avoid awkward distribution $f^{(m,n)}$.

in Section K-C.3. That is, (i) for given initial data ρ_r, u_i and T, compute $\hat{H}_i^r$ by Eqs. (85b)–(86d); (ii) construct the corresponding velocity distribution function $f^{(m,n)}$ of Chapman–Enskog type by Eqs. (87a)–(87e); (iii) determine the solution of the free-molecular equation (K-C.35) at the next time step with the velocity distribution function constructed in the step (ii) as the initial condition; (iv) compute ρ_r from the resulting velocity distribution function by Eqs (87a)–(87c); (v) return to the step (i) and repeat the process. Then, we can obtain the solution of the Navier–Stokes equations (K-C.34a)–(K-C.34c).

To determine the solution of the free-molecular equation in the step (iii) of the preceding paragraph, we need the boundary condition compatible with the solution of Chapman–Enskog type. For the Navier–Stokes equations, u_i and T are generally specified on the boundary of bodies, but ρ is not specified. The boundary value of ρ is determined with ρ and u_i at the previous time step by Eq. (K-C.34a). Then, we know the boundary data ρ_r. Together with the ρ, u_i, and T inside gas obtained by the free-molecular solution starting at the preceding time step, we can determine the boundary data of $\hat{H}_i^r$ by Eqs. (85b)–(86d). Then, the boundary data $f^{(m,n)}$ compatible with the solution of Chapman–Enskog type are determined by Eqs. (87a)–(87e).

In the process of solution of the free-molecular equation (K-C.35) by a finite difference method, one can make use of a simplifying process introduced in the lattice-Boltzmann-equation approach. Then, the step (iii) in the above procedure is simplified. Let $\boldsymbol{X}$ be a lattice point in x_i space, and let Δt be the time step of computation. We arrange the lattice points $\sum \boldsymbol{X}'s$ and $\boldsymbol{\xi}^{(m)}{}'s$ in such a way that $\boldsymbol{X}+\boldsymbol{\xi}^{(m)}\Delta t$ is also some lattice point in x_i. Then, $f^{(m,n)}$ at $t+\Delta t$ is obtained by shifting $f^{(m,n)}$ at t among the lattice points $\boldsymbol{X}$'s. For example, take rectangular lattices for x_i, where the lattice points $\boldsymbol{X}$'s are expressed as $\boldsymbol{X} = (m_1, m_2, m_3)\Delta x$ with m_i being an integer $(0, \pm 1, 2, \cdots)$. Then, we choose $\boldsymbol{\xi}^{(m)} = (k_1, k_2, k_3)\Delta x/\Delta t$, where k_i is, for example, $k_i = 0, \pm 1$. Then,

$$f(m_i \Delta x, k_i \xi, t + \Delta t) = f((m_i - k_i)\Delta x, k_i \xi, t),$$

where $\xi = \Delta x/\Delta t$.[41] With the above choice of $\boldsymbol{\xi}^{(m)}$, $(m_i - k_i)\Delta x$ is a lattice point of x_i or $\boldsymbol{X}$. Other examples of the lattice point system are found, for example, in Qian, Succi & Orszag [1995].

In Section K-C.3, we discussed the process and validity of solving the conservation equations, or the validity of the procedure of solution given in this section, by making use of the free molecule equation on the basis of the discussion in Section K-C.2. In this discussion, we implicitly assumed the stability of solution of Eq. (K-C.6) with Eq. (K-C.7). Further, it is also assumed that $|\hat{H}_i^r(\rho_r, \nabla\rho_r) - \hat{H}_i^r(\rho_r^c, \nabla\rho_r^c)| = O((\Delta t)^2)$ for $\hat{H}_i^r$ at the step (i) after step (iv) (or at time $t + \Delta t$) when $|\rho_r - \rho_r^c| = O((\Delta t)^2)$ [Eq. (K-C.40)] for the present Navier–Stokes equations.[42] More rigorously, the condition about the size used

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[41]The appropriate lattice ratio of Δx and Δt (thus, ξ) depends on the conservation equations (i.e., the Navier–Stokes equations) under consideration.

[42]This assumption corresponds to the assumption on the space variation of ρ_r and ρ_r^c. For the Euler equations, this assumption is not required because the space-derivative terms are absent in $\hat{H}_i^r$.

in the discussions in Section K-C.3 should be uniformly bounded in some neighborhood of the solution of Eq. (K-C.6). The conditions are mainly determined by the property of solution of the Navier–Stokes equations [generally, Eqs (K-C.12) and (K-C.13)]. Some examples of the numerical computation of the Navier–Stokes equations by the present method are carried out by Kataoka [private communication].

(Section 5.1: Version 3-00)

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