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<tr>
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*Note: The text in the table is in Japanese.*
Kinetic Theory and Fluid Dynamics

Yoshio Sone

(Birkhäuser, Boston, 2002)

Supplementary Notes and Errata

Yoshio Sone

Version 2-02 (11 March 2010)

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Bibliography

Update of bibliography

• [161] Sone, Y., M. Handa, and H. Sugimoto [2002]:
  (to be published) → 299–332

Errata

• p.11, the 10th line
  specular condition → specular reflection

• p.33, the first line in Eq. (3.31c):
  \[
  \frac{\partial S_{ijG0}}{\partial x_i} \rightarrow \frac{\partial S_{ijG0}}{\partial x_j}
  \]

• p.33, the first line in Eq. (3.31d):
  \[
  \frac{\partial S_{ijG1}}{\partial x_i} \rightarrow \frac{\partial S_{ijG1}}{\partial x_j}
  \]

• p.36, Footnote 7:
  Footnote 7 should be replaced by the following statement:
In some cases, the boundary data \( u_{wi}, \tau_w, \) or \( \omega_w \) (thus, \( P_w \)) are unknown beforehand and may depend on the Knudsen number, for example, the surface temperature of a particle set freely in a gas. To include the case, the data \( u_{wi}, \tau_w, \omega_w, \) and \( P_w \) also are expanded in power series of \( k \) in the following discussion. Even when they are independent of \( k \), the \( \phi_w \) depends on \( k \) because \( \phi_w \) generally depends on \( \phi (\zeta_i n_i < 0) \). 

- p.104, the second line in Footnote 18: only \( \Omega_8 \) and \( \Theta_8 \) \( \rightarrow \) \( (\Omega_8, \Theta_8), (\Omega_9, \Theta_9), \) and \( (\Omega_{10}, \Theta_{10}) \)
- p.146, the fifth line: \( \hat{Y}_1(\tilde{\eta}) = \hat{Y}_1(\tilde{\eta}) \rightarrow \hat{Y}_1(\tilde{\eta}) = Y_1(\tilde{\eta}) \)
- p.177, Eq. (6.53):
The following note should be made to Eq. (6.53):

Equation (6.54), which is derived from Eqs. (6.50), (6.52), and \( \hat{v}_{wi} n_i = 0 \), is used in the derivation of Eq. (6.53) from the solvability condition. See the second paragraph in page 130 of Y. Sone, Molecular Gas Dynamics (Birkhäuser, Boston, 2007).

To improve the indefinite expression, the third paragraph on p.177, which starts from its 9th line from the bottom and ends at the second line on p.178, should be replaced by the following paragraph:\footnote{In the new paragraph, the numbering (6.54) comes earlier than (6.53) to avoid the correction of the quotations in the rest of the book.}

Now return to the discussion of \( \hat{f}_{V_m} \). The first relation of the solvability condition (6.48) with \( m = 1 \), where

\[
H_1 = \zeta_i n_i (\partial \hat{f}_{V_0} / \partial y),
\]

reduces to

\[
\frac{\partial (\hat{p}_{V_0} \hat{v}_{iV_0} n_i)}{\partial y} = 0,
\]

from which with the aid of Eqs. (6.50) and \( \hat{v}_{wi} n_i = 0 \), we have

\[
\hat{v}_{iV_0} n_i = 0.
\]

With the use of Eq. (6.54), the scalar product of the second relation of Eq. (6.48) and \( n_i \) reduces to

\[
\partial \hat{p}_{V_0} / \partial y = 0.
\]

Thus, \( \hat{p}_{V_0} \) is a function of \( \chi_1 \) and \( \chi_2 \) only, i.e.,

\[
\hat{p}_{V_0} = \hat{p}_{V_0}(\chi_1, \chi_2).
\]

The remaining three relations \([\text{or the remaining two of the second relation of Eq. (6.48) and its third relation]}\) reduce to identities with the aid of Eq. (6.54).
Supplementary Notes

In the present supplementary notes, the letter K is attached to the labels of equations and sections, etc., in the book “Kinetic Theory and Fluid Dynamics” and the letter M is attached to those in Y. Sone, Molecular Gas Dynamics (Birkhäuser, Boston, 2007) to avoid confusion.

1 Chapter K-4

1.1 Notes on basic equations in classical fluid dynamics

1.1.1 Euler and Navier–Stokes sets

For the convenience of discussions, the basic equations in the classical fluid dynamics are summarized here.

The mass, momentum, and energy-conservation equations of fluid flow are given by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0, \tag{1}
\]

\[
\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial X_j}(\rho v_i v_j + p_{ij}) = 0, \tag{2}
\]

\[
\frac{\partial}{\partial t} \left[ \rho \left( e + \frac{1}{2} v_i^2 \right) \right] + \frac{\partial}{\partial X_j} \left[ \rho v_j \left( e + \frac{1}{2} v_i^2 \right) + v_i p_{ij} + q_i \right] = 0, \tag{3}
\]

where \( \rho \) is the density, \( v_i \) is the flow velocity, \( e \) is the internal energy per unit mass, \( p_{ij} \), which is symmetric with respect to \( i \) and \( j \), is the stress tensor, and \( q_i \) is the heat-flow vector. The pressure \( p \) and the internal energy \( e \) are given by the equations of state as functions of \( T \) and \( \rho \), i.e.,

\[
p = p(T, \rho), \quad e = e(T, \rho). \tag{4}
\]

Especially, for a perfect gas,

\[
p = R\rho T, \quad e = e(T). \tag{5}
\]
Equations (2) and (3) are rewritten with the aid of Eq. (1) in the form
\[ \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} + \frac{\partial p_{ij}}{\partial X_j} = 0, \quad (6) \]
\[ \rho \frac{\partial}{\partial t} \left( e + \frac{1}{2} v_i^2 \right) + \rho v_j \frac{\partial}{\partial X_j} \left( e + \frac{1}{2} v_i^2 \right) + \frac{\partial}{\partial X_j} (v_i p_{ij} + q_i) = 0. \quad (7) \]

The operator \( \partial/\partial t + v_j \partial/\partial x_j \), which expresses the time variation along the fluid particle, is denoted by \( D/Dt \), i.e.,
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial X_j}. \]

Multiplying Eq. (6) by \( v_i \) we obtain the equation for the variation of kinetic energy as
\[ \frac{\rho}{D} \frac{D}{Dt} \left( \frac{1}{2} v_i^2 \right) = -v_i \frac{\partial p_{ij}}{\partial X_j}. \quad (8) \]

Another form of Eq. (3), where Eq. (8) is subtracted from Eq. (7), is given as
\[ \frac{D e}{D t} = -p_{ij} \frac{\partial v_i}{\partial X_j} - \frac{\partial q_i}{\partial X_j}. \quad (9) \]

Noting the thermodynamic relation
\[ \frac{D e}{D t} = T \frac{D s}{D t} + \frac{p}{\rho^2} \frac{D \rho}{D t}, \]
where \( s \) is the entropy per unit mass, and Eq. (1), Eq. (9) is rewritten as
\[ \rho \frac{D s}{D t} = -\frac{1}{T} \left[ (p_{ij} - \rho \delta_{ij}) \frac{\partial v_i}{\partial X_j} + \frac{\partial q_i}{\partial X_j} \right]. \quad (11) \]

Equation (11) expresses the variation of the entropy of a fluid particle.

Equations (1)–(4) contain more variables than the number of equations. Thus, in the classical fluid dynamics, the stress tensor \( p_{ij} \) and the heat-flow vector \( q_i \) are assumed in some ways. The Navier–Stokes set of equations (or the Navier–Stokes equations) is Eqs. (1)–(4) where \( p_{ij} \) and \( q_i \) are given by
\[ p_{ij} = p \delta_{ij} - \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial \rho}{\partial X_k} \delta_{ij} \right) - \mu_B \frac{\partial v_k}{\partial X_k} \delta_{ij}, \quad (12) \]
\[ q_i = -\lambda \frac{\partial T}{\partial X_i}, \quad (13) \]
where \( \mu, \mu_B, \) and \( \lambda \) are, respectively, called the viscosity, bulk viscosity, and thermal conductivity of the fluid. They are functions of \( T \) and \( \rho \). The Euler set of equations (or the Euler equations) is Eqs. (1)–(4) where \( p_{ij} \) and \( q_i \) are given by
\[ p_{ij} = p \delta_{ij}, \quad q_i = 0, \quad (14) \]
or the Navier–Stokes equations with \( \mu = \mu_B = \lambda = 0 \).

For the Navier–Stokes equations, in view of the relations (12) and (13), the entropy variation is expressed in the form\(^2\)

\[
\frac{\rho Ds}{Dt} = \frac{1}{T} \left[ \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \mu_B \left( \frac{\partial v_k}{\partial X_k} \right)^2 + \frac{\partial}{\partial X_i} \left( \lambda \frac{\partial T}{\partial X_i} \right) \right].
\]

(15)

For the Euler equations, for which \( p_{ij} \) and \( q_i \) are given by Eq.(14), the entropy of a fluid particle is invariant, i.e.,

\[
\frac{\rho Ds}{Dt} = 0.
\]

(16)

For an incompressible fluid, the first relation of Eq.(4) is replaced by

\[
\frac{3}{D\rho}{D}{Dt} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0.
\]

(17)

Thus, from Eqs.(1) and (17),

\[
\frac{\partial v_i}{\partial X_i} = 0.
\]

(18)

Equation (12) for the Navier–Stokes-stress tensor reduces to

\[
p_{ij} = p\delta_{ij} - \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right).
\]

(19)

The first term on the right-hand side of Eq.(9) reduces to

\[
-p_{ij} \frac{\partial v_i}{\partial X_j} = \left[ p\delta_{ij} - \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right] \frac{\partial v_i}{\partial X_j}
\]

\[
= \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2.
\]

Thus, Eq.(9) reduces to

\[
\omega = \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left( \lambda \frac{\partial T}{\partial X_j} \right).
\]

(20)

\(^2\)Note the following transformation:

\[
\frac{\partial v_i}{\partial X_j} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \frac{1}{3} \frac{\partial v_i}{\partial X_j} \delta_{ij} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right).
\]

The second term in the last expression is easily seen to vanish.

\(^3\)The density is invariant along fluid-particle paths. If \( \rho \) is of uniform value \( \rho_0 \) initially, it is a constant, i.e.,

\[
\rho = \rho_0.
\]

In a time-independent (or steady) problem, the density is constant along streamlines.

---

2Note the following transformation:

\[
\frac{\partial v_i}{\partial X_j} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right) = \frac{1}{2} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right)^2 + \frac{1}{3} \frac{\partial v_i}{\partial X_j} \delta_{ij} \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} - \frac{2}{3} \frac{\partial v_k}{\partial X_k} \delta_{ij} \right).
\]

3The density is invariant along fluid-particle paths. If \( \rho \) is of uniform value \( \rho_0 \) initially, it is a constant, i.e.,

\[
\rho = \rho_0.
\]
To summarize, the Navier–Stokes equations for incompressible fluid are

\[
\frac{\partial v_i}{\partial X_i} = 0, 
\]

\[
\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial X_j} = -\frac{\partial p}{\partial X_i} + \frac{\partial}{\partial X_j} \left[ \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right) \right], 
\]

\[
\rho \frac{\partial e}{\partial t} + \rho v_j \frac{\partial e}{\partial X_j} = \mu \left( \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 + \frac{\partial}{\partial X_j} \left( \lambda \frac{\partial T}{\partial X_j} \right), 
\]

with the incompressible condition (17) being supplemented, i.e.,

\[
\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial X_j} = 0. 
\]

### 1.1.2 Boundary condition for the Euler set

In Chapter K-7, we discussed the asymptotic behavior for small Knudsen numbers of a gas around its condensed phase where evaporation or condensation with a finite Mach number is taking place, and derived the Euler equations and their boundary conditions that describe the overall behavior of the gas in the limit that the Knudsen number tends to zero. The number of boundary conditions on the evaporating condensed phase is different from that on the condensing one. We will try to understand the structure of the Euler equations giving the non-symmetric feature of the boundary conditions by a simple but nontrivial case.

Consider, as a simple case, the two-dimensional boundary-value problem of the time-independent Euler equations in a bounded domain for an incompressible ideal fluid of uniform density. The mass and momentum-conservation equations of the Euler set are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, 
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, 
\]

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y}, 
\]

where \( \rho \) is the density, which is uniform, \( (u, v) \) is the flow velocity, and \( p \) is the pressure. Owing to Eq. (23), the stream function \( \Psi \) can be introduced as

\[
u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. 
\]
Eliminating $p$ from Eqs. (24) and (25), we have\(^4\)
\[
u \frac{\partial \Omega}{\partial x} + v \frac{\partial \Omega}{\partial y} = 0,
\]
(27)
where $\Omega$ is the vorticity, i.e.,
\[
\Omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}.
\]
(28)
From Eqs. (26) and (27),
\[
\frac{\partial \Psi \frac{\partial \Omega}{\partial y}}{\partial x} - \frac{\partial \Psi \frac{\partial \Omega}{\partial x}}{\partial y} = 0.
\]
(29)
This equation shows that $\Omega$ is a function of $\Psi$,\(^5\) i.e.,
\[
\Omega = F(\Psi).
\]
(30)
\(^4\)The following equation is formed from them:
\[
\partial \text{Eq. (24)}/\partial y - \partial \text{Eq. (25)}/\partial x = 0.
\]
\(^5\)This can be seen with the aid of theorems on implicit functions (see Bronshtein & Semendyayev [1997], Buck [1965], Takagi [1961]):
\[
\Omega = \Omega(x, y), \quad \Psi = \Psi(x, y).
\]
\((*)\)
Solving the second equation with respect to $x$, we have
\[
x = \dot{x}(\Psi, y).
\]
\((***)\)
With this relation into Eq. (*),
\[
\Omega = \Omega(\dot{x}(\Psi, y), y) = \Omega(\Psi, y),
\]
\((\sharp a)\)
\[
\Psi = \Psi(\dot{x}(\Psi, y), y) = \dot{\Psi}(\Psi, y).
\]
\((\sharp b)\)
That is, $\Omega$ is expressed as a function of $\Psi$ and $y$. From Eqs. $(\sharp a)$ and $(\sharp b)$,
\[
\frac{\partial \Omega(\Psi, y)}{\partial y} = \frac{\partial \Omega(\dot{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Omega(x, y)}{\partial x} \frac{\partial \dot{x}(\Psi, y)}{\partial y} + \frac{\partial \Omega(x, y)}{\partial y},
\]
\((\sharp\sharp a)\)
\[
\frac{\partial \Psi(\Psi, y)}{\partial y} = 0.
\]
\((\sharp\sharp b)\)
On the other hand,
\[
\frac{\partial \Psi(\Psi, y)}{\partial y} = \frac{\partial \Psi(\dot{x}(\Psi, y), y)}{\partial y} = \frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \dot{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y}.
\]
Thus,
\[
\frac{\partial \Psi(x, y)}{\partial x} \frac{\partial \dot{x}(\Psi, y)}{\partial y} + \frac{\partial \Psi(x, y)}{\partial y} = 0.
\]
\((\dagger)\)
From Eqs. (29), $(\sharp\sharp a)$ and $(\dagger)$, we have
\[
\frac{\partial \Omega(\Psi, y)}{\partial y} = 0, \quad \text{or} \quad \Omega = \Omega(\Psi).
\]
This functional relation between \( \Omega \) and \( \Psi \) is a local relation, and therefore \( F \) may be a multivalued function of \( \Psi \). From Eqs. (28) and (30),

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = F(\Psi). \tag{31}
\]

Consider a boundary-value problem in a simply-connected bounded domain, where \( \Psi \) is given on the boundary \((\Psi = \Psi_B)\). Introduce a coordinate \( s \) \((0 \leq s < S)\) along the boundary in the direction encircling the domain counterclockwise. Then, the fluid flows into the domain on the boundary where \( \partial \Psi_B / \partial s < 0 \), and the fluid flows out from the domain on the boundary where \( \partial \Psi_B / \partial s > 0 \). When \( F \) is given, the problem is a standard boundary-value problem. In the present problem, we have a freedom to choose \( F \) on the part where \( \partial \Psi_B / \partial s < 0 \) or \( \partial \Psi_B / \partial s > 0 \). For example, take the case where \( \partial \Psi_B / \partial s < 0 \) for \( 0 < s < S_m \) and \( \partial \Psi_B / \partial s > 0 \) for \( S_m < s < S \), and choose the distribution \( \Omega_B(s) \) of \( \Omega \) along the boundary for the part \( 0 < s < S_m \). By the choice of \( \Omega_B \), the function \( F(\Psi) \) is determined in the following way. Inverting the relation \( \Psi = \Psi_B(s) \) between \( \Psi \) and \( s \) on the part \( 0 < s < S_m \), i.e., \( s(\Psi) \), and noting the relation (30), we find that \( F \) is given by

\[
F(\Psi) = \Omega_B(s(\Psi)). \tag{32}
\]

Then, the boundary-value problem is fixed. That is, Eq. (31) is fixed as \(^6\)

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \Omega_B(s(\Psi)), \tag{33}
\]

and the boundary condition is given as \( \Psi = \Psi_B(s) \). This system is a standard from the point of counting of the number of boundary conditions. Obviously, from Eq. (28), the solution of the above system automatically satisfies condition \( \Omega = \Omega_B(s) \) along the boundary for \( 0 < s < S_m \). We cannot choose the distribution of \( \Omega \) on the boundary for \( S_m < s < S \).

The energy-conservation equation of the incompressible Euler set is given by Eq. (20) with \( \mu = \lambda = 0 \), i.e.,

\[
u \frac{\partial e}{\partial x} + v \frac{\partial e}{\partial y} = 0, \quad \text{or} \quad \frac{\partial \Psi}{\partial y} \frac{\partial e}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial e}{\partial y} = 0, \tag{34}
\]

where \( e \) is the internal energy. Thus, \( e \) is a function of \( \Psi \), i.e.,

\[
e = F_1(\Psi). \tag{35}
\]

In the above boundary-value problem, therefore, \( e \) can be specified on the the part \((0 < s < S_m)\) of the boundary, but no condition can be specified on other part \((S_m < s < S)\) and vice versa.\(^7\)

\(^6\)There is still some ambiguity. The case where there is a region with closed stream lines \( \Psi(x, y) = \text{const} \) inside the domain is not excluded.

\(^7\)From the second relation on \( e \) of Eq. (4) and the uniform-density condition, the condition on \( e \) can be replaced by the condition on the temperature \( T \).
To summarize, we can specify three conditions for $\Psi$, $\Omega$, and $e$ on the part $\partial \Psi_B / \partial s < 0$ ($\partial \Psi_B / \partial s > 0$) of boundary but one condition for $\Psi$ on the other part $\partial \Psi_B / \partial s > 0$ ($\partial \Psi_B / \partial s < 0$). The number of the boundary conditions is not symmetric and consistent with that derived by the asymptotic theory.

1.1.3 Equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small

It is widely said that the set of equations derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small is the incompressible Navier–Stokes set. This statement should be made precise. The difference is briefly explained in the book “Kinetic Theory and Fluid Dynamics” in connection with the equations derived by the $S$ expansion from the Boltzmann equation in Section K-4.3. Here, we explicitly show the process of analysis from the compressible Navier–Stokes set. In Section K-4.3, the time-independent case is discussed. Here, we discuss the problem without this restriction. The following discussion is applied to the former case simply by eliminating the time-derivative terms.

Take a monatomic perfect gas, for which the internal energy per unit mass is $3RT/2$. The corresponding Navier–Stokes set of equations is written in the nondimensional variables introduced by Eq.(K-2.57) in Section K-2.10 as follows:

\[
Sh \frac{\partial \omega}{\partial t} + \frac{\partial (1 + \omega) u_i}{\partial x_i} = 0, \tag{36}
\]

\[
Sh \frac{\partial (1 + \omega) u_i}{\partial t} + \frac{\partial}{\partial x_j} \left( (1 + \omega) u_i u_j + \frac{1}{2} P_{ij} \right) = 0, \tag{37}
\]

\[
Sh \frac{\partial}{\partial t} \left[ (1 + \omega) \left( \frac{3}{2} (1 + \tau) + u_i^2 \right) \right] + \frac{\partial}{\partial x_j} \left[ (1 + \omega) u_j \left( \frac{3}{2} (1 + \tau) + u_i^2 \right) + u_i (\delta_{ij} + P_{ij}) + Q_j \right] = 0. \tag{38}
\]

The nondimensional stress tensor $P_{ij}$, and heat-flow vector $Q_i$, are expressed as

\[
P_{ij} = P \delta_{ij} - \frac{\mu_0 (2RT_0)^{1/2}}{p_0 L} (1 + \tilde{\mu}) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right), \tag{39a}
\]

\[
Q_i = - \frac{\lambda_0 T_0}{L p_0 (2RT_0)^{1/2}} (1 + \tilde{\lambda}) \frac{\partial \tau}{\partial x_i}. \tag{39b}
\]

Here, $\tilde{\mu}$ and $\tilde{\lambda}$ are, respectively, the nondimensional perturbed viscosity and thermal conductivity defined by

\[
\mu = \mu_0 (1 + \tilde{\mu}), \quad \lambda = \lambda_0 (1 + \tilde{\lambda}),
\]

\footnote{For a monatomic gas, the bulk viscosity vanishes, i.e., $\mu_B = 0$.}
where $\mu_0$ and $\lambda_0$ are, respectively, the values of the viscosity $\mu$ and the thermal conductivity $\lambda$ at the reference state. The $\bar{\mu}$ and $\bar{\lambda}$ are functions of $\tau$ and $\omega$. The first relation of the equation of state [Eq. (5)] is expressed as

$$P = \omega + \tau + \omega \tau.$$  \hspace{1cm} (40)

Take a small parameter $\varepsilon$, and consider the case where

$$u_i = O(\varepsilon), \quad \omega = O(\varepsilon), \quad \tau = O(\varepsilon), \quad \text{Sh} = O(\varepsilon),$$  \hspace{1cm} (41a)

$$\frac{\mu_0 (2RT_0)^{1/2}}{p_0 L} = \gamma_1 \varepsilon, \quad \frac{\lambda_0 T_0}{L p_0 (2RT_0)^{1/2}} = \frac{5}{4} \gamma_2 \varepsilon,$$  \hspace{1cm} (41b)

thus,

$$P = O(\varepsilon), \quad \bar{\mu} = O(\varepsilon), \quad \bar{\lambda} = O(\varepsilon).$$

According to the definition of $u_i$ in Eq. (K-2.57), $\varepsilon$ is of the order of the Mach number. In view of this and the definition of the Prandtl number $Pr = 5 R \mu / 2 \lambda$ (see Section K-3.9), $\gamma_1$ and $\gamma_2$ are, respectively, of the orders of $1/Re$ and $1/PrRe$ ($Re$: the Reynolds number). According to Eq. (K-2.41a), the condition $\text{Sh} = O(\varepsilon)$ in Eq. (41a) means that the time scale $t_0$ of the variation of variables is of the order of $L/(2RT_0)^{1/2} \varepsilon$, which is of the order of time scale of viscous diffusion. Thus, we are considering the case where the Mach number is small, the Reynolds and Prandtl numbers are of the order of unity, and the time scale of variation of the system is of the order of the time scale of viscous diffusion. We can take $t_0 = L/(2RT_0)^{1/2} \varepsilon$ without loss of generality. Then,

$$\text{Sh} = \varepsilon.$$  \hspace{1cm} (42)

Corresponding to the above situation, $u_i, \omega, P,$ and $\tau$ are expanded in power series of $\varepsilon$, i.e.,

$$u_i = u_{i1} \varepsilon + u_{i2} \varepsilon^2 + \cdots,$$  \hspace{1cm} (43a)

$$\omega = \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \cdots,$$  \hspace{1cm} (43b)

$$P = P_1 \varepsilon + P_2 \varepsilon^2 + \cdots,$$  \hspace{1cm} (43c)

$$\tau = \tau_1 \varepsilon + \tau_2 \varepsilon^2 + \cdots,$$  \hspace{1cm} (43d)

$$\bar{\mu} = \bar{\mu}_1 \varepsilon + \bar{\mu}_2 \varepsilon^2 + \cdots,$$  \hspace{1cm} (43e)

$$\bar{\lambda} = \bar{\lambda}_1 \varepsilon + \bar{\lambda}_2 \varepsilon^2 + \cdots,$$  \hspace{1cm} (43f)

$$P_{ij} = P_{ij1} \varepsilon + P_{ij2} \varepsilon^2 + \cdots,$$  \hspace{1cm} (43g)

$$Q_i = Q_{i1} \varepsilon^2 + \cdots.$$  \hspace{1cm} (43h)

Substituting Eqs. (43a)–(43h) with Eqs. (41b) and (42) into Eqs. (36)–(38) with Eqs. (39a) and (39b), and arranging the same order terms of $\varepsilon$, we have

$$\frac{\partial u_{i1}}{\partial x_i} = 0, \quad \frac{\partial P_1}{\partial x_i} = 0, \quad \frac{\partial u_{i1}}{\partial x_i} = 0.$$
\[
\begin{align*}
\frac{\partial \omega_1}{\partial t} + \frac{\partial \omega_1 u_{i1}}{\partial x_i} + \frac{\partial u_{i2}}{\partial x_i} &= 0, \\
\frac{\partial u_{i1}}{\partial t} + \frac{\partial u_{i1} u_{j1}}{\partial x_j} + \frac{1}{2} \frac{\partial P_2}{\partial x_i} - \frac{\gamma_1}{2} \frac{\partial}{\partial x_i} \left( \frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{k1}}{\partial x_k} \delta_{ij} \right) &= 0, \\
\frac{3}{2} \frac{\partial P_1}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{5}{2} u_{j2} + \frac{5}{2} P_1 u_{j1} - \frac{5}{4} \frac{\partial \tau_1}{\partial x_j} \right) &= 0,
\end{align*}
\]

and so on. At the leading order, the equations derived from Eqs. (36) and (38) degenerate into the same equation \( \frac{\partial u_{i1}}{\partial x_i} = 0 \). Owing to this degeneracy, in order to solve the variables from the lowest order successively, the equations should be rearranged by combination of equations of staggered orders. Thus, we rearrange the equations as follows:

\[
\begin{align*}
\frac{\partial P_1}{\partial x_i} &= 0, \quad (44) \\
\frac{\partial u_{i1}}{\partial x_i} &= 0, \quad (45a) \\
\frac{\partial u_{i1}}{\partial t} + u_{j1} \frac{\partial u_{i1}}{\partial x_j} &= - \frac{1}{2} \frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2} \frac{\partial^2 u_{i1}}{\partial x_j^2}, \quad (45b) \\
5 \frac{\partial \tau_1}{\partial t} - \frac{\partial P_1}{\partial t} + 5 \frac{u_{i1}}{2} \frac{\partial \tau_1}{\partial x_i} &= \frac{5}{4} \frac{\partial^2 \tau_1}{\partial x_i^2}, \quad (45c)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_{i2}}{\partial x_i} &= - \frac{\partial \omega_1}{\partial t} - \frac{\partial \omega_1 u_{i1}}{\partial x_i}, \quad (46a) \\
\frac{\partial u_{i2}}{\partial t} + u_{j1} \frac{\partial u_{i2}}{\partial x_j} + u_{j2} \frac{\partial u_{i1}}{\partial x_j} &= \frac{1}{2} \left( \frac{\partial P_3}{\partial x_i} - \omega_1 \frac{\partial P_2}{\partial x_i} \right) + \frac{\gamma_1}{2} \frac{\partial}{\partial x_i} \left( \frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{k2}}{\partial x_k} \delta_{ij} \right) \\
&\quad - \frac{\gamma_1}{2} \frac{\partial^2 u_{i1}}{\partial x_j^2} + \frac{\gamma_1}{2} \frac{\partial}{\partial x_j} \left[ \rho_1 \left( \frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} \right) \right], \quad (46b) \\
3 \frac{\partial P_2}{\partial t} + \frac{3}{2} \frac{u_{j1}}{\partial x_j} \frac{\partial P_2}{\partial x_j} + \frac{5}{2} \left( P_1 \frac{\partial u_{j2}}{\partial x_j} - \frac{\partial u_{i2}}{\partial t} - \frac{\partial (\omega_1 u_{j2} + \omega_2 u_{j1})}{\partial x_j} \right) &= \frac{5 \gamma_2}{4} \frac{\partial}{\partial x_i} \left( \frac{\partial \tau_2}{\partial x_i} + \lambda_1 \frac{\partial \tau_1}{\partial x_i} \right) + \frac{\gamma_1}{2} \left( \frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i} \right)^2, \quad (46c)
\end{align*}
\]

where

\[ P_1 = \omega_1 + \tau_1, \quad P_2 = \omega_2 + \tau_2 + \omega_1 \tau_1. \quad (47) \]

These equations are very similar to Eqs. (M-3.265)–(M-3.268) [or Eqs. (K-3.87)–(K-3.90) for the time-independent case] obtained by the S expansion of the Boltzmann equation in Section M-3.7.2 (or Section K-3.2.2 for the time independent case).
In order to compare Eqs. (45a)–(45c) and (47) with the incompressible Navier–Stokes equations (18)–(22), we will rewrite the latter equations for the situation where the former equations are derived. The starting equations are Eqs. (36)–(39b) and the nondimensional form of Eq. (17), i.e.,

\[ \text{Sh} \frac{\partial \omega}{\partial \hat{t}} + u_i \frac{\partial \omega}{\partial x_i} = 0, \]  

(48)

instead of Eq. (40).\(^9\) The analysis is carried out in a similar way and the equations corresponding to Eqs. (45a)–(45c) are

\[ \frac{\partial u_{1i}}{\partial x_i} = 0, \]  

(49a)

\[ \frac{\partial u_{11}}{\partial \hat{t}} + u_{i1} \frac{\partial u_{11}}{\partial x_j} = \frac{1}{2} \frac{\partial P_2}{\partial x_i} + \frac{\gamma_1}{2} \frac{\partial^2 u_{11}}{\partial x_j^2}, \]  

(49b)

\[ \frac{3}{2} \frac{\partial \tau_{1i}}{\partial \hat{t}} + \frac{3}{2} u_{i1} \frac{\partial \tau_{1i}}{\partial x_i} = \frac{5}{4} \frac{\partial^2 \tau_{1i}}{\partial x_j^2}. \]  

(49c)

Equations (49a) and (49b) are, respectively, of the same form as Eqs. (45a) and (45b). Equation (45c) is rewritten with the aid of Eqs. (44) and (47) as

\[ \frac{3}{2} \frac{\partial \tau_{1i}}{\partial \hat{t}} + \frac{3}{2} u_{i1} \frac{\partial \tau_{1i}}{\partial x_i} \left( \frac{\partial \omega_{11}}{\partial \hat{t}} + u_1 \frac{\partial \omega_{11}}{\partial x_i} \right) = \frac{5}{4} \frac{\partial^2 \tau_{1i}}{\partial x_j^2}. \]  

(50)

The difference of Eq. (45c) or (50) from Eq. (49c) is

\[ \frac{\partial \omega_{11}}{\partial \hat{t}} + u_1 \frac{\partial \omega_{11}}{\partial x_i}, \]

which vanishes for an incompressible fluid. The work \( W \) done per unit time on unit volume of fluid by pressure, given by \(-p_0(2RT_0)^{1/2}L^{-1} \partial(1 + P)u_i/\partial x_i\), is transformed with the aid of Eqs. (44), (45a), and (46a) in the following way:

\[ \frac{W}{p_0(2RT_0)^{1/2}L^{-1}} = - \frac{\partial (1 + P)u_i}{\partial x_i} \]

\[ = - \frac{\partial u_{1i}}{\partial x_i} \varepsilon - \left( \frac{P_1 u_{i1}}{\partial x_i} + u_1 \frac{\partial P_1}{\partial x_i} + \frac{\partial u_{12}}{\partial x_i} \right) \varepsilon^2 + \cdots \]

\[ = - \frac{\partial u_{12}}{\partial x_i} \varepsilon^2 + \cdots \]

\[ = \left( \frac{\partial \omega_{11}}{\partial \hat{t}} + u_1 \frac{\partial \omega_{11}}{\partial x_i} \right) \varepsilon^2 + \cdots. \]

The work vanishes up to the order considered here for an incompressible fluid, because \( \partial u_i/\partial x_i = 0 \) and \( \partial P_1/\partial x_i = 0 \) (see Footnotes 10 and 11). That is, Eq. (45c) differs from Eq. (49c) by the amount of the work done by pressure.

---

9 As the internal energy \( \varepsilon \), \( 3RT/2 \) \([= 3RT_0(1 + \tau)/2]\) is chosen for consistency.

10 From Eqs. (36) and (48), we have \( \partial u_{1i}/\partial x_i = 0 \).

11 We also obtain \( \partial P_1/\partial x_i = 0 \).
To summarize, the mass and momentum-conservation equations of the set derived from the compressible Navier–Stokes set when the Mach number and the temperature variation are small are of the same form as the corresponding equations for the incompressible Navier–Stokes set, but the energy-conservation equation differs by the work done by pressure.\textsuperscript{12}

Finally, it may be noted that the solenoidal condition for \( u_{i1} \), i.e., Eq. (45a) or (49a), is derived only from the mass conservation equation (36) without the help of the incompressible condition (48) under the situation (41a). The density \( \omega_1 \) obtained from Eqs. (44)–(45c) with the first relation of Eq. (47) does not generally satisfy the incompressible condition (48) with \( \omega = \omega_1 \) and \( u_i = u_{i1} \).\textsuperscript{13}

2 Appendix K-A

2.1 Boundary condition for Euler equations

In Appendix K-A.10, we discussed the boundary condition for the linearized Euler equations for simple examples. Related discussion is given in Section 1.1.2 in this notes.

References


\textsuperscript{12}When the density \( \rho \) is uniform initially, for which \( \rho \) is a constant for an incompressible fluid, the viscosity and thermal conductivity are constants, and heat production by viscosity is neglected, Eqs. (49a)–(49c) can be compared directly with Eqs. (45a)–(45c) and (47), without carrying expansion, and the same results are obtained.

\textsuperscript{13}It is easily seen that the velocity \( u_{i1} \) vanishes, the pressure \( P_1 \) is a constant, and the temperature \( \tau_1 \) (thus, the density \( \omega_1 \)) varies with time in initial-value problems where the velocity is zero and temperature is nonuniform (strictly, non-harmonic) initially, and the pressure is time independent at infinity. See also the example given in Section K-4.10.3, where the velocity vanishes and the density varies with time.