DILUTE SOLUTION PROPERTIES OF SEMIFLEXIBLE STAR POLYMERS

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1 Introduction

1.1 Background

Dilute solution behavior of polymers is mainly governed by average chain dimensions of the solute polymers, which are determined by the following three basic factors: primary molecular structures, e.g., linear or branched, chain stiffness, and interactions between segments constituting the polymer chains, the last factor being the so-called excluded-volume effects. Quite a number of theoretical and experimental studies have therefore long been made of the relation between the chain dimension and the three basic factors, and an almost complete understanding has already been built up of the contribution of each individual factor to the chain dimension.\textsuperscript{1,2} However, in order to obtain a comprehensive understanding of solution behavior of real polymers having a variety of molecular structures, it is desirable to pursue further the study for cases in which two or more of the factors contribute to the chain dimension. In this thesis, we consider the case in which both the primary molecular structure and chain stiffness are simultaneously concerned, i.e., dilute solution behavior of semiflexible branched polymers.

Although quite a few studies have been made so far of the relation between the primary molecular structure and observable quantities relating to the average dimension of unperturbed chains without excluded volume, e.g., the mean-square radius of gyration $<S^2>$ and intrinsic viscosity $[\eta]$,\textsuperscript{1,3–9} almost all of them were of flexible polymers except for the theoretical study by Mansfield and Stockmayer,\textsuperscript{7} the Monte Carlo (MC) one by Zimm,\textsuperscript{8} and the experimental one by Goodson and Novak.\textsuperscript{9} Mansfield and Stockmayer calculated $<S^2>$ of the Kratky–Porod (KP) wormlike\textsuperscript{2,10} star chain (without excluded volume) and examined the behavior of its ratio $g_S$ to $<S^2>$ of the corresponding linear one having the same total length of the chain contour (molecular weight). Zimm evaluated $[\eta]$ of “wormlike” four- and six-arm stars by the use of his rigid-body ensemble approximation.\textsuperscript{11} Unfortunately, however, his calculation is a preliminary survey, as mentioned by himself. Goodson and Novak synthesized three-arm star poly($n$-hexyl isocyanate) (PHIC), which is a typical semiflexible polymer, by living titanium-catalized coordination polymerization,\textsuperscript{9} and then investigated the behavior of $<S^2>$, $[\eta]$, and $g_S$ as functions of molecular weight. However, they did not examine the behavior of the ratio $g_\eta$ of $[\eta]$ of the star polymer to that of
the corresponding linear one, since there is no available theory of $[\eta]$ or $g_\eta$ for semiflexible regular star polymers. Recall that a star polymer with uniform arm length is generally referred to as the regular star (polymer).

As for (perturbed) polymer chains in good solvents, short-ranged repulsive interactions work between segments, expanding the individual chains and yielding an effective volume $V_E$ excluded to one chain by the presence of another. The volume $V_E$ is another measure of the average chain dimension, although well-defined only in a good solvent system or in the perturbed state, and is proportional to the second virial coefficient $A_2$, which may be determined simultaneously with $\langle S^2 \rangle$ from light scattering measurements. Of course, there have been several theoretical and experimental studies$^{5,6,12,13}$ of $A_2$ of the flexible star polymers, but none of them has considered the effects of chain stiffness on $A_2$ for the star polymers.

Under such present situations mentioned above, our attention is necessarily focused on $[\eta]$ and $A_2$ of the semiflexible regular three-arm star polymers and we make an extensive theoretical and computer simulation study of the effects of chain stiffness on $g_\eta$ in the unperturbed state and the ratio $g_{A_2}$ of $A_2$ of the regular three-arm star chain to that of the corresponding linear one in the perturbed state.

For this purpose, we should choose an appropriate model which may well reproduce dimensional behavior of the real regular three-arm star polymers with various chain stiffness in both the unperturbed and perturbed states. Recently, MC studies$^{14,15}$ were made of $\langle S^2 \rangle$ and $A_2$ for flexible linear polymers in dilute solution using the freely rotating chain$^{1,2}$ of bond angle $\theta = 109^\circ$ (\(\sim\) tetrahedral bond angle) and with the Lennard–Jones (LJ) 6-12 intra- and intermolecular potential$^{16}$ between beads in a cutoff version. It was shown that the behavior of the MC data for $\langle S^2 \rangle$ and $A_2$ in the unperturbed and perturbed states may be described by the theory on the basis of the KP or helical wormlike$^2$ chain model as well as the cases of real flexible polymers. We therefore use this model with various values of $\theta$, and calculate $g_\eta$ and $g_{A_2}$ for the regular three-arm stars with various chain stiffness by MC simulation. Further, it seems to be convenient and useful for an analysis of experimental data to construct a theoretical expression for $g_\eta$ like the Mansfield–Stockmayer expression for $g_S$.$^7$ We then construct an interpolation formula for $g_\eta$ of the semiflexible regular three-arm star chain on the basis of the KP chain without excluded volume.

The evaluation of $[\eta]$ requires some comments. As is well known, there is no method for the exact theoretical evaluation of $[\eta]$ since the hydrodynamic interaction between the segments composing polymer chains cannot be treated properly. We then use the Kirkwood–Riseman (KR) approximation$^{1,17}$ in the MC and theoretical calculations in this study as used in many theories$^{1,2,4}$ for polymer hydrodynamics. However, the KR
approximation may not always give a good approximate value of $\eta$ because of the preaveraging of the Oseen hydrodynamic tensor. Therefore, in the MC simulation for $\eta$ in this study, we also evaluate $\eta$ without the preaveraging of the Oseen tensor on the basis of the Zimm rigid-body ensemble approximation\textsuperscript{11} and the Fixman method.\textsuperscript{18,19} The Zimm method gives an upper bound\textsuperscript{18,20} to $[\eta]$ and the Fixman method gives a lower bound.\textsuperscript{18} On the basis of the two bounds to $[\eta]$ for the regular three-arm star and linear chains having the same total chain length and chain stiffness, an upper and lower bounds of $g_\eta$ may be evaluated. Comparing the KR value of $g_\eta$ with them, we examine the validity of the KR approximation in the evaluation of $g_\eta$ of semiflexible regular three-arm stars.

The second virial coefficient $A_2$ of semiflexible or stiff chains having an interaction potential with an attractive tail, e.g., the LJ potential also requires some comments. Schoot and Odijk\textsuperscript{21} pointed out that two rigid rods with an attractive tail, which are close to each other, prefer to be parallel rather than perpendicular to each other and their $A_2$ may therefore be suppressed. This may be also the case with the present MC model chains (the freely rotating chains interacting with the LJ 6-12 potential in a cutoff version) when the chains become stiffer. We therefore examine effects of the attractive tail on $A_2$ by comparing results for the MC chains with the LJ potential and the hard-sphere one.

### 1.2 Outline

The plan of this thesis is as follows.

In Chapter 2, Monte Carlo (MC) results of the intrinsic viscosity $[\eta]$ and also of the mean-square radius of gyration $\langle S^2 \rangle$ are presented for regular three-arm star freely rotating chains of bond angles $\theta = 109^\circ$, $165^\circ$, and $175^\circ$ and with the Lennard–Jones (LJ) 6-12 potentials between beads having the parameter values corresponding to the $\Theta$ temperature, in the range of the total number $n$ of bonds in the chain from 60 to 300. Three kinds of approximate values of $[\eta]$ are calculated by the use of the Kirkwood–Riseman (KR) approximation, the Zimm rigid-body ensemble approximation which gives an upper bound $[\eta]\textsuperscript{(U)}$ to $[\eta]$, and by the Fixman method which gives a lower bound $[\eta]\textsuperscript{(L)}$, the KR value of $[\eta]$ being designated $[\eta]\textsuperscript{(KR)}$. On the basis of the three kinds of MC values of $[\eta]$ so obtained, the behavior of the ratio $g_\eta$ of $[\eta]$ for the star chain to that for the linear one, both having the same $n$, is examined as a function of the reduced contour length $\lambda L$ as defined as the total contour length $L$ of the corresponding Kratky–Porod (KP) wormlike chain divided by its stiffness parameter $\lambda^{-1}$, the values of $\lambda L$ being determined from an analysis of the present and previous MC data for $\langle S^2 \rangle$ on the basis of the KP chain. It is found that the KR value $g_\eta\textsuperscript{(KR)}$ of $g_\eta$ as defined by $[\eta]\textsuperscript{(KR)}(\text{star})/[\eta]\textsuperscript{(KR)}(\text{linear})$ lies between the values of an upper bound $g_\eta\textsuperscript{(U)}$ and a lower one $g_\eta\textsuperscript{(L)}$, which are defined
by \( [\eta]^{(U)}(\text{star})/\eta^{(L)}(\text{linear}) \) and \( [\eta]^{(L)}(\text{star})/\eta^{(U)}(\text{linear}) \), respectively, irrespective of the values of \( \lambda L \). Further, the difference between the two bounds becomes very small for small \( \lambda L \), indicating that \( g_n^{(KR)} \) is a good approximate value for semiflexible or stiff chains.

In Chapter 3, the ratio \( g_\eta \) of \( [\eta] \) of the KP wormlike regular three-arm star touched-bead model to that of the KP linear one, both having the same (reduced) total contour length \( L \) and (reduced) bead diameter \( d_b \), is numerically evaluated in the KR approximation. Prior to the evaluation of \( g_\eta \), an interpolation formula for the mean reciprocal of the end-to-end distance of the once-broken KP chain, which is necessary for the theoretical calculation in the KR approximation, is constructed on the basis of the asymptotic forms derived by the use of the Daniels method near the random-coil limit and the \( \epsilon \) method near the rod limit and also on the basis of the Monte Carlo results. From an examination of the behavior of \( g_\eta \) as a function of \( L \) and \( d_b \), it is found that the ratio \( g_\eta/g_\eta^0 \) of \( g_\eta \) to the rod-limiting value \( g_\eta^0 \) of \( g_\eta \) monotonically increases from 1 to 2.03 with increasing \( L \) and is almost independent of \( d_b \) for \( d_b \lesssim 0.2 \), although the behavior of \( g_\eta \) itself as a function of \( L \) remarkably depends on \( d_b \). An empirical interpolation formula is then constructed for \( g_\eta/g_\eta^0 \) as a function of \( L \), which is considered to be useful for practical purposes.

In Chapter 4, MC results of the second virial coefficient \( A_2 \) and also \( \langle S^2 \rangle \) are presented for regular three-arm star and linear freely-rotating chains of bond angles \( \theta \) ranging from 109° (typical flexible chain) to 175° (typical semiflexible or stiff chain) with the LJ 6-12 potentials between beads corresponding to a good solvent system, the range of the total number \( n \) of bonds from 30 to 900. On the basis of the MC values of \( A_2 \) so obtained, an examination is made of effects of chain stiffness on the ratio \( g_{A_2} \) of \( A_2 \) of the star chain to that of the linear one, both chains having the same \( n \) and \( \theta \). It is then found that \( g_{A_2} \) is rather insensitive to change in \( \theta \) (chain stiffness) in contrast to the cases of the ratios \( g_S \) and \( g_\eta \) related to \( \langle S^2 \rangle \) and \( [\eta] \), respectively, which remarkably decrease with increasing \( \theta \).

In Chapter 5, MC results of \( \langle S^2 \rangle \) and \( A_2 \) are presented for the two freely rotating chains with the LJ 6-12 potential and the hard-sphere (HS) one in the range of the bond angle \( \theta \) from 109° (typical flexible chain) to 175° (typical semiflexible or stiff chain) and in the range of the number \( n \) of bonds from 6 to 1000. It is shown that a value may be properly assigned to the collision diameter of the HS potential so that \( \langle S^2 \rangle \) of the chain with the HS potential agrees well with that of the chain with the LJ one whose parameter values correspond to a good-solvent condition, irrespective of the chain stiffness. It is then found that \( A_2 \) of the latter chain becomes remarkably smaller than that of the former as the chain stiffness is increased. The result implies that the binary cluster approximation does not seem to work well for typical semiflexible and stiff polymers.
References

1 Introduction
2 Intrinsic Viscosity of Semiflexible Three-Arm Stars: A Monte Carlo Study

2.1 Introduction

In this chapter, we examine effects of chain stiffness on the ratio $g_\eta$ of the intrinsic viscosity $[\eta]$ of the regular three-arm star chain to that of the corresponding linear one, both having the same total chain length by Monte Carlo (MC) simulation using the freely rotating chain\textsuperscript{1,2} with a cutoff version of the Lennard–Jones (LJ) 6-12 potential\textsuperscript{3} between beads, which is essentially the same as that used in previous MC studies\textsuperscript{4,5} of $\langle S^2 \rangle$ and the second virial coefficient of linear chains. On the basis of sets of sample configurations generated by the use of the Metropolis method\textsuperscript{6} with the pivot algorithm\textsuperscript{7–9} for the regular three-arm star and linear chains, both having the same total chain length, the values of $[\eta]$ for the two kinds of chains and then $g_\eta$ may be evaluated.

As mentioned in Chapter 1, there is no method for the exact evaluation of $[\eta]$ since the hydrodynamic interaction between the segments composing polymer chains cannot be treated properly. We therefore evaluate $[\eta]$ in the three approximate ways: the Kirkwood–Riseman (KR) approximation\textsuperscript{1,10} the Zimm rigid-body ensemble approximation\textsuperscript{11} and the Fixman method\textsuperscript{12,13}. We note that the Zimm method gives an upper bound to $[\eta]$ and the Fixman method gives a lower bound. On the basis of the three kinds of MC values, we examine the behavior of $[\eta]$ and $g_\eta$ as functions of the total chain length and the chain stiffness.

2.2 Models and Methods

2.2.1 Models

The MC model used in this study is the regular three-arm star freely rotating chain, each arm composed of $m$ successive bonds of length unity, so that it is composed of $3m$ ($= n$) bonds, in total, and $3m + 1$ beads whose centers are located at $3m - 3$ junctions of two successive bonds on the arms, at the three terminal ends, and at the branch point (center), as illustrated in Figure 2.1. The angle between each pair of the bonds connected to the center is fixed to be $120^\circ$, so that those bonds are on the same plane. For convenience, the
three arms are designated the first, second, and third ones and the beads on the $i$th arm ($i = 1, 2, 3$) are numbered $(i - 1)m + 1, (i - 1)m + 2, \ldots, im$ from the center to the end, with the center bead numbered 0. The $i$th bond vector $l_i$ ($1 \leq i \leq 3m; i \neq m + 1, 2m + 1$) connects the centers of the $(i - 1)$th and $i$th beads with its direction from the $(i - 1)$th to $i$th bead, and $l_{m+1}$ and $l_{2m+1}$ are from the 0th to the $(m + 1)$th and $(2m + 1)$th beads, respectively. All the $3m - 3$ bond angles $\theta$ (not supplement) except for those around the center are fixed, so that the configuration of the entire chain may be specified by the set of $3m - 3$ rotation angles $\{\phi_{3m-3}\} = (\phi_1, \ldots, \phi_{m-1}, \phi_{m+1}, \ldots, \phi_{2m-1}, \phi_{2m+1}, \ldots, \phi_{3m-1})$ apart from its position and orientation in an external Cartesian coordinate system, where $\phi_i$ is the internal rotation angle around $l_i$.

The linear chain model, the counterpart of the above star one, is the same as that used in refs 4 and 5, i.e., the freely rotating chain composed of $n$ bonds of length unity and $n + 1$ beads, whose centers are located at the $n - 1$ junctions of two successive bonds and at the two terminal ends. We set $n$ equal to $3m$. The beads are numbered 0, 1, 2, $\ldots, n$ from one end to the other, and $l_i$ ($1 \leq i \leq n$) connects the centers of the $(i - 1)$th and $i$th beads with its direction from the $(i - 1)$th to the $i$th bead. All the $n - 1$ bond angles are fixed at $\theta$, so that the configuration of the linear chain may be specified by the set of $n - 2$ internal rotation angles $\{\phi_{n-2}\} = (\phi_2, \phi_3, \ldots, \phi_{n-1})$ apart from its position and orientation in the external Cartesian coordinate system.

The total potential energy $U$ of the regular three-arm star chain as a function of
\{\phi_{3m-3}\} may be given by

\[
U(\{\phi_{3m-3}\}) = \sum_{i=0}^{1} \sum_{j=i+1}^{2} \sum_{k,l=1}^{m} h(k+l-4) u(R_{(im+k)(jm+l)})
+ \sum_{i=0}^{2} \sum_{j=1}^{n-4} \sum_{k=j+4}^{m} u(R_{(im+j)(im+k)}) + \sum_{i=0}^{2} \sum_{j=4}^{m} u(R_{0(im+j)})
\] (regular three-arm star) \ (2.1)

and that of the linear chain as a function of \{\phi_{n-2}\} by

\[
U(\{\phi_{n-2}\}) = \sum_{i=0}^{n-4} \sum_{j=i+4}^{n} u(R_{ij}) \quad \text{linear} \ (2.2)
\]

where \(h(x)\) is a unit step function such that \(h(x) = 1\) for \(x \geq 0\) and \(h(x) = 0\) for \(x < 0\) and \(R_{ij}\) is the distance between the centers of the \(i\)th and \(j\)th beads. We note that in eqs 2.1 and 2.2 the interactions between the third-neighbor beads along the chain have been neglected, since they seem to make the chain locally take the \(cis\) conformation to excess. We adopt as the pair potential \(u(R)\) (of mean force) the cutoff version of the LJ 6-12 potential given by

\[
u(R) = \infty \quad \text{for} \ 0 \leq R < c\sigma
= u^{LJ}(R) \quad \text{for} \ c\sigma \leq R < 3\sigma
= 0 \quad \text{for} \ 3\sigma \leq R
\] \quad \text{(2.3)}

where \(u^{LJ}(R)\) is the LJ potential given by

\[
u^{LJ}(R) = 4\epsilon \left[ \left( \frac{\sigma}{R} \right)^{12} - \left( \frac{\sigma}{R} \right)^{6} \right]
\] \quad \text{(2.4)}

with \(\sigma\) and \(\epsilon\) the collision diameter and the depth of the potential well at the minimum of \(u^{LJ}(R)\), respectively. We note that \(u(R)\) given by eqs 2.3 is the LJ potential cut off at the upper bound \(3\sigma\). The lower bound \(c\sigma\) in eqs 2.3 has been introduced for numerical convenience; the factor \(c\) is properly chosen so that the Boltzmann factor \(e^{-u^{LJ}/k_B T}\) may be regarded as numerically vanishing compared to unity, where \(k_B\) is the Boltzmann constant and \(T\) is the absolute temperature. In practice, in double-precision computation, we put

\[
c = \left[ 2/(1 + \sqrt{1 + 36T^*}) \right]^{1/6}
\] \quad \text{(2.5)}

so that \(e^{-u^{LJ}/k_B T} \lesssim 2 \times 10^{-16}\) for \(0 \leq R < c\sigma\), where \(T^*\) is the reduced temperature defined by \(T^* = k_B T/\epsilon\). Further, we put \(\sigma = 1\) for simplicity, as previously done.
2.2.2 MC Sampling

The MC simulation algorithm used in this study is the same as that used in refs 4 and 5 and is essentially the same as the Stellman–Gans\textsuperscript{8} version of the pivot algorithm\textsuperscript{7,9} for a sequential generation of chain configurations. A brief description is given of the algorithm for the regular three-arm star chain, which is in principle the same as that for the linear chain described previously.\textsuperscript{4}

First, we generate an initial configuration \( \{l_{3m}\} = (l_1, l_2, \cdots, l_{3m}) \) by trial and error. A trial set of the rotation angles \( \{\phi_{3m-3}\} \) are randomly generated in the interval \([-\pi, \pi]\).

The branch point of the star chain is located at the origin of the external Cartesian coordinate system and the first bond vector \( l_{m+1} \) on the \((i + 1)\)th arm is laid in the \(xz\) plane such that \( l_{1}^T = (0, 0, 1), l_{m+1}^T = (\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}) \), and \( l_{2m+1}^T = (-\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}) \), with the superscript \( T \) indicating the transpose. The succeeding bond vector \( l_{im+j} \) on the \((i + 1)\)th arm may be given by

\[
l_{im+j} = C_i \cdot A(\theta, \phi_{im+1}) \cdot A(\theta, \phi_{im+2}) \cdots A(\theta, \phi_{im+j-1}) \cdot (0, 0, 1)^T
\]

where \( C_i \) \((i = 0, 1, 2)\) is the orthogonal transformation matrix given by

\[
C_i = \begin{pmatrix}
-1/2 & 0 & \sqrt{3}/2 \\
0 & 1 & 0 \\
-\sqrt{3}/2 & 0 & -1/2
\end{pmatrix}^i
\]

and \( A(\theta, \phi_{i-1}) \) is the orthogonal transformation matrix from the \(i\)th to the \((i - 1)\)th localized Cartesian coordinate system\textsuperscript{2} given by

\[
A(\theta, \phi) = \begin{pmatrix}
\cos \theta \cos \phi & \sin \phi & -\sin \theta \cos \phi \\
\cos \theta \sin \phi & -\cos \phi & -\sin \theta \sin \phi \\
-\sin \theta & 0 & -\cos \theta
\end{pmatrix}
\]

The \(i\)th localized system associated with \( l_i \) \((2 \leq i \leq 3m; i \neq m + 1, 2m + 1)\) is defined as follows. The \(z_i\) axis is taken along \( l_i\), the \(x_i\) axis is in the plane of \( l_{i-1} \) and \( l_i \) with its direction chosen at an obtuse angle with \( l_{i-1} \), and the \(y_i\) axis completes the right-handed system, so that \( \phi_i = 0 \) in the \textit{trans} conformation. Since the branch point is located at the origin, the vector position \( r_{im+j} \) \((i = 0, 1, 2; j = 1, 2, \cdots, m)\) of the center of the \(j\)th bead on the \((i + 1)\)th arm is given by

\[
r_{im+j} = \sum_{k=1}^{j} l_{im+k}
\]

If all the distances between the centers of beads are greater than or equal to \( c\sigma \), the above-given trial configuration is adopted as the initial configuration. If not, this trial is repeated until an initial satisfactory configuration is obtained.
Next the initial configuration \{l_{3m}\} so obtained is sequentially changed by the pivot algorithm. A trial configuration is generated by rotating the chain with a given present configuration by an angle \(\Delta \phi\) randomly chosen in the interval \([-\pi, \pi]\) around a bond randomly chosen from the \(3m - 3\) bonds \(l_i\) (\(1 \leq i \leq 3m - 1; i \neq m, 2m\)). If the \(p\)th bond is chosen, the set of rotation angles \(\{\phi_{3m-3}'\}\) in the trial configuration are given by

\[
\phi'_i = \phi_i + \delta_{ip} \Delta \phi \quad (1 \leq i \leq 3m - 1; i \neq m, 2m)
\]

with \(\delta_{ip}\) the Kronecker delta. The trial configuration \(\{l_{3m}'\}\) is generated from \(\{\phi_{3m-3}'\}\) by the use of the numerical procedure described previously. Practically, the bond vectors (not the vector positions of the centers of beads) are rotated around \(l_j\) in order to suppress a roundoff error characteristic of computer work. (Note that \(l_i\) with \(l_i = (T_i, \pi, \pi)\) is chosen, the set of rotation angles \(f_m\) randomly chosen from the 3 bonds \(\pi, \pi\) around a bond.) The trial configuration \(l_i\) is the unit matrix and the rotation matrix \(R(l_i; \Delta \phi)\) is given by

\[
R(l_i; \Delta \phi) = (\cos \Delta \phi)I + (1 - \cos \Delta \phi) \begin{pmatrix}
  l_x^2 & l_x l_y & l_x l_z \\
l_y l_x & l_y^2 & l_y l_z \\
l_z l_x & l_z l_y & l_z^2
\end{pmatrix}
\]

with \(l_i^T = (l_x, l_y, l_z)\). With this rotation, \(l_i'(p < i \leq jm)\) is renormalized to \(l_i'_{(corr)}\) so that \(|l_i'_{(corr)}| = 1\), i.e.,

\[
l_i'_{(corr)} = l_i'/\sqrt{(l_i'_{ix}^2 + l_i'_{iy}^2 + l_i'_{iz}^2)}^{1/2}
\]

\[
\simeq \left[1 - \frac{1}{2}(l_i'_{ix}^2 + l_i'_{iy}^2 + l_i'_{iz}^2 - 1)\right]l_i' \quad \text{for } p < i \leq jm
\]

in order to suppress a roundoff error characteristic of computer work. (Note that \(|l_i' - l_i'_{(corr)}| \ll 1\).) The vector position of the centers of beads in the trial configuration are calculated from eq 2.9 with the corrected \(\{l_{3m}'\}\).

Then, the adoption of the trial configuration so generated as the next one is determined by the Metropolis method of importance sampling on the basis of the total potential energy given by eq 2.1 for the trial and present configurations, i.e., the trial configuration is adopted as the next one with the (transition) probability \(\tau(\{\phi_{3m-3}'\}|\{\phi_{3m-3}\})\) defined by

\[
\tau(\{\phi_{3m-3}'\}|\{\phi_{3m-3}\}) = \min[1, e^{-(U(\{\phi_{3m-3}'\}) - U(\{\phi_{3m-3}\}))/k_BT}]
\]
If the trial configuration is discarded, the present one is again adopted as the next one. The practical procedure is as follows. If the distance between the centers of beads of a pair is smaller than \( c\sigma \), the trial configuration is discarded and the present one is adopted as the next one. If not, the total potential energy \( U(\{\phi'_{3m-3}\}) \) is calculated from eq 2.1 and the Boltzmann factor in eq 2.13 is evaluated. Then the probability \( \tau \) is compared with a random number \( x \) for \( 0 \leq x < 1 \). If \( \tau \) is greater than the random number, the trial configuration is adopted as the next one. If not, the present configuration is adopted as the next one. In computing \( U(\{\phi'_{3m-3}\}) \), the “zippering” method\(^8,14\) has been used for speedy calculation of the double sum in eq 2.1.

By the use of the model and algorithm described above for the regular three-arm star chain and previously for the linear chain,\(^4\) we sample one configuration at every \( M_{\text{nom}} \) (nominal) pivot steps, and \( N \) configurations in total, for both chains. Then, the ensemble average \( \langle A \rangle \) of a variable \( A \) may be evaluated on the basis of the \( N \) sample configurations so generated. If \( A \) is independent of the orientation of the chain like the squared radius of gyration \( S^2 \), it is a function only of \( \{\phi_\alpha\} \) with \( \alpha \) being equal to \( 3m - 3 \) for the star chain and \( n - 2 \) for the linear one, so that \( \langle A \rangle \) may be evaluated from

\[
\langle A \rangle = N^{-1} \sum_{\{\phi_\alpha\}} A(\{\phi_\alpha\}) \tag{2.14}
\]

where the sum is taken over the \( N \) sample configurations. If \( A \) is dependent on the orientation, on the other hand, the above average must be evaluated after randomizing the orientation for each sample configuration. Note that \( N \times M_{\text{nom}} \) pivot steps are required to carry out a MC run in both cases. All the MC runs have been carried out at the reduced temperature \( T^* = 3.72 \) (\( \Theta \)) at which \( \langle S^2 \rangle / n \) for the linear chains with \( \sigma = 1 \) and \( \theta = 109^\circ \) becomes a constant independent of \( n \) for very large \( n \).\(^4\)

All numerical work has been done by the use of a personal computer with an AMD Athlon XP 2600+ CPU. A source program coded in C has been compiled by the GNU C compiler version 2.95.3 with real variables of double precision. For a generation of pseudo-random numbers, we have used the subroutine package MT19937 supplied by Matsumoto and Nishimura\(^15\) instead of the subroutine RAND included in the standard C library.

### 2.2.3 Mean-Square Radius of Gyration

The mean-square radius of gyration \( \langle S^2 \rangle \) has been evaluated from eq 2.14, where \( S^2 \) for each sample chain has been calculated from

\[
S^2 = \frac{1}{n+1} \sum_{i=0}^{n} S_i^2 \tag{2.15}
\]
where $S_i$ is the magnitude of the vector distance $\mathbf{S}_i$ of the center of the $i$th bead from the center of mass of the chain, i.e.,

$$S_i = \mathbf{r}_i - \mathbf{r}_{\text{c.m.}}$$  \hspace{1cm} (2.16)

with $\mathbf{r}_{\text{c.m.}}$ the vector position of the center of mass of the sample chain given by

$$\mathbf{r}_{\text{c.m.}} = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{r}_i$$  \hspace{1cm} (2.17)

### 2.2.4 Intrinsic Viscosity

The intrinsic viscosity $[\eta]$ of the bead model without the contribution of each single-bead (i.e., without consideration of a distribution of frictional force on the surface of each bead) may be written in terms of the (total) frictional force $\mathbf{F}_i^T = (F_{ix}, F_{iy}, F_{iz})$ exerted by the $i$th bead on a surrounding solvent and $\mathbf{S}_i^T = (S_{ix}, S_{iy}, S_{iz})$ defined by eq 2.16 as follows,$^1$

$$[\eta] = N_A \frac{2M}{\eta_0 g} \sum_{i=0}^{n} (F_{iy}S_{ix} + F_{ix}S_{iy})$$  \hspace{1cm} (2.18)

where $N_A$ is the Avogadro constant, $M$ is the molecular weight, $\eta_0$ is the viscosity coefficient of the solvent, and $g$ is the velocity gradient of a simple shear flow field. The frictional force $\mathbf{F}_i$ satisfies the following hydrodynamic equations,

$$\mathbf{F}_i = \zeta(\mathbf{u}_i - \mathbf{v}_i^0) - \zeta \sum_{j=0}^{n} \mathbf{T}_{ij} \cdot \mathbf{F}_j \hspace{1cm} (i = 0, 1, 2, \cdots, n)$$  \hspace{1cm} (2.19)

where $\zeta$ is the friction coefficient of the bead given by the Stokes relation $\zeta = 3\pi\eta_0d_b$ with $d_b$ the hydrodynamic diameter of the bead, $\mathbf{u}_i$ is the velocity of the $i$th bead, $\mathbf{v}_i^0$ is the unperturbed velocity of the solvent at the center of the $i$th bead, and $\mathbf{T}_{ij}$ is the Oseen tensor representing the hydrodynamic interaction between the $i$th and $j$th beads.

In the numerical evaluation of $[\eta]$, the linear simultaneous equations 2.19 are first solved for $\mathbf{F}_i$ for the polymer chain immersed in the solvent having the unperturbed flow field,

$$\mathbf{v}_i^0 = ge_x e_y \cdot \mathbf{r}_i$$  \hspace{1cm} (2.20)

in the external Cartesian coordinate system $(e_x, e_y, e_z)$, and then the ensemble averages on the right-hand side of eq 2.18 are evaluated by the use of the values of $\mathbf{F}_i$ so obtained. Since $[\eta]$ cannot be evaluated exactly, as mentioned already, it has been approximately evaluated in the three ways: the KR approximation,$^1,10$ the Zimm rigid-body ensemble approximation,$^11$ and the Fixman method.$^{12,13}$ In what follows, we consider the touched-bead model so that $d_b = 1$. 

2.2.4.1 KR Approximation

In the KR approximation, the polymer chain in the unperturbed flow filed given by eq 2.20 is assumed to move with the translational velocity equal to that of the unperturbed flow at its center of mass and to rotate around its center of mass with the angular velocity $-\frac{1}{2}g\mathbf{e}_z$, so that $\mathbf{u}_i$ in eq 2.19 may be given by

$$\mathbf{u}_i = \frac{1}{2}g(\mathbf{e}_x\mathbf{e}_y - \mathbf{e}_y\mathbf{e}_x) \cdot \mathbf{S}_i$$

(2.21)

Further, the Oseen tensor defined by

$$\mathbf{T}_{ij} = \frac{1}{8\pi \eta_0} R_{ij} \left( \mathbf{I} + \frac{\mathbf{R}_{ij}\mathbf{R}_{ij}}{R_{ij}^2} \right)$$

(2.22)

with $\mathbf{I}$ the unit tensor and $\mathbf{R}_{ij} = \mathbf{S}_j - \mathbf{S}_i$ is replaced by its ensemble average $\langle \mathbf{T}_{ij} \rangle$, which may be written in the form,

$$\langle \mathbf{T}_{ij} \rangle = \frac{1}{6\pi \eta_0} \langle R_{ij}^{-1} \rangle \mathbf{I}$$

(2.23)

In practice, $[\eta]$ in the KR approximation, which we designate $[\eta]^{(KR)}$, has been evaluated numerically as follows: First evaluate the average $\langle R_{ij}^{-1} \rangle$ by the use of the $N$ sample chains, then solve the linear simultaneous equations 2.19 with $\langle \mathbf{T}_{ij} \rangle$ so obtained in place of $\mathbf{T}_{ij}$ for each sample chain having a random orientation with respect to the external system, and finally calculate $[\eta]^{(KR)}$ from eq 2.18. We note that this numerical recipe correctly gives $[\eta]^{(KR)}$, although it seems apparently somewhat different from the original one. $^{1,10}$

2.2.4.2 Rigid-Body Ensemble Approximation

The rigid-body ensemble approximation has been proposed by Zimm$^{11}$ in order to evaluate $[\eta]$ by MC simulation, in which each sample chain having a rigid conformation immersed in the shear flow given by eq 2.20 is assumed to move with an unknown translational velocity $\mathbf{U}$ and to rotate around its center of mass with the angular velocity $-\frac{1}{2}g\mathbf{e}_z$ under the condition that the total frictional force $\mathbf{F}$ exerted by the chain on the solvent vanishes. Then $\mathbf{u}_i$ in eq 2.19 may be given by

$$\mathbf{u}_i = \mathbf{U} + \frac{1}{2}g(\mathbf{e}_x\mathbf{e}_y - \mathbf{e}_y\mathbf{e}_x) \cdot \mathbf{S}_i$$

(2.24)

In this approximation, $\mathbf{F}_i$ and also $\mathbf{U}$ are evaluated for each sample chain having a random orientation by solving the linear simultaneous equations 2.19 with $\mathbf{u}_i$ and $\mathbf{v}_i^0$ given by eqs 2.24 and 2.20, respectively, along with the condition,

$$\mathbf{F} = \sum_{i=0}^{n} \mathbf{F}_i = \mathbf{0}$$

(2.25)
Then $[\eta]$ may be calculated from eq 2.18 with $F_i$ so evaluated.

In the practical evaluation of $[\eta]$, we have used the modified Oseen tensor\textsuperscript{16,17} $T_{m,ij}$ defined by

$$T_{m,ij} = T_{ij} + \frac{1}{16\pi\eta_0 R_{ij}} \left( \frac{d_{ib}}{R_{ij}} \right)^2 \left( \frac{1}{3} I - \frac{R_{ij} R_{ij}}{R_{ij}^2} \right) \quad (2.26)$$

in place of the (original) Oseen tensor $T_{ij}$ in order to avoid possible numerical troubles, i.e., the Zwanzig singularities\textsuperscript{18,19} associated with the touched-bead model. We note that $\langle T_{m,ij} \rangle$ becomes identical with $\langle T_{ij} \rangle$.

Since the rigid-body ensemble approximation gives an upper bound to $[\eta]$, as pointed out by Wilemski and Tanaka\textsuperscript{20} and by Fixman,\textsuperscript{12} $[\eta]$ obtained in this approximation is designated $[\eta]^{(U)}$.

### 2.2.4.3 Fixman Method

Fixman\textsuperscript{12} has proposed an MC method for evaluating a lower bound $[\eta]^{(L)}$ to $[\eta]$. On the basis of a formal expression for $[\eta]$ by the use of the formal solution of the diffusion equation for the polymer chain, he has derived the following inequality,

$$[\eta] \geq \frac{N_A}{M\eta_0 g^2 k_B T} \left[ 2 \langle V^T \cdot D_a^{-1} \cdot V \rangle - (V^T \cdot D_a^{-1} \cdot D \cdot D_a^{-1} \cdot V) \right] \quad (2.27)$$

where $V$ is the column matrix with the $i$th element $V_i$ given by

$$V_i = \frac{1}{2} g(e_x e_y + e_y e_x) \cdot S_i \quad (2.28)$$

$D$ is the $3(n+1) \times 3(n+1)$ (true) diffusion matrix whose $ij$ element is the $3 \times 3$ matrix $D_{ij}$ defined by

$$D_{ij} = k_B T[\varsigma^{-1} \delta_{ij} I + (1 - \delta_{ij}) T_{m,ij}] \quad (2.29)$$

and $D_a$ is an approximate expression for $D$, which is desired to be easier to handle. Both the two tensors $D$ and $D_a$ are required to be positive definite, symmetric, and divergenceless, and the former given by eq 2.29 satisfies the requirements. We note that in eq 2.27 equality holds if $D_a = D$.

For any given expression for $D_a$ satisfying the above requirements, the two kinds of averages on the right-hand side of eq 2.27 may be evaluated on the basis of the MC sample configurations. The remaining problem is to find a tractable expression for $D_a$, for which Fixman has proposed the following expression,\textsuperscript{13}

$$D_a^{-1} = \alpha(D_p^{-1} + D'_p^{-1}) \quad (2.30)$$

In this expression, $D_p$ and $D'_p$ the $3(n+1) \times 3(n+1)$ matrices whose $ij$ elements are the $3 \times 3$ matrices given by

$$D_{p,ij} = k_B T[\varsigma^{-1} \delta_{ij} I + (1 - \delta_{ij}) T_{ij}] \quad (2.31)$$

$$D'_{p,ij} = k_B T[(\beta \varsigma)^{-1} \delta_{ij} I + (1 - \delta_{ij}) T_{ij}] \quad (2.32)$$
respectively, $\alpha$ is a constant chosen so that the right-hand side of eq 2.27 becomes the maximum, and $\beta$ is a constant properly chosen in the range of $0 < \beta < 1$. Then the desired expression for $[\eta]^{(L)}$ is given by the maximum of the the right-hand side of eq 2.27 with eq 2.30 and may be written in the form,

$$[\eta]^{(L)} = \frac{N_A}{M\eta_0 g^2 k_B T} \left[ \frac{\langle V^T \cdot (D_p^{-1} + D_p'^{-1}) \cdot V \rangle^2}{\langle V^T \cdot (D_p^{-1} + D_p'^{-1}) \cdot D \cdot (D_p^{-1} + D_p'^{-1}) \cdot V \rangle} \right]$$

(2.33)

As for the value of the constant $\beta$, it has been shown by Freire and Rey\textsuperscript{21} that in the case of Gaussian star chains $[\eta]^{(L)}$ given by eq 2.33 is insensitive to the value in the range of $0 < \beta < 1$ and therefore $\beta$ may be set equal to 0.5. Following them, we have put $\beta = 0.5$.

2.3 Results and Discussion

We have carried out MC runs to generate sample configurations for the regular three-arm star and linear freely rotating chains with $n = 3m = 60–300$ and $\theta = 109^\circ$, 165$^\circ$, and 175$^\circ$, with the parameters $\sigma$ and $T^*$ for the LJ potential being fixed to be 1 and 3.72, respectively. The freely rotating chain with $\theta = 109^\circ$ corresponds to a flexible polymer and that with $\theta = 165^\circ$ or 175$^\circ$ to a semiflexible or stiff polymer. The value 3.72 of $T^*$ was previously\textsuperscript{4} determined so that $\langle S^2 \rangle/n$ became a constant independent of $n$ for very large $n$, and therefore it corresponds to the $\Theta$ temperature, as mentioned in the previous section. For each chain, 5 MC runs have been carried out to obtain $10^4$ sample configurations, so that $5 \times 10^4$ sample configurations have been generated in total. We note that we have adopted $N = 10^4$ in the present study of $[\eta]$, which is one order of magnitude smaller than $10^5$ adopted in the previous study of $\langle S^2 \rangle$,\textsuperscript{4} since the computation time necessary for each sample configuration is much longer for $[\eta]$ than for $\langle S^2 \rangle$.

On the basis of the sample configurations so generated, we first examine the behavior of the MC values of $\langle S^2 \rangle$ as a function of $n$ and compare them with the theoretical values for the ideal freely rotating chain without interactions between beads, which we call the ideal chain hereafter, and for the corresponding Kratky–Porod (KP) wormlike chain.\textsuperscript{2,22} Then we examine the behavior of $g_S$ as a function of the reduced contour length $\lambda L$, i.e., the total contour length $L$ of the corresponding KP chain divided by the stiffness parameter $\lambda^{-1}$ having the dimension of length. Finally, we examine the behavior of $g_\eta$ as a function of $\lambda L$, which is the main purpose of the present study.

2.3.1 Mean-Square Radius of Gyration

In Table 2.1 are given the MC values of $\langle S^2 \rangle/n$ for both the regular three-arm star and linear chains along with the values of the acceptance fraction, i.e., the mean number of
2.3 Results and Discussion

<table>
<thead>
<tr>
<th>$\theta = 109^\circ$</th>
<th>three-arm star</th>
<th>linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$\langle S^2 \rangle/n$ (error %)</td>
<td>Acceptance</td>
</tr>
<tr>
<td>60</td>
<td>0.2964 (0.5)</td>
<td>59/200</td>
</tr>
<tr>
<td>120</td>
<td>0.3072 (0.4)</td>
<td>174/1000</td>
</tr>
<tr>
<td>180</td>
<td>0.3090 (0.4)</td>
<td>243/2000</td>
</tr>
<tr>
<td>240</td>
<td>0.3090 (0.7)</td>
<td>459/5000</td>
</tr>
<tr>
<td>300</td>
<td>0.3105 (0.9)</td>
<td>363/5000</td>
</tr>
<tr>
<td>$\theta = 165^\circ$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>1.886 (0.1)</td>
<td>199/200</td>
</tr>
<tr>
<td>120</td>
<td>3.028 (0.2)</td>
<td>995/1000</td>
</tr>
<tr>
<td>180</td>
<td>3.816 (0.1)</td>
<td>1979/2000</td>
</tr>
<tr>
<td>240</td>
<td>4.380 (0.1)</td>
<td>4919/5000</td>
</tr>
<tr>
<td>300</td>
<td>4.801 (0.2)</td>
<td>4888/5000</td>
</tr>
<tr>
<td>$\theta = 175^\circ$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>2.292 (0.0)</td>
<td>199/200</td>
</tr>
<tr>
<td>120</td>
<td>4.342 (0.0)</td>
<td>999/1000</td>
</tr>
<tr>
<td>180</td>
<td>6.289 (0.0)</td>
<td>1999/2000</td>
</tr>
<tr>
<td>240</td>
<td>8.141 (0.0)</td>
<td>4999/5000</td>
</tr>
<tr>
<td>300</td>
<td>9.898 (0.1)</td>
<td>4999/5000</td>
</tr>
</tbody>
</table>

changes in configuration in the $M_{nom}$ pivot steps divided by $M_{nom}$. Specifically, for example, for the regular three-arm star chain with $n = 60$ ($m = 20$), 5 independent MC runs have been repeated, in each of which $200 \times 10^4$ pivot steps have resulted in $59 \times 10^4$ changes in configuration. The values of $\langle S^2 \rangle/n$ and its statistical error given in the second column for the star and the fourth for the linear are those of the mean and the standard deviation, respectively, of the results of the 5 independent MC runs.

Figure 2.2 shows double-logarithmic plots of $\langle S^2 \rangle/n$ against $n$ with the present MC data given in Table 2.1 for the regular three-arm star (⊙) and linear (△) chains with $\theta = 109^\circ$ along with the previous MC data for the linear chains (▲). Although the present number of sample configurations is smaller than the previous one, the unfilled and filled triangles seem to form a single-composite curve, indicating that the present data are so accurate as the previous ones.

In the figure, the lower and upper solid line segments connect the theoretical values for the regular three-arm star and linear ideal chains, respectively. The former values have
been calculated from
\[ \langle S^2 \rangle = \frac{7}{54} \frac{1 - \cos \theta}{1 + \cos \theta} n + \frac{1}{54} \frac{1 + 19 + 49 \cos \theta + 71 \cos^2 \theta + 5 \cos^3 \theta}{(1 + \cos \theta)^2} \]
\[ + \frac{1}{18} \frac{1 + 54}{n + 1} \frac{1}{(1 + \cos \theta)^3} \]
\[ - \frac{2(1 + 2 \cos \theta)}{(1 + \cos \theta)^3} \frac{1 - (-\cos \theta)^{n/3+1}}{n + 1} \]
\[ + \frac{1}{27} \frac{1}{(1 + \cos \theta)^3} \frac{(1 - \cos \theta)(37 - 16 \cos \theta + \cos^2 \theta)}{(n + 1)^2} \]
\[ - \frac{2(2 + \cos^2 \theta)}{(1 + \cos \theta)^4} \frac{1 - (-\cos \theta)^{n/3+1}}{n + 1} \]
\[ + \frac{3}{(1 + \cos \theta)^4} \left[ 1 - (-\cos \theta)^{n/3+1} \right]^2 \frac{1}{(n + 1)^2} \]
(regular three-arm star) \quad (2.34)

which is a special case of the theoretical expression obtained by Guenza et al.\textsuperscript{23} for the regular f-arm star freely rotating chain, and the latter from
\[ \langle S^2 \rangle = \frac{1}{6} \frac{1 - \cos \theta}{1 + \cos \theta} n + \frac{1}{6} \frac{1 + 6 \cos \theta - \cos^2 \theta}{(1 + \cos \theta)^2} \]
\[ + \frac{1}{6} \frac{1 - 1 - 7 \cos \theta + 7 \cos^2 \theta + \cos^3 \theta}{(1 + \cos \theta)^3} \frac{1}{n + 1} \]
\[ - \frac{2 \cos^2 \theta}{(1 + \cos \theta)^4} \frac{1 - (-\cos \theta)^{n+1}}{(n + 1)^2} \]
(linear) \quad (2.35)

It is seen for both the regular three-arm star and linear chains that the MC values are ca. 20% larger than the corresponding ideal-chain values in the limit of \( n \to \infty \). This is due to the fact that the unperturbed (\( \Theta \)) dimension of a polymer chain may be considerably enlarged by nonbonded interactions.\textsuperscript{24}

In Figure 2.2 are also shown the theoretical values for the KP regular three-arm star and linear chains, the lower and upper dashed curves representing the respective values. The former values have been calculated from
\[ \langle S^2 \rangle = \frac{7L}{54\lambda} - \frac{1}{3\lambda^2} + \frac{1}{4\lambda^3L} + \frac{1}{4\lambda^3L}(1 - e^{-2\lambda L/3}) - \frac{3}{16\lambda^4L^2}(1 - e^{-4\lambda L/3}) \]
(KP regular three-arm star) \quad (2.36)

which is a special case of the theoretical expression obtained by Mansfield and Stockmayer\textsuperscript{25} for a general KP star chain, and the latter from\textsuperscript{26}
\[ \langle S^2 \rangle = \frac{L}{6\lambda} - \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2L} - \frac{1}{8\lambda^2L^2}(1 - e^{-2\lambda L}) \]
(KP linear) \quad (2.37)

In order to plot the KP theoretical values in the figure, we have converted \( L \) to \( n \) by the use of the relation
\[ \log n = \log(\lambda L) + \log(\lambda^{-1}n_L) \]  
(2.38)
Figure 2.2. Double-logarithmic plots of $\langle S^2 \rangle/n$ against $n$ for the regular three-arm star and linear chains, both with $\theta = 109^\circ$ and $\sigma = 1$ at $T^* = 3.72$. The unfilled circles and triangles represent the present MC data for the regular three-arm star and the linear chains, respectively, and the filled triangles represent the previous MC data for the latter. The lower and upper solid line segments connect the theoretical values of the regular three-arm star and linear ideal freely rotating chains, respectively. The lower and upper dashed curve represent the theoretical values of the corresponding KP regular three-arm star and linear chains, respectively.
Figure 2.3. Double-logarithmic plots of \( \langle S^2 \rangle / n \) against \( n \) for the regular three-arm star and linear chains, both with \( \theta = 165^\circ \) and \( \sigma = 1 \) at \( T^* = 3.72 \). All the symbols, line segments, and curves have the same meaning as those in Figure 2.2.
Figure 2.4. Double-logarithmic plots of $(S^2)/n$ against $n$ for the regular three-arm star and linear chains, both with $\theta = 175^\circ$ and $\sigma = 1$ at $T^* = 3.72$. All the symbols, line segments, and curves have the same meaning as those in Figure 2.2.
Figure 2.5. Plots of $g_s$ against log($\lambda L$) for the regular three-arm star. The three sets of the five unfilled circles represent the MC values with the indicated values of $\theta$. The solid line segments associated with each set of the MC data points connect the corresponding theoretical values of the ideal freely rotating and the dashed curve represents the KP theoretical values. The upper and lower dotted horizontal lines represent the asymptotic value 7/9 in the (ideal) random-coil limit and 4/9 in the rigid-rod limit, respectively.
Table 2.2. Values of the KP model parameters

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\lambda^{-1}$</th>
<th>$n_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>109°</td>
<td>3.01$^a$</td>
<td>1.24$^a$</td>
</tr>
<tr>
<td>165°</td>
<td>5.77$ \times $10</td>
<td>1.00</td>
</tr>
<tr>
<td>175°</td>
<td>5.26$ \times $10²</td>
<td>1.00</td>
</tr>
</tbody>
</table>

$^a$ These values have been determined in the previous study.⁴

where $n_L = n/L$ is the number of bonds per unit contour length, and used the values 3.01 and 1.24 of $\lambda^{-1}$ and $n_L$, respectively, which values were previously⁴ determined in such a way that the theoretical values of the KP linear chain could most closely reproduce the previous MC values for the linear chain (▲) with $n \geq 50$.

Figures 2.3 and 2.4 show similar plots for the regular three-arm star and linear chains with $\theta = 165°$ and 175°, respectively, which values of $\theta$ correspond to a semiflexible or stiff chain. All the symbols, line segments, and curves in the figures have the same meaning as those in Figure 2.2. It is seen that the MC values in each figure are almost identical with the corresponding ideal-chain values in contrast to the case of $\theta = 109°$ which corresponds to a flexible chain. This may be arising from the fact that effects of intramolecular interactions between beads become small with increasing chain stiffness if the total chain length is not very long. We therefore determine the values of the parameters $\lambda^{-1}$ and $n_L$ of the corresponding KP chain in each case of $\theta$ from an analysis of the linear ideal-chain values with $n \geq 10^2$. In each of Figures 2.3 and 2.4, the upper dashed curve represents the best-fit KP values calculated from eqs 2.37 and 2.38 with the parameter values given in Table 2.2. It is seen that the theoretical values for the KP regular three-arm star (the lower dashed curve in each figure) calculated from eqs 2.36 and 2.38 with those parameter values may well reproduce the corresponding MC values. We note that the $\lambda^{-1}$ value so determined is 1.3% larger than the $\lambda^{-1}$ value calculated from the (ideal) relation $\lambda^{-1} = 2/(1 + \cos \theta)$ in the case of $\theta = 109°$ and becomes identical with the latter value for $\theta \geq 165°$.

Now we proceed to examine the behavior of $g_S$ as a function of $\lambda L$. Values of $g_S$ are plotted against $\log(\lambda L)$ in Figure 2.5. The three sets of the five open circles represent the MC values with the indicated values of $\theta$, calculated from the $\langle S^2 \rangle$ values for the regular three-arm star and linear chains given in Table 2.1 with $n$ converted to $\lambda L$ by the use of eq 2.38 with the values of $\lambda^{-1}$ and $n_L$ given in Table 2.2. The solid line segments associated with each set of the MC data points connect the corresponding ideal-chain values calculated from eqs 2.34 and 2.35 with $n$ converted to $\lambda L$ in the above-mentioned manner. The dashed curve represents the KP theoretical values calculated from eq 2.36 and 2.37, with the upper and lower dotted horizontal lines representing the asymptotic
values $7/9$ in the (ideal) random-coil limit and $4/9$ in the rigid-rod limit, respectively.

In the cases of $\theta = 165^\circ$ and $175^\circ$ (semiflexible chain), the MC and ideal-chain values are almost identical with each other as a necessary consequence of the good agreement between the MC and ideal-chain values of $\langle S^2 \rangle$ (see Figures 2.3 and 2.4). A remarkable point is that the two kinds of $g_S$ values for $\theta = 109^\circ$ semiquantitatively agree with each other in spite of the rather large disagreement between the MC and ideal-chain values of $\langle S^2 \rangle$ (see Figure 2.2). Both the MC and ideal-chain values for $\theta = 109^\circ$ monotonically decrease with increasing $n$ and approach the asymptotic value $7/9$ in the limit of $n \to \infty$, while the KP value monotonically increases.

### 2.3.2 Intrinsic Viscosity

It is convenient to present MC results for $[\eta]$ in terms of the dimensionless quantity $X_\eta$ defined by

$$[\eta] = \frac{3\pi N_A n^{3/2}}{2M} X_\eta$$

(2.39)

As in the case of $[\eta]$, the superscript (KR), (U), and (L) are used to indicate that the $X_\eta$ values have been evaluated in the KR approximation, the Zimm rigid-body ensemble approximation, and the Fixman method, respectively. In Table 2.3 are given the values of $X_\eta^{(KR)}$, $X_\eta^{(U)}$, and $X_\eta^{(L)}$ along with their statistical errors, which are the mean and the standard deviation, respectively, of the results of the 5 independent MC runs, for both the regular three-arm star and linear chains.

Corresponding to the three kinds of $X_\eta$ (or $[\eta]$), we may consider the three kinds of $g_\eta$, i.e., $g_\eta^{(KR)}$, $g_\eta^{(Z)}$, and $g_\eta^{(F)}$ defined by

$$g_\eta^{(KR)} = \frac{[\eta]^{(KR)}(\text{star})}{[\eta]^{(KR)}(\text{linear})}, \quad g_\eta^{(Z)} = \frac{[\eta]^{(U)}(\text{star})}{[\eta]^{(U)}(\text{linear})}, \quad g_\eta^{(F)} = \frac{[\eta]^{(L)}(\text{star})}{[\eta]^{(L)}(\text{linear})}$$

(2.40)

We note that $g_\eta^{(Z)}$ is not an upper bound to $g_\eta^{(F)}$, nor is $g_\eta^{(F)}$ a lower bound. Figure 2.6 shows plots of $g_\eta^{(KR)}$ ($\bigcirc$), $g_\eta^{(Z)}$ ($\triangle$), and $g_\eta^{(F)}$ ($\triangledown$) against $\log(\lambda L)$. Three sets of the data points represent the MC values with the indicated values of $\theta$, calculated from eqs 2.39 and 2.40 with the three kinds of $X_\eta$ values given in Table 2.3 and with $n$ converted to $\lambda L$ in the above-mentioned manner. It is interesting to see that the $g_\eta^{(Z)}$ and $g_\eta^{(F)}$ values almost completely agree with each other irrespective of the value of $\lambda L$, i.e., chain stiffness ($\theta$) and total chain length ($n$), although the $X_\eta^{(U)}$ and $X_\eta^{(L)}$ values themselves do not. As for $g_\eta^{(KR)}$, its value agrees well with the $g_\eta^{(Z)}$ and $g_\eta^{(F)}$ values for very small $\lambda L$, i.e., for $\theta = 175^\circ$ with small $n$, but deviates upward from those with increasing $\lambda L$. In the case of $\theta = 109^\circ$ (flexible chain), $g_\eta^{(KR)}$ values are ca. 5% larger than the corresponding $g_\eta^{(Z)}$ and $g_\eta^{(F)}$ values.
Table 2.3. Values of $X_{\eta}^{(KR)}$, $X_{\eta}^{(U)}$, and $X_{\eta}^{(L)}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$X_{\eta}^{(KR)}$ (error %)</th>
<th>$X_{\eta}^{(U)}$ (error %)</th>
<th>$X_{\eta}^{(L)}$ (error %)</th>
<th>$X_{\eta}^{(KR)}$ (error %)</th>
<th>$X_{\eta}^{(U)}$ (error %)</th>
<th>$X_{\eta}^{(L)}$ (error %)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>three-arm star</td>
<td>linear</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.3041 (0.3)</td>
<td>0.2724 (0.5)</td>
<td>0.2506 (0.4)</td>
<td>0.3333 (0.3)</td>
<td>0.3069 (0.3)</td>
<td>0.2830 (0.3)</td>
</tr>
<tr>
<td>120</td>
<td>0.3212 (0.4)</td>
<td>0.2844 (0.4)</td>
<td>0.2559 (0.4)</td>
<td>0.3528 (0.4)</td>
<td>0.3211 (0.4)</td>
<td>0.2898 (0.5)</td>
</tr>
<tr>
<td>180</td>
<td>0.3290 (0.4)</td>
<td>0.2883 (0.4)</td>
<td>0.2566 (0.6)</td>
<td>0.3632 (0.5)</td>
<td>0.3293 (0.6)</td>
<td>0.2931 (0.6)</td>
</tr>
<tr>
<td>240</td>
<td>0.3329 (0.5)</td>
<td>0.2903 (0.6)</td>
<td>0.2554 (0.8)</td>
<td>0.3697 (0.2)</td>
<td>0.3350 (0.3)</td>
<td>0.2951 (0.3)</td>
</tr>
<tr>
<td>300</td>
<td>0.3360 (0.7)</td>
<td>0.2933 (0.8)</td>
<td>0.2555 (0.9)</td>
<td>0.3743 (0.5)</td>
<td>0.3387 (0.5)</td>
<td>0.2957 (0.6)</td>
</tr>
<tr>
<td></td>
<td>$\theta = 109^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>1.763 (0.3)</td>
<td>1.778 (0.2)</td>
<td>1.726 (0.3)</td>
<td>2.911 (0.5)</td>
<td>2.961 (0.5)</td>
<td>2.880 (0.5)</td>
</tr>
<tr>
<td>120</td>
<td>3.217 (0.2)</td>
<td>3.237 (0.3)</td>
<td>3.119 (0.2)</td>
<td>5.022 (0.3)</td>
<td>5.135 (0.3)</td>
<td>4.959 (0.3)</td>
</tr>
<tr>
<td>180</td>
<td>4.455 (0.3)</td>
<td>4.478 (0.2)</td>
<td>4.287 (0.3)</td>
<td>6.634 (0.3)</td>
<td>6.765 (0.4)</td>
<td>6.491 (0.4)</td>
</tr>
<tr>
<td>240</td>
<td>5.510 (0.4)</td>
<td>5.521 (0.2)</td>
<td>5.264 (0.4)</td>
<td>7.946 (0.7)</td>
<td>8.071 (0.7)</td>
<td>7.721 (0.7)</td>
</tr>
<tr>
<td>300</td>
<td>6.413 (0.3)</td>
<td>6.379 (0.3)</td>
<td>6.085 (0.3)</td>
<td>9.062 (0.5)</td>
<td>9.161 (0.5)</td>
<td>8.745 (0.5)</td>
</tr>
<tr>
<td></td>
<td>$\theta = 165^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>2.129 (0.1)</td>
<td>2.127 (0.1)</td>
<td>2.087 (0.1)</td>
<td>3.988 (0.6)</td>
<td>4.020 (0.5)</td>
<td>3.899 (0.5)</td>
</tr>
<tr>
<td>120</td>
<td>4.572 (0.2)</td>
<td>4.595 (0.1)</td>
<td>4.468 (0.2)</td>
<td>8.708 (0.3)</td>
<td>8.832 (0.4)</td>
<td>8.516 (0.4)</td>
</tr>
<tr>
<td>180</td>
<td>7.263 (0.2)</td>
<td>7.327 (0.2)</td>
<td>7.092 (0.2)</td>
<td>13.804 (0.4)</td>
<td>14.076 (0.3)</td>
<td>13.523 (0.3)</td>
</tr>
<tr>
<td>240</td>
<td>10.08 (0.2)</td>
<td>10.23 (0.2)</td>
<td>9.848 (0.3)</td>
<td>19.060 (0.2)</td>
<td>19.490 (0.3)</td>
<td>18.700 (0.3)</td>
</tr>
<tr>
<td>300</td>
<td>13.01 (0.2)</td>
<td>13.24 (0.2)</td>
<td>12.71 (0.3)</td>
<td>24.228 (0.3)</td>
<td>24.934 (0.4)</td>
<td>23.874 (0.4)</td>
</tr>
<tr>
<td></td>
<td>$\theta = 175^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Available theoretical asymptotic values of \( \eta \) in the (ideal) random-coil limit \((n \to \infty \text{ or } \lambda L \to \infty)\) are 0.90\(\tau\) obtained by Zimm and Kilb\(^{28}\) for the (dynamic) Gaussian spring-bead model with the preaveraged Oseen tensor and 0.90 obtained by Irurzun\(^{29}\) for the Gaussian chain in the KR approximation. In Figure 2.6, the dotted horizontal line represents the Zimm–Kilb value 0.90\(\tau\). Naturally, the \(g^{(KR)}_\eta\) values for \(\theta = 109^\circ\) agree well with the Zimm–Kilb value and also with the Irurzun one. If those MC values were appreciably different from the theoretical asymptotic values, the sample configurations of the regular three-arm star might not have been properly generated.

The approximate relation \(g_\eta \simeq g_S^{1/2}\) proposed by Zimm and Kilb\(^{28}\) for Gaussian branched chains has been often used in analyses of experimental data. The dashed curve in Figure 2.6 represents the values of \(g_S^{1/2}\) for the KP regular three-arm star chain calculated from eqs 2.36 and 2.37. It is seen that the approximate relation overestimates \(g_\eta\) for semiflexible or stiff polymers.

Finally, we consider upper and lower bounds to \(g_\eta\). From the inequality \([\eta]^{(L)} \leq [\eta] \leq [\eta]^{(U)}\) that holds for both the regular three-arm star and linear chains, we have the following two inequalities,

\[
g_\eta \leq g_\eta^{(U)} = \frac{[\eta]^{(U)}(\text{star})}{[\eta]^{(L)}(\text{linear})}, \quad g_\eta \geq g_\eta^{(L)} = \frac{[\eta]^{(L)}(\text{star})}{[\eta]^{(U)}(\text{linear})}
\] (2.41)

Figure 2.7 shows the plots of \(g_\eta^{(KR)}(\bigcirc)\) against log(\(\lambda L\)) reproduced from Figure 2.6, where the upper and lower ends of each vertical error bar represent the \(g_\eta^{(U)}\) and \(g_\eta^{(L)}\) values, respectively, calculated from eqs 2.39 and 2.41 with the values of \(X_\eta^{(U)}\) and \(X_\eta^{(L)}\) given in Table 2.3 and with \(n\) converted to \(\lambda L\) in the above-mentioned manner. The difference between the upper and lower bounds increases with increasing \(\lambda L\). Although the differences are unexpectedly large for flexible chains (\(\theta = 109^\circ\)), they are sufficiently small for semiflexible or stiff chains (\(\theta = 165^\circ, 175^\circ\)) to let us conclude that \(g_\eta^{(KR)}\) may give a good approximate value for the latter chains.

### 2.4 Concluding Remarks

We have examined the behavior of the ratio \(g_\eta\) of the intrinsic viscosity \([\eta]\) for the regular three-arm star chain to that for the linear one, both in the \(\Theta\) state and having the same total chain length, by MC simulation of freely rotating chains with the LJ 6-12 potentials between beads in a cutoff version. On the basis of the values of the upper bound \([\eta]^{(U)}\) and lower one \([\eta]^{(L)}\) to \([\eta]\) evaluated by the Zimm rigid-body approximation and by the Fixman method, respectively, the values of the upper bound \(g_\eta^{(U)}\) and lower one \(g_\eta^{(L)}\) to \(g_\eta\) have been estimated. It has then been found that the KR value \(g_\eta^{(KR)}\) of \(g_\eta\) lies between the two bounds irrespective of the number \(n\) of bonds and the bond angle \(\theta\) (or the reduced
2.4 Concluding Remarks

Figure 2.6. Plots of $g_{\eta}^{(KR)}$, $g_{\eta}^{(Z)}$, and $g_{\eta}^{(F)}$ against $\log(\lambda L)$. Three sets of the five unfilled circles, the five unfilled triangles, and the five unfilled inverted triangles represent the MC data of $g_{\eta}^{(KR)}$, $g_{\eta}^{(Z)}$, and $g_{\eta}^{(F)}$, respectively, with the indicated values of $\theta$. The dotted horizontal line represents the Zimm–Kilb$^{28}$ value 0.90 for the Gaussian chain and the dashed curve the values of $g_{S}^{1/2}$ for the KP chain.
Figure 2.7. Plots of $g^{(KR)}_\eta$, $g^{(U)}_\eta$, and $g^{(L)}_\eta$ against log($\lambda L$). The unfilled circles have the same meaning as those in Figure 2.6. The upper and lower ends of each vertical error bar represent the MC values of $g^{(U)}_\eta$ and $g^{(L)}_\eta$, respectively (see the text).
contour length $\lambda L$). Although the differences between the two bounds are unexpectedly large for flexible chains ($\theta = 109^\circ$), they are sufficiently small for semiflexible or stiff chains ($\theta = 165^\circ, 175^\circ$) to let us conclude that $g^{(KR)}_n$ gives a good approximate value for the latter chains. We therefore proceed to make a theoretical study of $g_n$ for the unperturbed KP chain in the KR approximation and to construct an interpolation formula useful for an analysis of experimental data.

References

3 Intrinsic Viscosity of Semiflexible Three-Arm Stars: A Theoretical Study

3.1 Introduction

In Chapter 2, effects of chain stiffness on the ratio \( g_\eta \) of the intrinsic viscosity \([\eta]\) of a regular three-arm star polymer chain to that of the corresponding linear one, both having the same total chain length and chain stiffness, have been examined by Monte Carlo (MC) simulation on the basis of the freely rotating chain with the Lennard–Jones 6-12 intramolecular potential between beads.\(^1\) Since there is no theoretical method of obtaining \([\eta]\) without any approximations, we have evaluated it in the three approximate ways: the Kirkwood–Riseman (KR) approximation,\(^2,3\) the Zimm rigid-body ensemble approximation,\(^4\) and the Fixman method.\(^5,6\) We note that upper and lower bounds to \([\eta]\) may be obtained by the Zimm and Fixman methods, respectively, and we have ignored the contribution of the Einstein spheres.\(^7,8\) By combining the two bounds to \([\eta]\) so obtained, we have evaluated upper and lower bounds to \( g_\eta \) and then shown that the KR approximation may give a good approximate \( g_\eta \) value for semiflexible or stiff polymer chains. It is therefore convenient and useful for an analysis of experimental data to construct a theoretical expression for \( g_\eta \) in the KR approximation on the basis of a proper model for semiflexible and stiff polymer chains. The purpose in this chapter is to construct an interpolation formula for \( g_\eta \) of the semiflexible regular three-arm star chain in the KR approximation on the basis of the Kratky–Porod (KP) wormlike chain\(^7,9\) without excluded volume.

For an evaluation of \([\eta]\) for both the KP regular three-arm star and linear chains, we adopt the touched-bead hydrodynamic model so that \([\eta]\) may be written as a sum of the solution \([\eta]_{KR}\) of the KR equation and the contribution \([\eta]_E\) of the Einstein spheres.\(^7,8\) In order to complete the necessary KR equation, i.e., a set of linear simultaneous equations for the hydrodynamic force balance at each bead with the preaveraged Oseen hydrodynamic interaction tensor,\(^2,3\) we need an appropriate approximate expression for the mean reciprocal \( (R^{-1}) \) of the distance between the centers of two beads, where \( \langle \cdots \rangle \) denotes an equilibrium configurational average. Although such an expression is available for a pair of beads on the KP linear chain,\(^10,11\) there are none for a pair of beads on different two arms of the KP star. Prior to the evaluation of \([\eta]\) and \( g_\eta \), we must therefore derive an approx-
imate expression for $\langle R^{-1} \rangle$ between the two beads on different parts of the once-broken KP chain as a function of the two contour distances from the broken point to the centers of the respective beads. We carry out the task by the use of the two asymptotic methods, i.e., the Daniels method$^{7,12–14}$ near the random-coil limit and the $\epsilon$ method$^{7,15,16}$ near the rod limit, and also of the MC results for the once-broken KP chain.

3.2 Basic Equations

Consider a regular three-arm star touched-bead model composed of $3m + 1$ identical spherical beads of (hydrodynamic) diameter $d_b$ whose centers are located on the KP regular three-arm star chain contour, as illustrated in Figure 3.1. For convenience, the three arms are designated the first, second, and third ones and the $m$ beads on the $i$th ($i = 1, 2, 3$) arm are numbered $(i-1)m + 1$, $(i-1)m + 2$, \ldots, $im$ from the branch point (center) to the terminal end, with the center bead numbered 0. The angle between each pair of the unit vectors tangent to the KP contours at the branch point is fixed to be $120^\circ$, so that the three vectors are on the same plane. The linear touched-bead model, the counterpart of the above star one, is the KP touched-bead model composed of $n + 1$ identical beads of diameter $d_b$ whose centers are located on the KP linear chain contour. We set $n + 1$ equal to $3m + 1$, so that $n = 3m$. The $n + 1$ beads are numbered 0, 1, 2, \ldots, $n$ from one end to the other. For both the star and linear touched-bead models, the

Figure 3.1. Illustration of the KP regular three-arm star touched-bead model.
contour distance between the two adjacent beads is set equal to \( d_b \).

The intrinsic viscosity \([\eta]\) of the touched-bead model composed of \( n + 1 \) identical spherical beads of diameter \( d_b \) may be written as the sum of the solution \([\eta]_{KR}\) of the KR equation and the contribution \([\eta]_E\) of the Einstein spheres,\(^{7,8}\) i.e.,

\[
[\eta] = [\eta]_{KR} + [\eta]_E \tag{3.1}
\]

where

\[
[\eta]_{KR} = \frac{N_A}{M} \sum_{i=0}^{n} \phi_{ii} \tag{3.2}
\]

\[
[\eta]_E = \frac{5\pi N_A(n + 1)d_b^3}{12M} \tag{3.3}
\]

with \( N_A \) the Avogadro constant and \( M \) the polymer molecular weight. In eq 3.2, \( \phi_{ij} \) is the solution of the following linear simultaneous equations,

\[
\phi_{ij} + \frac{d_b}{2} \sum_{k=0}^{n} \langle R_{ik}^{-1} \rangle \phi_{kj} = \frac{\pi d_b}{2} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \tag{3.4}
\]

where \( \langle R_{ij}^{-1} \rangle \) is the mean reciprocal of the distance between the centers of the \( i \)th and \( j \)th beads and \( \mathbf{S}_i \) is the vector distance from the center of mass of the chain to the center of the \( i \)th bead. In order to complete the KR equation 3.4, we need an appropriate approximate expression for \( \langle R_{ij}^{-1} \rangle \) along with the theoretical expression for the equilibrium average \( \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \). We first give the latter and then consider the former. In what follows, all lengths are measured in units of the stiffness parameter \( \lambda^{-1} \) of the KP chain unless otherwise specified.

### 3.3 Average \( \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \)

As in the previous study of \([\eta]\) of the linear touched-bead model,\(^9\) where we have adopted the value of \( \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \) of the (continuous) KP chain of total contour length \( L = (n + 1)d_b \) as that of the (discrete) touched-bead model, we adopt the value of \( \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \) of the KP regular three-arm star chain of total contour length \( L = (3m + 1)d_b \) as that of the regular three-arm star touched-bead model. Here, \( L \) has consistently been set equal to the total number \( n + 1 \) or \( 3m + 1 \) of beads multiplied by \( d_b \) for both the linear and star chains. The relation \( L = (3m + 1)d_b \) leads to the relation \( L_a = (m + \frac{1}{3})d_b \) between the contour length \( L_a = \frac{1}{3}L \) of each arm and \( d_b \). Although the latter relation seems somewhat awkward because of the inclusion of the non-integer term \( \frac{1}{3} \), it only reflects the fact that the center bead belongs to all the three arms.
For the \([i-1]m+k\)th and \([j-1]m+l\)th beads \((i, j = 1, 2, 3; k, l = 1, 2, \cdots, m)\) of the regular three-arm star chain, i.e., the \(k\)th bead on the \(i\)th arm and the \(l\)th bead on the \(j\)th arm, respectively, \(\langle S(i-1)m+k \cdot S(j-1)m+l \rangle\) may then be given by

\[
\langle S(i-1)m+k \cdot S(j-1)m+l \rangle = \langle S(t_k^{(i)}) \cdot S(t_l^{(j)}) \rangle
\]

where \(S(t_k^{(i)})\) is the vector distance from the center of mass of the KP regular three-arm star chain to the contour point on the \(i\)th arm with the contour distance \(t_k^{(i)}\) from the branch point, so that

\[
t_k^{(i)} = kd_h
\]

The average \(\langle S(t_k^{(i)}) \cdot S(t_l^{(j)}) \rangle\) may be given by (see Appendix 3.A)

\[
\langle S(t_k^{(i)}) \cdot S(t_l^{(j)}) \rangle = \frac{L}{2L} + \frac{(t_k^{(i)})^2 + (t_l^{(j)})^2}{2L} + \frac{t_k^{(i)} + t_l^{(j)}}{6} - \frac{u_{kl}^{(ij)}}{2}
\]

\[+ \frac{1}{24}(1 - 3\delta_{ij})\left[1 - \exp(-2t_k^{(i)}) - \exp(-2t_l^{(j)})\right] - \frac{1}{8}(1 + \delta_{ij})\exp(-2u_{kl}^{(ij)})
\]

\[+ \frac{1}{4L} e^{-2L/3} \left[\cosh(2t_k^{(i)}) + \cosh(2t_l^{(j)})\right] + \frac{3}{16L^2}(1 - e^{-4L/3})
\]

where \(\delta_{ij}\) is the Kronecker delta and \(u_{kl}^{(ij)}\) is defined by

\[
u_{kl}^{(ij)} = \left[\left(t_k^{(i)} - t_l^{(j)}\right)^2 + 2(1 - 2\nu_{ij})t_k^{(i)} t_l^{(j)}\right]^{1/2}
\]

We note that \(\langle S(i-1)m+k \cdot S_0 \rangle = \langle S(t_k^{(i)}) \cdot S(0) \rangle\) and \(\langle S_0 \cdot S_0 \rangle = \langle S(0) \cdot S(0) \rangle\) are also given by eq 3.7 with \(u_{kl}^{(ij)} = t_k^{(i)}\) and 0, respectively. In the rod limit, i.e., the limit of \(L\) (in units of \(\lambda^{-1}\)) \(\to 0\), eq 3.7 reduces to

\[
\langle S(t_k^{(i)}) \cdot S(t_l^{(j)}) \rangle = \frac{1}{2} (3\delta_{ij} - 1) t_k^{(i)} t_l^{(j)} \quad \text{(rod limit)}
\]

As for the linear touched-bead model, \(\langle S_i \cdot S_j \rangle (i, j = 0, 1, 2, \cdots, n)\) may be given by

\[
\langle S_i \cdot S_j \rangle = \langle S(t_i) \cdot S(t_j) \rangle
\]

where \(S(t_i)\) is the vector distance from the center of mass of the KP linear chain to its contour point with the contour distance \(t_i\) from one end, so that

\[
t_i = (i + \frac{1}{2}) d_h
\]

The average \(\langle S(t) \cdot S(t') \rangle\) is given by eq 19 in ref 17 and may be written in the form,

\[
\langle S(t) \cdot S(t') \rangle = \frac{L}{3} + \frac{t^2 + t'^2}{2L} - \frac{t + t'}{2} - \frac{|t - t'|}{2} - \frac{e^{-2|t-t'|}}{4}
\]

\[+ \frac{1}{8L} \left[2 - e^{-2t} - e^{-2t'} - e^{-2(l-t)} - e^{-2(l-t')}\right] + \frac{1}{8L^2}(1 - e^{-2L})
\]

In the rod limit, it reduces to

\[
\langle S(t) \cdot S(t') \rangle = \frac{1}{4} (2t - L)(2t' - L) \quad \text{(rod limit)}
\]
3.4 Mean Reciprocal $\langle R^{-1} \rangle$ of the End-to-End Distance

The mean reciprocal $\langle R_{[(i-1)m+k][(j-1)m+l]}^{-1} \rangle$ of the distance $R_{[(i-1)m+k][(j-1)m+l]}$ between the centers of the $k$th bead on the $i$th arm and $l$th bead on the $j$th arm may be given by

$$\langle R_{[(i-1)m+k][(j-1)m+l]}^{-1} \rangle = \langle R^{-1}(t_k^{(i)}, t_l^{(j)}) \rangle$$

where $\langle R^{-1}(t_k^{(i)}, t_l^{(j)}) \rangle$ is the mean reciprocal of the distance between the contour points $t_k^{(i)}$ and $t_l^{(j)}$ on the KP regular three-arm star chain.

We first consider asymptotic forms for the mean reciprocal $\langle R^{-1}(t_1, t_2, \theta) \rangle$ of the end-to-end distance of the once-broken KP chain of total contour length $t_1 + t_2$ such that two KP subchains 1 and 2 of contour lengths $t_1$ and $t_2$, respectively, are connected with a bending angle $\theta$, i.e., the angle between the unit vectors $u_1$ and $u_2$ tangent to the contours of the subchains 1 and 2, respectively, at the broken point, as illustrated in Figure 3.2.

We then construct an interpolation formula for the case of $\theta = 120^\circ$ on the basis of the asymptotic forms so obtained and also of MC results for the once broken KP chain.
3.4.1 Asymptotic Forms

The asymptotic forms for \( \langle R^{-1}(t_1, t_2, \theta) \rangle \) in the cases of \( t_1 \gg 1, t_2 \gg 1, t_1 \gg 1 \) and \( t_2 \ll 1, t_1 \ll 1 \) and \( t_2 \gg 1, t_1 \gg 1 \) and \( t_1 \ll 1 \) and \( t_2 \ll 1 \) may be written in the form (see Appendix 3.B),

\[
\langle R^{-1}(t_1, t_2, \theta) \rangle = \langle R^2 \rangle^{-1/2} f_{DD}(t_1, t_2, \theta) \quad \text{for } t_1 \gg 1, t_2 \gg 1
\]

\[
= \langle R^2 \rangle^{-1/2} f_{Dc}(t_1, t_2, \theta) \quad \text{for } t_1 \gg 1, t_2 \ll 1
\]

\[
= \langle R^2 \rangle^{-1/2} f_{cD}(t_1, t_2, \theta) \quad \text{for } t_1 \ll 1, t_2 \gg 1
\]

\[
= \langle R^2 \rangle^{-1/2} f_{\alpha}(t_1, t_2, \theta) \quad \text{for } t_1 \ll 1, t_2 \ll 1
\]

\[
(3.15)
\]

where

\[
f_{DD}(t_1, t_2, \theta) = \left( \frac{6}{\pi} \right)^{1/2} \left\{ 1 - \frac{11}{40} \left( \frac{R^2}{\langle R^2 \rangle} \right)^2 - \frac{1}{80} \left( \frac{R^2}{\langle R^2 \rangle} \right)^2 \right\}
\]

\[
(3.16)
\]

\[
f_{Dc}(t_1, t_2, \theta) = \left( \frac{6}{\pi} \right)^{1/2} \left\{ 1 - \frac{11}{40} \left( \frac{R^2}{\langle R^2 \rangle} \right)^2 + \frac{1}{40} \left( \frac{R^2}{\langle R^2 \rangle} \right)^2 \right\}
\]

\[
(3.17)
\]

\[
f_{cD}(t_1, t_2, \theta) = 1 + \frac{3}{8} \left( \frac{R^4}{\langle R^2 \rangle^2} - 1 \right)
\]

\[
(3.18)
\]

and \( f_{cD}(t_1, t_2, \theta) = f_{Dc}(t_2, t_1, \theta) \). In eqs 3.15—3.18, \( \langle R^2 \rangle = \langle R^2(t_1, t_2, \theta) \rangle \) and \( \langle R^4 \rangle = \langle R^4(t_1, t_2, \theta) \rangle \) are the second and fourth moments, respectively, of the end-to-end distance of the once-broken KP chain given by

\[
\langle R^2(t_1, t_2, \theta) \rangle = t_1 + t_2 - \frac{1}{2} \left[ 1 - e^{-2(t_1+t_2)} \right] - \frac{1}{2} \left( 1 - e^{-t_1} \right) \left( 1 - e^{-t_2} \right) \left( 1 + \cos \theta \right)
\]

\[
(3.19)
\]

\[
\langle R^4(t_1, t_2, \theta) \rangle = \frac{5}{3} (t_1 + t_2)^2 - (t_1 + t_2) \left[ \frac{26}{9} + e^{-2(t_1+t_2)} \right]
\]

\[
+ 2 \left[ 1 - e^{-2(t_1+t_2)} - \frac{1}{54} (1 - e^{-6(t_1+t_2)}) \right]
\]

\[
- \frac{1}{2} \left[ 1 - e^{-2t_1} \right] \left[ 5 t_2 + t_2 e^{-2t_2} - \frac{3}{2} (1 - e^{-2t_2}) + \frac{1}{18} (1 - e^{-6t_2}) \right]
\]

\[
+ \frac{1}{2} \left[ 1 - e^{-2t_2} \right] \left[ 5 t_1 + t_1 e^{-2t_1} - \frac{3}{2} (1 - e^{-2t_1}) + \frac{1}{18} (1 - e^{-6t_1}) \right]
\]

\[
\left( 1 + \cos \theta \right)
\]

\[
(3.20)
\]

We note that \( f_{Dc}(t, 0, \theta) \) given by eq 3.17 with \( t_1 = t \) and \( t_2 = 0 \) [or \( f_{cD}(0, t, \theta) \)] is identical with the second Daniels approximate expression for the mean reciprocal of the end-to-end distance of the KP linear chain of total contour length \( t \) obtained by Yamakawa and
Fujii,\(^1\) up to \(\mathcal{O}(t^{-2})\), and also note that the expression for \(\langle R^2(t_1, t_2, \theta) \rangle\) given by eq 3.19 was first derived by Mansfield and Stockmayer.\(^8\) In the rod limit, eqs 3.15 (strictly the fourth equation) reduces to

\[
\langle R^{-1}(t_1, t_2, \theta) \rangle = [R_{\text{rod}}(t_1, t_2, \theta)]^{-1} \quad \text{(rod limit)}
\]  

(3.21)

where \(R_{\text{rod}}(t_1, t_2, \theta)\) is the end-to-end distance of the once-broken rod composed of the two straight rods of lengths \(t_1\) and \(t_2\) with the bending angle \(\theta\) and is given by

\[
R_{\text{rod}}(t_1, t_2, \theta) = (t_1^2 + t_2^2 - 2t_1t_2 \cos \theta)^{1/2}
\]  

(3.22)

Figures 3.3, 3.4, and 3.5 show plots of \(R_{\text{rod}}(t_1, t_2, 120^\circ)\langle R^{-1}(t_1, t_2, 120^\circ) \rangle\) against \(t_1\) for the once-broken KP chain with \(\theta = 120^\circ\) and with \(t_2 = 0.1, 1, \) and \(2, \) respectively. The dashed (DD), two dot-dashed (De and eD), and dotted (ee) curves represent the theoretical asymptotic values calculated from eqs 3.15 with eqs 3.16, 3.17, and 3.18, respectively, with \(\theta = 120^\circ\). In the figures, the unfilled circles and the solid curves represent the MC values and those calculated from an interpolation formula, which are obtained in the following two subsections.

### 3.4.2 MC Simulation

As seen from Figures 3.3, 3.4, and 3.5, it is difficult to construct an interpolation formula on the basis only of the theoretical asymptotic values because they cannot cover the whole range of \(t_1\). In order to obtain additional reference values of \(\langle R^{-1}(t_1, t_2, 120^\circ) \rangle\), we carry out MC simulation by the use of the ideal once-broken freely rotating chain such that two freely rotating subchains 1 and 2 are connected with a bending angle \(120^\circ\), the subchain \(i\) \((i = 1, 2)\) being composed of \(n_i\) bonds of length \(a\) (in units of \(\lambda^{-1}\)) joined with a complementary bond angle \(\alpha\). We note that the subchain \(i\) becomes identical with the KP chain of total contour length \(t_i\) in the limit of \(n_i \to \infty\) and \(a \to 0\) under the conditions \(t_i = n_i a\) and \(1 - \cos \alpha = 2a\). As done by Yamakawa and Fujii,\(^1\) we replace the KP subchains approximately by the freely rotating ones having very small but finite \(\alpha\) satisfying the above two conditions, i.e., \(a = 0.01\) and \(\cos \alpha = 0.98\).

By the use of the once-broken freely rotating chain so defined, we evaluate \(\langle R^{-1}(t_1, t_2, 120^\circ) \rangle\) as follows. Let \(\mathbf{a}_k^{(i)}\) \((i = 1, 2; k = 1, 2, \ldots, n_i)\) be the \(k\)th bond vector on the subchain \(i\), the bond vectors being numbered 1, 2, \ldots, \(n_i\) from the broken point to the end of the subchain \(i\), and let \(\mathbf{a}_1^{(1)} = (0, 0, a)^T\) and \(\mathbf{a}_1^{(2)} = (\sqrt{3}a/2, 0, -a/2)^T\) in an external Cartesian coordinate system with the superscript \(T\) indicating the transpose. In the external system, \(\mathbf{a}_k^{(i)}\) may then be written in the form,

\[
\mathbf{a}_k^{(i)} = [\mathbf{A}(120^\circ, 180^\circ)]^{i-1} \cdot \mathbf{A}(\alpha, \phi_1^{(i)}) \cdot \mathbf{A}(\alpha, \phi_2^{(i)}) \cdots \mathbf{A}(\alpha, \phi_{k-1}^{(i)}) \cdot (0, 0, a)^T
\]  

(3.23)
Figure 3.3. Plots of $R_{rod}(t_1, t_2, 120^\circ)(R^{-1}(t_1, t_2, 120^\circ))$ against $t_1$ for the once-broken KP chain with $\theta = 120^\circ$ and $t_2 = 0.1$. The unfilled circles represent the MC values. The solid curve represents the values of the interpolation formula and the dashed (DD), two dot-dashed ($\epsilon D$ and $D \epsilon$), and dotted ($\epsilon \epsilon$) curves represent the theoretical asymptotic values for $t_1 \gg 1$ and $t_2 \gg 1$, $t_1 \gg 1$ and $t_2 \ll 1$, $t_1 \ll 1$ and $t_2 \gg 1$, and $t_1 \ll 1$ and $t_2 \ll 1$, respectively (see the text).
Figure 3.4. Plots of $R_\text{rod}(t_1, t_2, 120^\circ)\langle R^{-1}(t_1, t_2, 120^\circ) \rangle$ against $t_1$ for the once-broken KP chain with $\theta = 120^\circ$ and $t_2 = 1$. All the symbols and curves have the same meaning as those in Figure 3.3.
Figure 3.5. Plots of $R_{rod}(t_1, t_2, 120^\circ)(R^{-1}(t_1, t_2, 120^\circ))$ against $t_1$ for the once-broken KP chain with $\theta = 120^\circ$ and $t_2 = 2$. All the symbols and curves have the same meaning as those in Figure 3.3.
3.4 Mean Reciprocal $\langle R^{-1} \rangle$ of the End-to-End Distance

where $\phi_k^{(i)}$ is the internal rotation angle around $a_k^{(i)}$ and $A(\alpha, \phi)$ is the orthogonal transformation matrix defined by

$$A(\alpha, \phi) = \begin{pmatrix} -\cos \alpha \cos \phi & \sin \phi & -\sin \alpha \cos \phi \\ -\cos \alpha \sin \phi & -\cos \phi & -\sin \alpha \sin \phi \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$ (3.24)

The configuration of the once-broken freely rotating chain may therefore be specified by the set of $n_1 + n_2 - 2$ internal rotation angles $\{\phi_{n_1+n_2-2}\} = \{\phi_1^{(1)}, \phi_2^{(1)}, \ldots, \phi_{n_1-1}^{(1)}, \phi_1^{(2)}, \phi_2^{(2)}, \ldots, \phi_{n_2-1}^{(2)}\}$. On the basis of $N$ sample configurations, i.e., $N$ sets of $n_1 + n_2 - 2$ internal rotation angles randomly chosen in the interval $[-\pi, \pi]$, $\langle R^{-1}(t_1, t_2, 120^\circ) \rangle$ may be evaluated from

$$\langle R^{-1}(t_1, t_2, 120^\circ) \rangle = N^{-1} \sum_{\{\phi_{n_1+n_2-2}\}} \left[ \sum_{i=1}^{n_1} a_i^{(1)} - \sum_{j=1}^{n_2} a_j^{(2)} \right]^{-1}$$ (3.25)

where the first sum is taken over the $N$ sample configurations.

In practice, we have adopted $N = 10^5$. All the numerical work has been done by the use of a personal computer with an Intel Pentium4 CPU with a clock rate of 3.00 GHz. A source program coded in C has been compiled by the GNU C compiler version 3.3.3 with real variables of double precision.

The MC values of $R_{rod}(t_1, t_2, 120^\circ)\langle R^{-1}(t_1, t_2, 120^\circ) \rangle$ so obtained for $t_2 = 0.1, 1, \text{ and } 2$ are also shown in Figures 3.3, 3.4, and 3.5, respectively (unfilled circles). It is seen from Figure 3.3 for the case of small $t_2 (= 0.1)$ that the MC values connect smoothly the $\epsilon \epsilon$ values which are valid for $t_1 \ll 1$ and $t_2 \ll 1$ and the $\epsilon \delta$ ones which are valid for $t_1 \gg 1$ and $t_2 \ll 1$. In the case of large $t_2 (= 2)$, on the other hand, it is seen from Figure 3.5 that the MC values connect smoothly the $\epsilon \delta$ values which are valid for $t_1 \ll 1$ and $t_2 \gg 1$ and the $\delta \delta$ ones which are valid for $t_1 \gg 1$ and $t_2 \gg 1$. In the case of intermediate $t_2 (= 1)$ shown in Figure 3.4, The MC values lie between the $\epsilon \epsilon$ and $\epsilon \delta$ values for $t_1 \lesssim 1$ and follow the $\delta \delta$ values for $t_1 \gtrsim 1$.

3.4.3 Interpolation Formula

Now we are in a position to construct an interpolation formula for $\langle R^{-1}(t_1, t_2, 120^\circ) \rangle$ on the basis of the theoretical asymptotic forms given by eqs 3.15 with eqs 3.16—3.18 along with the MC ones for the once-broken KP chain.

The asymptotic form $f_{\epsilon \epsilon}$ (or $f_{\epsilon \delta}$) is not symmetric with respect to the pair of variables $t_1$ and $t_2$ and is not convenient for further developments as it stands. We therefore construct a hybrid $f_{\epsilon \delta \epsilon}(t_1, t_2, 120^\circ)$ between $f_{\epsilon \delta}(t_1, t_2, 120^\circ)$ and $f_{\delta \epsilon}(t_1, t_2, 120^\circ)$ given by eqs 3.16 and 3.17, respectively, both with $\theta = 120^\circ$, prior to the construction of the
interpolation formula, which may be written in the form,

\[ f_{\text{DDe}}(t_1, t_2, 120^\circ) = \left( \frac{6}{\pi} \right)^{1/2} \left[ 1 - \frac{11}{40(R^2)} + \frac{431}{4480(R^2)^2} + \frac{C(t_1, t_2)}{(R^2)^2} \right] \] (3.26)

where \( C(t_1, t_2) \) is given by

\[ C(t_1, t_2) = \frac{108\bar{t}^2 + 77\bar{t}^4}{640(1 + \bar{t}^4)} \] (3.27)

with

\[ \bar{t}^2 = (t_1^{-2} + t_2^{-2})^{-1} \] (3.28)

We note that \( f_{\text{DDe}} \) so defined has the symmetry relation \( f_{\text{DDe}}(t_1, t_2, 120^\circ) = f_{\text{DDe}}(t_2, t_1, 120^\circ) \) and recovers \( f_{\text{DD}} \) given by eq 3.16 up to \( O(t_2^{-2}) \) in the case of \( t_2 \gg 1 \) and \( f_{\text{De}} \) given by eq 3.17 up to \( O(t_2^{-2}) \) in the case of \( t_2 \ll 1 \).

By combining \( f_{\text{xx}} \) and \( f_{\text{DDe}} \) given by eqs 3.18 and 3.26, respectively, we then assume that \( \langle R^{-1}(t_1, t_2, 120^\circ) \rangle \) may be well represented by the following interpolation formula,

\[ \langle R^{-1}(t_1, t_2, 120^\circ) \rangle = \langle R^2 \rangle^{-1/2}[1 + A_1 \langle R^2 \rangle^{A_2}]^{-1} [f_{\text{xx}}(t_1, t_2, 120^\circ) + A_3 \langle R^2 \rangle^{3/2} + A_4 t_1 t_2 (t_1 + t_2) e^{-A_5 t_1 t_2} + A_1 \langle R^2 \rangle^{A_2} f_{\text{DDe}}(t_1, t_2, 120^\circ)] \] (3.29)

where \( \langle R^2 \rangle = \langle R^2(t_1, t_2, 120^\circ) \rangle \) is given by eq 3.19 with \( \theta = 120^\circ \). In eq 3.29, \( A_1, A_2, \ldots, A_5 \) are numerical constants which must be determined so that the interpolation formula may well reproduce the MC values evaluated in the last subsection, their optimum values being given by

\[ A_1 = 5, \quad A_2 = \frac{11}{2}, \quad A_3 = 0.045, \quad A_4 = 0.40, \quad A_5 = 1 \] (3.30)

In Figures 3.3—3.5, the solid curves represent the approximate values calculated from eq 3.29 with eqs 3.18—3.20, 3.26—3.28, and 3.30 with \( \theta = 120^\circ \). It is seen that the interpolation formula so constructed may be well reproduced the MC values.

Finally, the necessary \( \langle R^{-1}((i-1)m+k)_{(j-1)m+l}^{-1} \rangle \) may be given by eq 3.14 with \( \langle R^{-1}(t_i^{(i)}, t_i^{(j)}) \rangle \) given by

\[ \langle R^{-1}(t_i^{(i)}, t_i^{(j)}) \rangle = \langle R^{-1}(t_i^{(i)}, t_i^{(j)}, 120^\circ) \rangle \quad \text{for } i \neq j \]

\[ = \langle R^{-1}([t_i^{(i)} - t_i^{(j)}], 0, 120^\circ) \rangle \quad \text{for } i = j \] (3.31)

We note that \( \langle R^{-1}(t, 0, 120^\circ) \rangle \) and/or \( \langle R^{-1}(0, t, 120^\circ) \rangle \) represent the mean reciprocal of the end-to-end distance of the KP linear chain of total contour length \( t \) and its approximate value calculated from eq 3.29 with eqs 3.18—3.20, 3.26—3.28, and 3.30 and with \( t_1 = t \) and \( t_2 = 0 \) agrees with that calculated from the Yamakawa–Fujii\textsuperscript{11} interpolation formula for the KP linear cylinder (with the cylinder diameter \( d = 0 \)) within 0.7% over the whole range of \( t \).
3.5 Results for $g_\eta$ Factor

We have calculated the KR contribution $[\eta]_{\text{KR}}$ to the intrinsic viscosity $[\eta]$ from eq 3.2 with the numerical solutions $\phi_{ii}$ of the linear simultaneous equations 3.4 for both the KP regular three-arm star and linear touched-bead models, in the ranges of the total number $n + 1$ of beads from 4 to 1501 and of the bead diameter $d_b$ from 0.001 to 0.4. Note that the total contour length $L$ of the chain is equal to $(n + 1)d_b$, as already mentioned. In eq 3.4, $\langle R^{-1}(t_k^{(i)}, t_l^{(j)}) \rangle$ is given by eq 3.31 with eqs 3.18—3.20 and 3.26—3.30 for the star chain and by the second of eq 3.31 for the linear one and $\langle S_i \cdot S_j \rangle$ is given by eq 3.7 with eq 3.8 for the former and by eq 3.12 for the latter. We then have calculated $[\eta]$ from eq 3.1 with the values of $[\eta]$ so obtained along with $[\eta]_E$ calculated from eq 3.3, for both the star and linear chains. Finally we have evaluated $g_\eta$ as a function of $L$ and $d_b$ from the defining equation,

$$g_\eta(L, d_b) = \frac{[\eta](\text{star})}{[\eta](\text{linear})}$$

(3.32)

with the values of $[\eta]$ so obtained for the star and linear chains having the same $L$ and $d_b$.

Figure 3.6 shows plots of $g_\eta$ against the logarithm of $L$. The unfilled circles represent the theoretical values of $g_\eta$, various directions of pips indicating different values of $d_b$ indicated, and the dashed curves connect smoothly the theoretical values at constant $d_b$. The solid curves represent the values calculated from an interpolation formula, which are obtained and discussed in a later subsection. The horizontal line segment indicates the value 0.90 obtained by Irurzun\textsuperscript{19} for the Gaussian regular three-arm star chain without excluded volume in the KR approximation. As $L$ is decreased, $g_\eta$ first decreases and then increases after passing through a minimum, in the range of $d_b$ investigated. It should be noted that the behavior of $g_\eta$ remarkably depends on $d_b$.

In the following subsections, we examine the behavior of $g_\eta$ in the random-coil and rod limits and then propose an interpolation formula for $g_\eta$.

3.5.1 Random-Coil Limit

The value of $g_\eta$ becomes a constant independent of $d_b$ in the random-coil limit, i.e., the limit of $L$ (in units of $\lambda^{-1}$) $\to \infty$. Figure 3.7 shows plots of $g_\eta(L, d_b)$ against $L^{-1/2}$ for $d_b = 0.1, 0.2, 0.3,$ and $0.4$. All the symbols have the same meaning as those in Figure 3.6. The dashed curves connect smoothly the theoretical values at constant $d_b$ and the solid straight lines indicate the respective initial tangents. It is seen that as $L^{-1/2}$ is decreased to 0 ($L \to \infty$), $g_\eta$ approaches the above-mentioned value\textsuperscript{19} 0.90 irrespective of the value
Figure 3.6. Plots of $g_\eta(L, d_b)$ against log $L$. The unfilled circles represent the theoretical values, various directions of pips indicating different values of $d_b$ indicated. The dashed curves connect smoothly the theoretical values at constant $d_b$. The horizontal line segment indicates the value 0.90 obtained by Irurzun\textsuperscript{19} for the Gaussian regular three-arm star chain. The solid curves represent the values calculated from the interpolation formula (see the text).
Figure 3.7. Plots of $g_{\eta}(L, d_b)$ against $L^{-1/2}$. All the symbols have the same meaning as those in Figure 3.6. The dashed curves connect smoothly the theoretical values at constant $d_b$ and the solid straight lines indicate the respective initial tangents.
of \(d_b\). On the basis of such numerical results, it may be concluded that

\[
\lim_{L \to \infty} g_\eta(L, d_b) = 0.90_b \quad \text{(random coil)}
\] (3.33)

We note that the values of \(g_\eta\) for smaller \(d_b\) have been omitted in Figure 3.7, since we cannot make \(L^{-1/2} = [(n + 1)d_b]^{-1/2}\) small enough to evaluate \(g_\eta\) at \(L^{-1/2} = 0\). We also note that the \(g_\eta\) value so obtained in the random-coil limit is somewhat smaller than the value 0.90\(_r\) obtained by Zimm and Kilb\(^{20}\) for the (dynamic) Gaussian spring–bead model\(^{21,22}\) without excluded volume and with the preaveraged Oseen tensor.

The coil-limiting value in the KR approximation requires some comments. Khasat et al.\(^{23}\) obtained the \(g_\eta\) value 0.87 for regular three-arm star polystyrene in cyclohexane at 34.5 °C (\(\Theta\)), which is somewhat smaller than Irurzun’s value 0.90. Further, the \(g_\eta\) values 0.75 and 0.63 obtained for regular four- and six-arm star polystyrenes, respectively, in cyclohexane at 34.5 °C (\(\Theta\))\(^{24,25}\) are ca. 10% smaller than the respective theoretical values 0.82 and 0.70 obtained by Irurzun\(^{19}\) for the Gaussian regular four- and six-arm star chains in the KR approximation. The KR approximation therefore seems to overestimate \(g_\eta\) because of the preaveraging of the Oseen tensor. It is interesting to refer the previous MC result\(^1\) (Chapter 2) that the \(g_\eta\) values for the regular three-arm star freely rotating chain of bond angle 109° without the preaveraging approximation are ca. 5% smaller than the corresponding KR one.

### 3.5.2 Rod Limit

In the rod limit, i.e., the limit of \(L \to 0\), \([\eta]\) in the limit of \(L/d_b \to \infty\) (thin rod limit) may be written in the form (see Appendix 3.C),

\[
\lim_{L/d_b \to 0} [\eta] = \frac{\pi N_A L^3}{54M \ln(L/d_b)} \quad \text{(thin rod limit, star)}
\] (3.34)

As for the linear chain, we have

\[
\lim_{L/d_b \to \infty} [\eta] = \frac{\pi N_A L^3}{24M \ln(L/d_b)} \quad \text{(thin rod limit, linear)}
\] (3.35)

which has been obtained from the result\(^{17}\) for the cylinder model of diameter \(d\) along with the relation\(^8\) \(d = 0.74d_b\).

In the rod limit, \(g_\eta\) should be a function only of \(L/d_b\), i.e.,

\[
\lim_{L \to 0} g_\eta(L, d_b) = g_\eta^0(L/d_b) \quad \text{(rod limit)}
\] (3.36)

From eq 3.32 with eqs 3.34, 3.35, and 3.36, we have

\[
\lim_{L/d_b \to \infty} g_\eta^0(L/d_b) = \frac{4}{9} \quad \text{(thin rod limit)}
\] (3.37)
It is interesting to note that the ratio \( g_s \) of the mean-square radius of gyration of the KP regular three-arm star chain to that of the linear one, both having the same \( L \), also becomes \( 4/9 \) in the rod limit.

We have also evaluated \( g_0^\eta(L/d_b) \) numerically in the same manner as in the evaluation of \( g_\eta(L, d_b) \) mentioned above using the expressions for \( \langle R_{ij}^{-1} \rangle \) given by eq 3.21 and \( \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \) given by eqs 3.9 and 3.13 in place of those for the KP chain. Figure 3.8 shows plots of \( g_\eta \) against \( \ln(L/d_b) \). The unfilled circles represent the values so obtained and the horizontal line segment indicates the asymptotic value \( 4/9 \) in the limit of \( \ln(L/d_b) \to 0 \) \((L/d_b \to \infty)\). As \( \ln(L/d_b) \to 0 \) is decreased \((L/d_b \text{ is increased})\), \( g_\eta^0 \) monotonically decreases to \( 4/9 \). For later convenience, we have constructed an interpolation formula for \( g_\eta^0(L/d_b) \) in the range of \( L/d_b \geq 10 \), which is given by

\[
g_\eta^0(x) = 4 \frac{1 - 2.551(\ln x)^{-1} + 2.946(\ln x)^{-2}}{9 \frac{1 - 2.913(\ln x)^{-1} + 2.965(\ln x)^{-2}}{}} \quad \text{for } x \geq 10 \tag{3.38}
\]

In Figure 3.8, the curve represents the values calculated from eq 3.38 with \( x = L/d_b \). The error in the value of \( g_\eta^0(L/d_b) \) in the range of \( L/d_b \geq 10 \) (solid part) does not exceed 0.2%.

### 3.5.3 Interpolation Formula

A simple interpolation formula, if it is available, seems to be useful for practical purposes. Figure 3.9 shows plots of the ratio \( g_\eta(L, d_b)/g_\eta^0(L/d_b) \) against the logarithm of \( L \), where \( g_\eta/g_\eta^0 \) has been evaluated by dividing the \( g_\eta \) values shown in Figure 3.6 by the \( g_\eta^0 \) values calculated from eq 3.38 with \( x = L/d_b \). All the symbols have the same meaning as those in Figure 3.6. It is seen that the circles form a single composite curve almost independently of \( d_b \) for \( d_b \lesssim 0.2 \) and the curve monotonically increases from 1 to 2.03 \([= 0.90d_b/(4/9)]\) with increasing \( L \). We have therefore assumed that \( g_\eta/g_\eta^0 \) for \( d_b \lesssim 0.2 \) may be represented by a function \( f(L) \) of \( L \) irrespective of the value of \( d_b \), and have constructed an empirical interpolation formula for \( f(L) \), which may be written in the form,

\[
f(L) = 1 + 0.1466L - 0.01233L^2 \quad \text{for } L \leq 3
\]

\[
= 2.03 \frac{1 + 0.7106(\ln L)^{-1} + 3.219(\ln L)^{-2}}{1 + 0.9402(\ln L)^{-1} + 5.713(\ln L)^{-2}} \quad \text{for } L > 3 \tag{3.39}
\]

The function \( f(L) \) given by eq 3.39 satisfies the asymptotic conditions \( \lim_{L \to 0} f(L) = 1 \) and \( \lim_{L \to \infty} f(L) = 2.03 \), which hold in the limit of \( L/d_b \to \infty \). In Figure 3.9, the solid curve represents the values calculated from eq 3.39. The factor \( g_\eta(L, d_b) \) may then be approximately expressed as

\[
g_\eta(L, d_b) = g_\eta^0(L/d_b) f(L) \tag{3.40}
\]
Figure 3.8. Plots of $g^0_{\eta}(L/d_b)$ against $[\ln(L/d_b)]^{-1}$. The unfilled circles represent the theoretical values. The horizontal line segment indicates the asymptotic value $4/9$ in the limit of $[\ln(L/d_b)]^{-1} \to 0$ ($L/d_b \to \infty$). The curve represents the values of interpolation formula, the solid part indicating the range of $L/d_b \gtrsim 10$ (see the text).
Figure 3.9. Plots of $g_{\eta}(L, d_h)/g_{\eta}^{0}(L/d_h)$ against $\log L$. All the symbols have the same meaning as those in Figure 3.6. The solid curve represents the values of the interpolation formula (see the text).
where \(g_0^0(L/d_b)\) and \(f(L)\) are given by eqs 3.38 and 3.39, respectively.

In Figure 3.6, the solid curves represent the approximate values calculated from eq 3.40 with eqs 3.38 and 3.39. It is seen that the interpolation formula for \(g_0(L, d_b)\) so proposed may well reproduce the numerical theoretical values in the ranges of \(d_b \lesssim 0.2\) and \(L/d_b \gtrsim 10\). The error in the value of \(g_0\) in those ranges of \(d_b\) and \(L/d_b\) does not exceed 2%.

### 3.6 Conclusion

We have evaluated the ratio \(g_\eta\) of [\(\eta\)] of the KP regular three-arm star touched-bead model to that of the KP linear one, both having the same (reduced) total contour length \(L\) and (reduced) bead diameter \(d_b\), in the KR approximation, and then examined its behavior as a function of \(L\) and \(d_b\). It is found that the ratio \(g_\eta/g_\eta^0\) of \(g_\eta\) to the rod-limiting value \(g_\eta^0\) of \(g_\eta\) monotonically increases from 1 to 2.03 with increasing \(L\) and is almost independent of \(d_b\) for \(d_b \lesssim 0.2\), although the behavior of \(g_\eta\) itself as a function of \(L\) remarkably depends on \(d_b\). Thus we have constructed the empirical interpolation formula for \(f(L) = g_\eta/g_\eta^0\) as a function of \(L\). By the use of the expression for \(f(L)\) along with that for \(g_\eta^0\), which has also been constructed, the value of \(g_\eta\) may then be easily calculated for given \(L\) and \(d_b\) in the ranges of \(d_b \lesssim 0.2\) and \(L/d_b \gtrsim 10\).

### Appendix 3.A Average \(\langle S(t^{(i)}) \cdot S(t'^{(j)}) \rangle\)

In this appendix, we consider the equilibrium configurational average \(\langle S(t^{(i)}) \cdot S(t'^{(j)}) \rangle\) with \(S(t^{(i)})\) the vector distance from the center of mass of the KP regular \(f\)-arm star chain to the contour point on the \(i\)th arm \((i = 1, 2, \cdots, f)\) with the contour distance \(t^{(i)}\) from the branch point. Let \(L_a\) be the contour length of the arms and \(\theta_{ij}\) the angle between the unit vectors tangent to the \(i\)th and \(j\)th arms at the branch point, so that the total contour length \(L\) of the chain becomes \(fL_a\) and \(0 \leq t^{(i)}, t'^{(j)} \leq L_a\).

The average may be written in the form,

\[
\langle S(t^{(i)}) \cdot S(t'^{(j)}) \rangle = \frac{1}{2L} \sum_{k=1}^{f} \int_{0}^{L_a} \left[ \langle R^2(t^{(i)}, s^{(k)}) \rangle + \langle R^2(t'^{(j)}, s^{(k)}) \rangle \right] ds^{(k)} \]

\[
- \frac{1}{2} \langle R^2(t^{(i)}, t'^{(j)}) \rangle - \langle S^2 \rangle
\]

(3.A.1)

where \(\langle R^2(t^{(i)}, t'^{(j)}) \rangle\) is the mean-square distance between the two contour points \(t^{(i)}\) and \(t'^{(j)}\) and is given by

\[
\langle R^2(t^{(i)}, t'^{(j)}) \rangle = \langle R^2(t^{(i)}, t'^{(j)}, \theta_{ij}) \rangle
\]

\[
= \left| t^{(i)} - t'^{(j)} \right| - \frac{1}{2} \left( 1 - e^{-2|t^{(i)}-t'^{(j)}|} \right)
\]

for \(i \neq j\)

\[
\langle R^2(t^{(i)}, t'^{(i)}) \rangle = \langle R^2(t^{(i)}, t'^{(i)}, \theta_{ii}) \rangle
\]

\[
= \left| t^{(i)} - t'^{(i)} \right| - \frac{1}{2} \left( 1 - e^{-2|t^{(i)}-t'^{(i)}|} \right)
\]

for \(i = j\)

(3.A.2)
with \( \langle R^2(t^{(i)}, t^{(j)}, \theta_{ij}) \rangle \) being given by eq 3.19. In eq 3.A.1, \( \langle S^2 \rangle \) is the mean-square radius of gyration of the KP regular \( f \) arm star chain and is given by
\[
\langle S^2 \rangle = \frac{3f - 2}{6f^2} - \frac{2f - 1}{4f} + \frac{1}{4L} + \left( \frac{f - 1}{4L} - \frac{f}{8L^2} \right) (1 - e^{-2L/f})
\]
\[
-\frac{1}{2f^2} \left[ 1 - \left( \frac{f}{L} - \frac{f^2}{2L^2} \right) (1 - e^{-2L/f}) - \frac{f^2}{4L^2} (1 - e^{-4L/f}) \right]
\]
\[
\times \sum_{i=1}^{f-1} \sum_{j=i+1}^{f} \cos \theta_{ij}
\]
(3.A.3)

Carrying out the integration over \( s^{(k)} \), we obtain for the sum in eq 3.A.1
\[
\sum_{k=1}^{f} \int_{0}^{L_a} \left( \langle R^2(t^{(i)}, s^{(k)}) \rangle + \langle R^2(t^{(j)}, s^{(k)}) \rangle \right) ds^{(k)} = f L_a^2 - (2f - 1)L_a + \frac{1}{2}(f + 1)
\]
\[
+(t^{(i)})^2 + (t^{(j)})^2 + (f - 2)L_a(t^{(i)} + t^{(j)}) + \frac{1}{2}(f - 1)L_a\left(e^{-2t^{(i)}} + e^{-2t^{(j)}}\right)
\]
\[
-\frac{1}{2}(f - 1)e^{-2L_a} - \frac{1}{4}(e^{-2t^{(i)}} + e^{-2t^{(j)}}) - \frac{1}{4}e^{-2L_a}(e^{2t^{(i)}} + e^{2t^{(j)}})
\]
\[
-\frac{1}{2} \left[ L_a - \frac{1}{2}(1 - e^{-2L_a}) \right] \left( 1 - e^{-2t^{(i)}} \right) \sum_{k=1}^{f} \cos \theta_{ik} + \left( 1 - e^{-2t^{(j)}} \right) \sum_{k=1}^{f} \cos \theta_{jk}
\]
(3.A.4)

Eq 3.A.1 with eqs 3.A.2–3.A.4 and also with eq 3.19 gives \( \langle \mathbf{S}(t^{(i)}) \cdot \mathbf{S}(t^{(j)}) \rangle \). For the special case of \( f = 3 \) and \( \theta_{ij} = 120^\circ \) for all \( i \) and \( j \), it reduces to eq 3.7 with \( t^{(i)} \) and \( t^{(j)} \) in place of \( t_k^{(i)} \) and \( t_l^{(j)} \), respectively.

## Appendix 3.B  Asymptotic Forms for \( \langle R^{-1} \rangle \) of the Once-Broken KP Chain

In this appendix, we derive asymptotic forms for the mean reciprocal \( \langle R^{-1}(t_1, t_2, \theta) \rangle \) of the end-to-end distance \( R \) of the once-broken KP chain of total contour length \( t_1 + t_2 \) such that two KP subchains 1 and 2 of contour lengths \( t_1 \) and \( t_2 \), respectively, are connected with a bending angle \( \theta \) (see Figure 3.2).

Considering the characteristic function \( I(k; t_1, t_2, \theta) \), i.e., the Fourier transform of the distribution function \( P(R; t_1, t_2, \theta) \) of \( R \), defined by
\[
I(k; t_1, t_2, \theta) = \int P(R; t_1, t_2, \theta)e^{ikR}dR
\]
(3.B.1)
and then integrating both sides of eq 3.B.1 over \( k \), we obtain the following expression for \( \langle R^{-1}(t_1, t_2, \theta) \rangle \),
\[
\langle R^{-1}(t_1, t_2, \theta) \rangle = \frac{2}{\pi} \int_0^\infty I(k; t_1, t_2, \theta)dk
\]
(3.B.2)
Then the problem is to obtain asymptotic forms for $I(k; t_1, t_2, \theta)$.

The distribution function $P(R; t_1, t_2, \theta)$ may be written in the form,

$$P(R; t_1, t_2, \theta) = \frac{1}{8\pi^2} \int [\int G(R_1|u_1; t_1) G(R_2|u_2; t_2)$$

$$\times \delta(R - R_1 + R_2) \, dR_1 \, dR_2] \, du_1 \, du_2$$  \hspace{1cm} (3.B.3)$$

where $G(R_p|u_p; t_p)$ ($p = 1, 2$) is the conditional distribution function (or the Green’s function) of the vector distance $R_p$ from the broken point to the terminal end of the $p$th subchain of contour length $t_p$ with the unit vector $u_p$ tangent to the $p$th contour at the broken point (see Figure 3.2) being fixed, and $\delta(R)$ is the three-dimensional Dirac delta function. In eq 3.B.3, $\int \cdots du_1 du_2$ means the integrations over $u_1$ and $u_2$ with $\theta$ being fixed. From eq 3.B.1 with eq 3.B.3, we have for $I(k; t_1, t_2, \theta)$,

$$I(k; t_1, t_2, \theta) = \frac{1}{8\pi^2} \int I^*(k|u_1; t_1)I(k|u_2; t_2) du_1 \, du_2$$  \hspace{1cm} (3.B.4)$$

where $I(k|u_p; t_p)$ is the characteristic function of $G(R_p|u_p; t_p)$ and the asterisk indicates the complex conjugate.

The characteristic function $I(k|u_p; t_p)$ may be expanded in terms of the (normalized) spherical harmonics $Y_l^m$ as follows,

$$I(k|u_p; t_p) = (4\pi)^{1/2} \sum_{l=0}^{\infty} \mathcal{I}_l(k; t_p) \sum_{m=-l}^{l} Y_l^m(\theta_p, \phi_p)Y_l^m(\chi, \omega)$$  \hspace{1cm} (3.B.5)$$

where $\mathcal{I}_l(k; t_p)$ is the expansion coefficient and $u_p = (1, \theta_p, \phi_p)$ and $k = (k, \chi, \omega)$ in spherical polar coordinates in an external Cartesian coordinate system. We note that the above expression for $I(k|u; t)$ is essentially identical with eq 5 of ref 15 and the coefficient $\mathcal{I}_l(k; t)$ in eq 3.B.5 is identical with $\mathcal{I}_{0l}^{00}(k; t)$ in eq 5 of ref 15, although the direction of $u_p$ was chosen to coincide with the $z$-axis of the external system in ref 15. In order to perform the integrations on the right-hand side of eq 3.B.4 over $u_1$ and $u_2$ with $\theta$ being fixed, we introduce a Cartesian coordinate system associated with $u_1$ and $u_2$ such that $u_p$ becomes equal to $(1, \tilde{\theta}_p, 0)$ in spherical polar coordinates in it with $\tilde{\theta}_1 = 0$ and $\tilde{\theta}_2 = \theta$. Let $\Omega = (\theta', \phi', \psi')$ be the Euler angles defining the orientation of the Cartesian coordinate system with respect to the external one. Then the integration $\int \cdots du_1 du_2$ may be replaced by

$$\int \cdots d\Omega \, (d\Omega = \sin \theta' d\theta' d\phi' d\psi')$$

and $Y_l^m(\theta_p, \phi_p)$ may be written in terms of $\Omega$ as

$$Y_l^m(\theta_p, \phi_p) = (-1)^{(m+|m|)/2} \left( \frac{8\pi^2}{2l + 1} \right)^{1/2} \sum_{j=-l}^{l} (-1)^{(j+|j|)/2} D_l^m(\Omega) Y_l^j(\tilde{\theta}_p, 0)$$  \hspace{1cm} (3.B.6)$$
where $D_{mj}^l(\Omega)$ is the (normalized) Wigner function of $\Omega$. Substituting eq 3.B.5 with eq 3.B.6 into eq 3.B.4 and performing the integration over $\Omega$, we obtain

$$I(k; t_1, t_2, \theta) = (4\pi)^{-1} \sum_{l=0}^\infty (2l + 1) I_l(k; t_1) I_l(k; t_2) P_l(\cos \theta)$$

(3.B.7)

where we have used the relations,

$$\int D_{mj}^l(\Omega) D_{m'}^{l'}(\Omega) d\Omega = \delta_{ll'} \delta_{mm'} \delta_{jj'}$$

(3.B.8)

$$l X_m = \sum_{m=-l}^{l} (2l + 1) I_l(k; t_1) I_l(k; t_2) P_l(\cos \theta)$$

(3.B.9)

and $P_l(x)$ is the Legendre polynomial.

The asymptotic form for $I_l(k; t_p)$ in the case of $t_p \gg 1$ may be obtained in the Daniels approximation, and its general expression (for the helical wormlike (HW) chain) is given by eq 44 of ref 14 with $l_1 = 0, l_2 = l_3 = l, m_1 = m_2 = 0$, and $j_1 = j_2 = 0$. On retaining terms up to $O(t_p^{-2})$ (i.e., the second Daniels approximation), the necessary $I_l(k; t_p)$'s with $l = 0, 1, \cdots, 4$ can be straightforwardly evaluated to be

$$I_l(k; t_p) = (4\pi)^{1/2} \frac{1}{2l+1} \exp \left( -\frac{1}{6} t_p k^2 \right) (ik)^l g_l(k; t_p)$$

(3.B.12)

with $i$ the imaginary unit and

$$g_0(k; t_p) = 1 + \frac{1}{12} k^2 + \left( \frac{107}{6480} - \frac{11}{1080} t_p \right) k^4 - \frac{607}{272160} t_p k^6 + \frac{121}{2332800} t_p^2 k^8$$

$$g_1(k; t_p) = \frac{1}{2} + \frac{13}{180} k^2 - \frac{11}{2160} t_p k^4$$

$$g_2(k; t_p) = \frac{1}{18} + \frac{85}{9072} k^2 - \frac{11}{19440} t_p k^4$$

$$g_3(k; t_p) = \frac{1}{360}$$

$$g_4(k; t_p) = \frac{1}{12000}$$

(3.B.13)

The asymptotic form for $I_l(k; t_p)$ in the case of $t_p \ll 1$, on the other hand, may be obtained by the $\epsilon$ method, and its general expression (for the HW chain) is given
by eq 10 of ref 16. On retaining terms up to $O(t_p^2)$, the necessary $I_l(k; t_p)$'s with $l = 0, 1, \cdots, 4$ may be explicitly written in the form,

$$I_l(k; t_p) = (4\pi)^{1/2}i\left[1 + (\delta_l)j_l(kt_p) - \frac{1}{2}kt_p(\langle \epsilon \rangle + (\delta_l\epsilon))j_{l+1}(kt_p) + \frac{1}{8}(kt_p)^2(\epsilon^2)j_{l+2}(kt_p)\right]$$  \hspace{1cm} (3.B.14)

where $j_l(x)$ is the spherical Bessel function of the first kind and $\epsilon$ and $\delta_l$ are relative deviations of the square end-to-end distance $R_p^2$ and of the quantity $R_p^4Y_l^0(\theta_p, \phi_p)Y_l^0(\Theta_p, \Phi_p)$ with $R_p = (R_p, \Theta_p, \Phi_p)$ in spherical polar coordinates in the external system, respectively, of the KP subchain $p$, from the respective rod-limiting values. We note that $\delta_l$ is identical with $\delta_{0l00}$ defined for the HW chain by eq 4 of ref 16. The equilibrium averages $\langle \epsilon \rangle$, $\langle \epsilon^2 \rangle$, $\langle \delta_l \rangle$, and $\langle \delta_l\epsilon \rangle$ in eq 3.B.14 are given by eqs 5, 6, and 7 of ref 16 and are explicitly given by

$$\langle \epsilon \rangle = -\frac{2}{3}t_p + \frac{1}{3}t_p^2$$  
$$\langle \epsilon^2 \rangle = \frac{28}{45}t_p^2$$  
$$\langle \delta_l \rangle = 4\pi \langle R_p^4Y_l^0(\theta_p, \phi_p)Y_l^0(\Theta_p, \Phi_p) \rangle t_p^{-l} - 1$$  
$$\langle \delta_l\epsilon \rangle = 4\pi \langle R_p^4Y_l^0(\theta_p, \phi_p)Y_l^0(\Theta_p, \Phi_p) \rangle t_p^{-l-2} - (1 + \langle \epsilon \rangle + \langle \delta_l \rangle)$$  \hspace{1cm} (3.B.15)

The moments $\langle R_p^4Y_l^0Y_l^0 \rangle$ and $\langle R_p^{l+2}Y_l^0Y_l^0 \rangle$ in eqs 3.B.15 may be evaluated analytically by the use of the operational method, \cite{7,27} and then we have

$$\langle R_p^4Y_l^0Y_l^0 \rangle = (4\pi)^{-1} \frac{(2l + 1)!}{2(2l + 1)^{1/2}}$$  
$$\times \int \left\{ Y_0^{0*} \left[ \prod_{j=1}^l \int_0^{s_j-1} ds_je^{\mathcal{L}_0(s_j-1-s_j)}\delta\mathcal{L} \right] e^{-l(l+1)s_lY_l^0} \right\} du_p'$$  \hspace{1cm} (3.B.16)

$$\langle R_p^{l+2}Y_l^0Y_l^0 \rangle = (4\pi)^{-1} \frac{(2l + 3)!}{2(2l + 1)^{1/2}(l + 1)!}$$  
$$\times \int \left\{ Y_0^{0*} \left[ \prod_{j=1}^{l+2} \int_0^{s_j-1} ds_je^{\mathcal{L}_0(s_j-1-s_j)}\delta\mathcal{L} \right] e^{-l(l+1)s_{l+2}Y_l^0} \right\} du_p'$$  \hspace{1cm} (3.B.17)

where $s_0 = t_p$, $u_p' = (1, \theta_p', \phi_p')$ is the unit vector tangent to the terminal end of the $p$th contour in spherical polar coordinates in the external system, $\mathcal{L}_0$ is the Laplacian operator given by

$$\mathcal{L}_0 = \nabla_u u_p^2 = \frac{1}{\sin \theta_p'} \frac{\partial}{\partial \theta_p'} \sin \theta_p' \frac{\partial}{\partial \theta_p'} + \frac{1}{\sin^2 \theta_p'} \frac{\partial^2}{\partial \phi_p'^2}$$  \hspace{1cm} (3.B.18)

and $\delta\mathcal{L}$ is given by

$$\delta\mathcal{L} = a_0 + a_{-1}$$  \hspace{1cm} (3.B.19)
with $a_{\mu}^0 (\mu = 1, 2)$ the creation and annihilation operator which operate on $Y_l^m$ as

$$a_{\mu}^0 Y_l^m = A_l^{[m]} (l_{1+1/2}) Y_l^{m_{1+\mu}} \quad (3.B.20)$$

In eq 3.B.20, the eigenvalue $A_l^m$ is given by

$$A_l^m = \left[ \frac{(l + m + 1)(l - m + 1)}{(2l + 1)(2l + 3)} \right]^{1/2} \quad (3.B.21)$$

Performing the integration over $u_p$ in eqs 3.B.16 and 3.B.17 using with eqs 3.B.18–3.B.21, we have the necessary expressions for $\langle R_p^l Y_l^0 Y_l^0 \rangle$ and $\langle R_p^{l+2} Y_l^0 Y_l^0 \rangle$ as follows,

$$\langle R_p^1 Y_l^0 Y_l^0 \rangle = \frac{1}{8\pi} (1 - e^{-2t_p})$$

$$\langle R_p^2 Y_2^0 Y_2^0 \rangle = \frac{1}{16\pi} (1 - e^{-2t_p}) - \frac{1}{48\pi} (1 - e^{-6t_p})$$

$$\langle R_p^3 Y_3^0 Y_3^0 \rangle = \frac{3}{160\pi} (1 - e^{-2t_p}) - \frac{1}{96\pi} (1 - e^{-6t_p}) + \frac{1}{480\pi} (1 - e^{-12t_p})$$

$$\langle R_p^4 Y_4^0 Y_4^0 \rangle = \frac{1}{240\pi} (1 - e^{-2t_p}) - \frac{1}{336\pi} (1 - e^{-6t_p}) + \frac{1}{960\pi} (1 - e^{-12t_p}) - \frac{1}{6720\pi} (1 - e^{-20t_p}) \quad (3.B.22)$$

Substituting eqs 3.B.15 with eqs 3.B.22 and 3.B.23 into eq 3.B.14, we can finally
obtain

\[ I_0(k; t_p) = (4\pi)^{1/2} \left[ j_0(kt_p) - \frac{1}{6} t_p^2 (t_p - 2) k j_1(kt_p) + \frac{7}{90} t_p^4 k^2 j_2(kt_p) \right] \]

\[ I_1(k; t_p) = (4\pi)^{1/2} \left[ \left( \frac{2}{3} t_p^2 - t_p + 1 \right) j_1(kt_p) - \frac{1}{30} t_p^2 (19t_p - 10) k j_2(kt_p) \right. \]

\[ + \left. \frac{7}{90} t_p^4 k^2 j_3(kt_p) \right] \]

\[ I_2(k; t_p) = (4\pi)^{1/2} \left[ \left( \frac{13}{3} t_p^2 - 8 + t_p + 1 \right) j_2(kt_p) + \frac{1}{90} t_p^2 (127t_p - 30) k j_3(kt_p) \right. \]

\[ - \left. \frac{7}{90} t_p^4 k^2 j_4(kt_p) \right] \]

\[ I_3(k; t_p) = (4\pi)^{1/2} \left[ \left( \frac{73}{5} t_p^2 - 5t_p + 1 \right) j_3(kt_p) + \frac{1}{6} t_p^2 (15t_p - 2) k j_4(kt_p) \right. \]

\[ - \left. \frac{7}{90} t_p^4 k^2 j_5(kt_p) \right] \]

\[ I_4(k; t_p) = (4\pi)^{1/2} \left[ \left( \frac{221}{5} t_p^2 - 8t_p + 1 \right) j_4(kt_p) - \frac{1}{30} t_p^2 (117t_p - 10) k j_5(kt_p) \right. \]

\[ + \left. \frac{7}{90} t_p^4 k^2 j_6(kt_p) \right] \]

(3.B.24)

Substituting eqs 3.B.7 with \( I_i(k; t_1) \) and \( I_i(k; t_2) \), both given by eq 3.B.12 with eqs 3.B.13, into eq 3.B.2 and performing the integration over \( k \), we have for the asymptotic form for \( \langle R^{-1}(t_1, t_2, \theta) \rangle \) in the case of \( t_1 \gg 1 \) and \( t_2 \gg 1 \)

\[
\langle R^{-1}(t_1, t_2, \theta) \rangle = \left[ \frac{6}{\pi(t_1 + t_2)} \right]^{1/2} \left\{ 1 - \frac{1}{40(t_1 + t_2)} - \frac{73}{4480(t_1 + t_2)^2} \right. \\
+ \left[ \frac{1}{4(t_1 + t_2)} + \frac{49}{160(t_1 + t_2)^2} \right] \left( 1 + \cos \theta \right) - \frac{1}{40(t_1 + t_2)^2} (1 - \cos^2 \theta) \right\} \\
\quad \text{for } (t_1 \gg 1, \ t_2 \gg 1) \quad (3.B.25)
\]

If we use \( I_i(k; t_2) \) given by eqs 3.B.24 in the above calculation, we have the asymptotic form in the case of \( t_1 \gg 1 \) and \( t_2 \ll 1 \)

\[
\langle R^{-1}(t_1, t_2, \theta) \rangle = \left( \frac{6}{\pi t_1} \right)^{1/2} \left[ 1 - \frac{1}{40t_1} - \frac{t_2}{2t_1} - \frac{73}{4480t_1^2} + \frac{3t_2^2}{80t_1^2} + \frac{3t_2^2}{8t_1^2} \right. \\
+ \left( \frac{1}{2t_1} - \frac{3}{80t_1^2} \right) (t_2 - t_2^2)(1 + \cos \theta) - \frac{3t_2^2}{20t_1^2} (1 - \cos^2 \theta) \left( 1 - \frac{1}{40t_1} \right) \\
\quad \text{for } (t_1 \gg 1, \ t_2 \ll 1) \quad (3.B.26)
\]

Considering the fact that \( \langle R^2(t_1, t_2, \theta) \rangle \) given by eq 3.19 may be expanded in the above
Appendix 3.C  Intrinsic Viscosity in the Rod Limit

In this appendix, we derive the asymptotic solution in the limit of $L/d_b \to \infty$ (thin or long rod limit) for $[\eta]$ of the KP regular three-arm star in the rod limit. In the thin rod limit, $[\eta]_E$ may be ignored, so that we only consider $[\eta]_{KR}$ given by eq 3.2. Further, the distance between the adjacent beads, i.e., $d_b$ for the touched-bead model, becomes very small compared to $L$, so that we may convert the sums in eqs 3.2 and 3.4 to integrals. Eq 3.2 with eq 3.4 may then be rewritten in the form,

$$[\eta] = \frac{3N_A L}{M} \int_0^1 \psi_0(x,x) dx$$

(3.C.1)

where $\psi_0(x,y)$ is the solution of the set of integral equations,

$$\int_0^1 [K_0(x,\xi)\psi_0(\xi,y) + 2K_1(x,\xi)\psi_1(\xi,y)]d\xi = g_0(x,y)$$

$$\int_0^1 [K_0(x,\xi)\psi_1(\xi,y) + K_1(x,\xi)\psi_0(\xi,y) + K_1(x,\xi)\psi_1(\xi,y)]d\xi = g_1(x,y)$$

(3.C.2)

In eq 3.C.2, $K_0(x,y)$ and $K_1(x,y)$ are the continuous versions of the mean reciprocal of the distance between the centers of two beads on the same arm and on different ones, respectively, and $g_0(x,y)$ and $g_1(x,y)$ are those of $\pi(S_i \cdot S_j)$ for the centers of two beads on the same arm and on different ones, respectively. They are explicitly given by

$$K_0(x,y) = \frac{3}{L} |x - y|^{-1} \quad \text{for } |x - y| \geq 3d_b/L$$

$$= 0 \quad \text{for } |x - y| < 3d_b/L$$

(3.C.3)

$$K_1(x,y) = \frac{3}{L} \left( x^2 + y^2 + xy \right)^{-1/2}$$

(3.C.4)
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\[ g_0(x, y) = \frac{\pi L}{9} xy \]  \hspace{1cm} (3.C.5)

\[ g_1(x, y) = \frac{\pi L}{18} xy \]  \hspace{1cm} (3.C.6)

Now we expand \( \psi_0(x, y), \psi_1(x, y), g_0(x, y), \) and \( g_1(x, y) \) in terms of the shifted Legendre polynomial \( \tilde{P}_l(x) \) as, for example,

\[ \psi_0(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_{0,ij} \tilde{P}_i(x) \tilde{P}_j(y) \]  \hspace{1cm} (3.C.7)

where \( \tilde{P}_l(x) \) (\( l = 0, 1, 2, \ldots \)) is defined by

\[ \tilde{P}_l(x) = (-1)^l P_l(2x - 1) \]  \hspace{1cm} (3.C.8)

We note that \( \tilde{P}_l(x) \) satisfies the following orthogonality relation,

\[ \int_0^1 \tilde{P}_l(x) \tilde{P}_l(x) dx = \frac{\delta_{ll}}{2l + 1} \]  \hspace{1cm} (3.C.9)

Then eq 3.C.1 may be written in terms of the expansion coefficients \( \psi_{0,ij} \) as

\[ [\eta] = \frac{3N_A L}{M} \sum_{i=0}^{\infty} \frac{\psi_{0,ii}}{2i + 1} \]  \hspace{1cm} (3.C.10)

and the integral equations 3.C.2 may be converted to a set of linear simultaneous equations, which may be written in the following matrix form,

\[ K \cdot \psi = g \]  \hspace{1cm} (3.C.11)

The three matrices \( K, \psi, \) and \( g \) in eq 3.C.11 have the same structure composed of 9 submatrices and may be explicitly written as, for example,

\[ K = \begin{pmatrix} K_0 & K_1 & K_1 \\ K_1 & K_0 & K_1 \\ K_1 & K_1 & K_0 \end{pmatrix} \]  \hspace{1cm} (3.C.12)

where \( K_0 \) and \( K_1 \) are the symmetric matrices whose \( ij \) elements are given by

\[ K_{k,ij} = \int_0^1 \int_0^1 K_k(x, y) \tilde{P}_i(x) \tilde{P}_j(y) dx dy \hspace{1cm} (k = 0, 1) \]  \hspace{1cm} (3.C.13)

Similarly, \( \psi (g) \) is composed of \( \psi_0 \) and \( \psi_1 \) (\( g_0 \) and \( g_1 \)) whose \( ij \) elements are the expansion coefficients \( \psi_{0,ij} \) and \( \psi_{1,ij} \) (\( g_{0,ij} \) and \( g_{1,ij} \)), respectively.

It can be shown in the limit of \( L/d_b \to \infty \) that the diagonal elements \( K_{0,ii} \) of \( K \) are proportional to \( L^{-1} \ln(L/d_b) \) while the off-diagonal ones \( K_{0,ij} (i \neq j) \) and \( K_{1,ij} \) are of
\[ \mathcal{O}(L^{-1}\ln(L/d_b)^{[0]} \right) \text{ at most. Further the elements } g_{0,ij} \text{ and } g_{1,ij} \text{ of } g_0 \text{ and } g_1, \text{ respectively, may be calculated to be} \\
\begin{align*}
g_{0,ij} &= \frac{\pi L}{36} (2i+1)(2j+1) \left( \delta_{i0} - \frac{1}{3} \delta_{i1} \right) \left( \delta_{j0} - \frac{1}{3} \delta_{j1} \right) \\
g_{1,ij} &= -\frac{\pi L}{72} (2i+1)(2j+1) \left( \delta_{i0} - \frac{1}{3} \delta_{i1} \right) \left( \delta_{j0} - \frac{1}{3} \delta_{j1} \right)
\end{align*}
(3.C.14) (3.C.15)

Then we have
\[ \psi_{ii} = \frac{\pi L}{36(2i+1)} [\delta_{i0} K_{0,00}^{-1} + \delta_{i1} K_{0,11}^{-1} + \mathcal{O}(L[\ln(L/d_b)]^{-2})] \] (3.C.16)
and \([\eta]\) may be written in the form
\[ \lim_{L/d_b \to \infty} [\eta] = \frac{\pi N_A L^2}{12 M} \left( K_{0,00}^{-1} + \frac{1}{9} K_{0,11}^{-1} \right) \] (3.C.17)
The necessary elements \(K_{0,00}\) and \(K_{0,11}\) of \(K\) are calculated to be
\[ \begin{align*}
K_{0,00} &= 6 \ln(L/d_b)/L \\
K_{0,11} &= 2 \ln(L/d_b)/L
\end{align*} \] (3.C.18)

Substitution of eqs 3.C.18 into eq 3.C.17 leads to eq 3.34.

References

4 Second Virial Coefficient of Semiflexible Three-Arm Stars

4.1 Introduction

In Chapters 2 and 3, we have examined effects of chain stiffness on the intrinsic viscosity $[\eta]$ of unperturbed regular three-arm star polymer chains by Monte Carlo (MC) simulations$^1$ or on the basis of the Kirkwood–Riseman theory.$^{2-4}$ The ratio $g_\eta$ of $[\eta]$ of an unperturbed regular three-arm star chain to that of an unperturbed linear one, both having the same molecular weight and chain stiffness, has been shown to become remarkably smaller than the value ca. 0.9 in the random-coil limit$^{5,6}$ as the chains become stiffer, as in the case of the ratio $g_S$ of the mean-square radius of gyration $\langle S^2 \rangle$ of the former chain to that of the latter.$^7$ We note that $g_S$ becomes 7/9 in the random-coil limit.$^8$ Since $[\eta]$ and $\langle S^2 \rangle$ are measures of the average chain dimension, the above-mentioned results for $g_\eta$ and $g_S$ indicate that the average dimension of the unperturbed regular three-arm star chain becomes much smaller than that of the corresponding linear one as the chains become stiffer.

As already mentioned in Chapter 1, short-ranged repulsive interactions work between segments constituting the polymer chains in good solvents (perturbed state), expanding the individual chains and yielding an effective volume $V_E$ excluded to one chain by the presence of another. The volume $V_E$ is another measure of the average chain dimension, although well-defined only in a good solvent system or in the perturbed state. The quantity $V_E$ may be defined from the second virial coefficient $A_2$ as follows,$^4$

$$A_2 = 4N_AV_E/M^2 \quad (4.1)$$

where $N_A$ is the Avogadro constant and $M$ is the molecular weight. It is then interesting to examine the effects of chain stiffness on the ratio $g_{A_2}$ of $A_2$ of a regular three-arm star chain to that of the corresponding linear one. In this chapter, we make an MC study of $g_{A_2}$ as a necessary continuation of the previous studies$^{1,2}$ (Chapters 2 and 3).

An available theoretical value of $g_{A_2}$ for the regular three-arm star chain is 0.968 in the random-coil limit obtained by Douglas and Freed$^9$ on the basis of the polymer renormalization group (RG) theory, being nearly equal to unity. Then the main question
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we must answer in this chapter is whether or not \( g_{A_2} \) becomes remarkably smaller than unity as the chains become stiffer, as in the cases of \( g_\eta \) and \( g_S \).

### 4.2 Models and Methods

The MC models used in this study are the same as those used in the previous one\(^1\) (Chapter 2) except for the value of a parameter of interactions between segments (beads).

The star chain model is the regular three-arm star freely rotating chain, each arm composed of \( m \) successive bonds of length unity, so that it is composed of \( 3m = n \) bonds, in total, and \( 3m + 1 \) beads whose centers are located at \( 3m - 3 \) junctions of two successive bonds on the arms, at the three terminal ends, and at the branch point (center) (see Figure 2.1 in Chapter 2). The angle between each pair of the bonds connected to the center is fixed to be \( 120^\circ \), so that those bonds are on the same plane. For convenience, the three arms are designated the first, second, and third ones and the beads on the \( i \)th arm \((i = 1, 2, 3)\) are numbered \((i - 1)m + 1, (i - 1)m + 2, \ldots, im\) from the center to the end, with the center bead numbered 0. The \( i \)th bond vector \((1 \leq i \leq 3m; i \neq m + 1, 2m + 1)\) connects the centers of the \((i - 1)\)th and \( i \)th beads with its direction from the \((i - 1)\)th to \( i \)th bead, and the \((m + 1)\)th and \((2m + 1)\)th bond vectors are from the 0th to the \((m + 1)\)th and \((2m + 1)\)th beads, respectively. All the \( 3m - 3 \) bond angles \( \theta \) (not supplement) except for those around the center are fixed, so that the configuration of the entire chain may be specified by the set of \( 3m - 3 \) rotation angles \( \{\phi_{3m-3}\} = (\phi_1, \ldots, \phi_{m-1}, \phi_{m+1}, \ldots, \phi_{2m-1}, \phi_{2m+1}, \ldots, \phi_{3m-1}) \) apart from its position and orientation in an external Cartesian coordinate system, where \( \phi_i \) is the internal rotation angle around the \( i \)th bond vector.

The linear chain model, the counterpart of the above star one, is the freely rotating chain composed of \( n \) bonds of length unity and \( n + 1 \) beads, whose centers are located at the \( n - 1 \) junctions of two successive bonds and at the two terminal ends. We set \( n \) equal to \( 3m \). The beads are numbered \( 0, 1, 2, \ldots, n \) from one end to the other, and the \( i \)th bond vector \((i = 1, 2, \ldots, n)\) connects the centers of the \((i - 1)\)th and \( i \)th beads with its direction from the \((i - 1)\)th to the \( i \)th bead. All the \( n - 1 \) bond angles are fixed at \( \theta \), so that the configuration of the linear chain may be specified by the set of \( n - 2 \) internal rotation angles \( \{\phi_{n-2}\} = (\phi_2, \phi_3, \ldots, \phi_{n-1}) \) apart from its position and orientation in the external Cartesian coordinate system.

The total potential energy \( U \) of the regular three-arm star chain as a function of
4.2 Models and Methods

\( \{ \phi_{3m-3} \} \) may be given by

\[
U(\{ \phi_{3m-3} \}) = \sum_{i=0}^{1} \sum_{j=i+1}^{2} \sum_{k,l=1}^{m} h(k + l - 4) u(R_{(im+k)(jm+l)})
\]

\[
+ \sum_{i=0}^{2} \sum_{j=1}^{m-4} \sum_{k=j+4}^{m} u(R_{(im+j)(im+k)}) + \sum_{i=0}^{2} \sum_{j=4}^{m} u(R_{0(im+j)})
\]

(regular three-arm star) \ (4.2)

and that of the linear one as a function of \( \{ \phi_{n-2} \} \) by

\[
U(\{ \phi_{n-2} \}) = \sum_{i=0}^{n-4} \sum_{j=i+4}^{n} u(R_{ij}) \quad \text{(linear)} \quad (4.3)
\]

where \( h(x) \) is a unit step function such that \( h(x) = 1 \) for \( x \geq 0 \) and \( h(x) = 0 \) for \( x < 0 \) and \( R_{ij} \) is the distance between the centers of the \( i \)th and \( j \)th beads. We note that in eqs 4.2 and 4.3 the interactions between the third-neighbor beads along the chain have been neglected, since they seem to make the chain locally take the cis conformation to excess.

Similarly, the total intermolecular (excluded-volume) potential energy \( U_{12}(1, 2) \) between two chains 1 and 2 as a function of all the coordinates of chains 1 and 2 may be given by

\[
U_{12}(1, 2) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} u(R_{i_1i_2}) \quad (4.4)
\]

where \( R_{i_1i_2} \) is the distance between the centers of the \( i_1 \)th bead of chain 1 and of the \( i_2 \)th one of chain 2. We note that we use the McMillan–Mayer symbolism\(^4,10\) to formulate \( A_2 \), here and hereafter. Then the \( i \)th bead \( (i = 0, 1, 2, \cdots, n) \) of chain \( \alpha \) \( (\alpha = 1, 2) \) is labeled as \( i_\alpha \), and the symbol \( (\alpha) \) \( (\alpha = 1, 2) \) denotes all the coordinates (external and internal) of chain \( \alpha \).

We adopt as the pair potential \( u(R) \) (of mean force) the cutoff version of the Lennard–Jones (LJ) 6-12 potential given by

\[
u(R) = \begin{cases} 
\infty & \text{for } 0 \leq R < c\sigma \\
u^{LJ}(R) & \text{for } c\sigma \leq R < 3\sigma \\
0 & \text{for } 3\sigma \leq R
\end{cases}
\quad (4.5)
\]

where \( u^{LJ}(R) \) is the LJ potential\(^1\) given by

\[
u^{LJ}(R) = 4\epsilon \left( \frac{\sigma}{R} \right)^{12} - \left( \frac{\sigma}{R} \right)^6 
\quad (4.6)
with $\sigma$ and $\epsilon$ the collision diameter and the depth of the potential well at the minimum of $u^{\text{LJ}}(R)$, respectively. We note that $u(R)$ given by eqs 4.5 is the LJ potential cut off at the upper bound $3\sigma$. The lower bound $c\sigma$ in eqs 4.5 has been introduced for numerical convenience; the factor $c$ is properly chosen so that the Boltzmann factor $e^{-u^{\text{LJ}}/k_B T}$ may be regarded as numerically vanishing compared to unity, where $k_B$ is the Boltzmann constant and $T$ is the absolute temperature. In practice, in double-precision computation, we put

$$c = \left[ 2/(1 + \sqrt{1 + 36T^*}) \right]^{1/6}$$

(4.7)

so that $e^{-u^{\text{LJ}}/k_B T} \lesssim 2 \times 10^{-16}$ for $0 \leq R < c\sigma$, where $T^*$ is the reduced temperature defined by $T^* = k_B T/\epsilon$. Further, we put $\sigma = 1$ for simplicity, as previously\(^1\) (Chapter 2) done. We note that the value $3.72$ of $T^*$ corresponds to the $\Theta$ temperature and $8.0$ to a good solvent system.\(^{12,13}\)

The mean-square radius of gyration $\langle S^2 \rangle$, i.e., the ensemble average of the square radius of gyration $S^2$ may be evaluated from

$$\langle S^2 \rangle = N_s^{-1} \sum_{\{\phi_k\}} S^2(\{\phi_k\})$$

(4.8)

where the sum is taken over $N_s$ sample configurations $\{\phi_k\}$ with $k$ being equal to $3m - 3$ for the star chain and $n - 2$ for the linear one, $\{\phi_k\}$ generated in an MC run by an application of the pivot algorithm\(^{14,15}\) and the Metropolis method of importance sampling\(^1\) as done in the previous MC studies.\(^1,12\) For each sample configuration, $S^2$ as a function of $\{\phi_k\}$ may be calculated from

$$S^2 = \frac{1}{n + 1} \sum_{i=0}^{n} |\mathbf{r}_i - \mathbf{r}_{\text{c.m.}}|^2$$

(4.9)

with $\mathbf{r}_i$ the vector position of the center of the $i$th bead and $\mathbf{r}_{\text{c.m.}}$ the vector position of the center of mass of the chain given by

$$\mathbf{r}_{\text{c.m.}} = \frac{1}{n + 1} \sum_{i=0}^{n} \mathbf{r}_i$$

(4.10)

In every MC run, an initial configuration is generated by trial and error so that all the distances between the centers of beads are greater than or equal to $c$. (Note that $\sigma = 1$.) One configuration is sampled at every $M_{\text{nom}}$ (nominal) pivot steps, so that $N_s \times M_{\text{nom}}$ pivot steps are required to obtain a set of $N_s$ sample configurations. The number of $M_{\text{nom}}$ has been chosen to be ca. $10n$ for the regular three-arm star chains and ca. $2n$ for the linear ones.

As for the second virial coefficient $A_2$, it may be evaluated from

$$A_2 = \frac{2\pi N_A}{M^2} \int_0^{\infty} \left\{1 - \exp \left[ - \frac{U_{12}(r)}{k_B T} \right]\right\} r^2 dr$$

(4.11)
where $\overline{U}_{12}(r)$ is the averaged intermolecular potential as a function of the distance $r = |r|$ between the centers of mass of chains 1 and 2 defined by

$$
\overline{U}_{12}(r) = -k_B T \ln \left\langle \exp \left[ -\frac{U_{12}(1,2)}{k_B T} \right] \right\rangle_r
$$

(4.12)

In eq 4.12, $\langle \cdots \rangle_r$ indicates the conditional equilibrium average taken over the configurations of the two chains with $r$ fixed by the use of the single-chain distribution function for each with the intramolecular excluded-volume potential for the star and linear chains given by eqs 4.2 and 4.3, respectively. In practice, the conditional average may be calculated by the use of a set of $N_s$ sample configurations $\{\phi_k\}$ generated above, as follows. First, we randomly sample a pair of sample configurations (chains 1 and 2) from the set and calculate the intermolecular potential $U_{12}$ from eq 4.4 at given $r$ after randomizing the orientations of the two configurations in the external coordinate system. Numerical evaluation of $U_{12}$ may be carried out following the procedure used in the previous study of $A_2$ with the use of the “zippering” method. Then the average on the right-hand side of eq 4.12 may be evaluated from

$$
\left\langle \exp \left[ -\frac{U_{12}(1,2)}{k_B T} \right] \right\rangle_r = N_p^{-1} \sum'_{(1,2)} \exp \left[ -\frac{U_{12}(1,2)}{k_B T} \right]
$$

(4.13)

where $\sum'_{(1,2)}$ indicates summation over $N_p$ pairs of sample configurations (1,2) at given $r$. With the values of $\overline{U}_{12}(r)$ so evaluated for various values of $r$ at given $T^*$, the quantity $A_2M^2$ for given $n$ may then be calculated from eq 4.11 by numerical integration with the use of the trapezoidal rule formula.

In the practical evaluation of $\langle S^2 \rangle$ and $A_2$ for the regular three-arm star and linear freely rotating chains at $T^* = 8.0$ (good solvent system), we have generated 10 sets of $10^5$ ($= N_s$) sample configurations for $n = 30, 60, 90, 150, \text{ and } 300$, 5 sets of those for $n = 600$, and 2 sets of those for $n = 900$. In the evaluation of $\overline{U}_{12}$ (and $A_2$), $10^6$ or $10^7$ ($= N_p$) sample pairs have been taken from each set. Then the total number of sample pairs is equal to the number $N_p$ of sample pairs in each set multiplied by the number of sets.

All the numerical work has been done by the use of a personal computer with an Intel Core2 Duo E6600 CPU. A source program coded in C has been compiled by the GNU C compiler version 3.4.6 with real variables of double precision. For a generation of pseudorandom numbers, the subroutine package MT19937 supplied by Matsumoto and Nishimura has been used instead of the subroutine RAND included in the standard C library.
4.3 Results and Discussion

4.3.1 Mean-Square Radius of Gyration

We have carried out MC runs to generate sample configurations for the regular three-arm star and linear freely rotating chains with \( n = 3m \) = 30, 60, 90, 150, 300, 600, and 900 and \( \theta = 109^\circ, 120^\circ, 135^\circ, 165^\circ, \) and \( 175^\circ \) at \( T^* = 8.0 \) (good solvent system). The chain of \( \theta = 109^\circ \) corresponds to a typical flexible chain, and the chain becomes stiffer with increasing \( \theta \) from \( 109^\circ \) to \( 175^\circ \).

The MC values of \( \langle S^2 \rangle /n \) for the star and linear chains are given in the second and fourth columns, respectively, of Table 4.1, the number in the parentheses attached to each value indicating its statistical error. The \( \langle S^2 \rangle /n \) value and its error for each chain are the mean and the standard deviation, respectively, of independent MC results.

Figure 4.1 shows double-logarithmic plots of \( \langle S^2 \rangle /n \) against \( n \) at \( T^* = 8.0 \) (good solvent system). The open and closed symbols represent the MC values for the star and linear chains, respectively, of \( \theta = 109^\circ \) (\( \sigma, \bullet \)), \( 120^\circ \) (\( \square, \blacksquare \)), \( 135^\circ \) (\( \bigotimes, \bigtriangleup \)), \( 165^\circ \) (\( \bigtriangleup, \bigtriangleup \)), and \( 175^\circ \) (\( \nabla, \nabla \)). The solid and dashed curves represent the theoretical values of \( \langle S^2 \rangle /n \) for the ideal regular three-arm star and linear freely rotating chains, respectively, without interactions between beads, which have been calculated from

\[
\langle S^2 \rangle = \frac{7}{54} \left( 1 - \cos \theta \right) \frac{1}{n} + \frac{1}{54} \frac{31 + 90 \cos \theta - 13 \cos^2 \theta}{(1 + \cos \theta)^2} + \frac{19}{18} \frac{19 + 49 \cos \theta + 71 \cos^2 \theta + 5 \cos^3 \theta}{(1 + \cos \theta)^3} + \frac{1}{n + 1}
\]

\[
\langle S^2 \rangle = \frac{1}{6} \left( 1 - \cos \theta \right) \frac{1}{n} + \frac{1}{6} \frac{1 + 6 \cos \theta - \cos^2 \theta}{(1 + \cos \theta)^2} + \frac{1}{6} \frac{1 - 7 \cos \theta + 7 \cos^2 \theta + \cos^3 \theta}{(1 + \cos \theta)^3} + \frac{1}{n + 1}
\]

and

\[
\langle S^2 \rangle = \frac{1}{18} \left( 1 - \cos \theta \right) \frac{1}{n} + \frac{2(1 + 2 \cos \theta)}{(1 + \cos \theta)^3} \frac{1 - (- \cos \theta)^{n/3} + 1}{n + 1} + \frac{1}{27} \frac{(1 - \cos \theta)(37 - 16 \cos \theta + \cos^2 \theta)}{(1 + \cos \theta)^3} \frac{1}{(n + 1)^2} - \frac{2(2 + \cos^2 \theta)}{(1 + \cos \theta)^4} \frac{1 - (- \cos \theta)^{n/3} + 1}{n + 1} + \frac{3}{(1 + \cos \theta)^4} \frac{[1 - (- \cos \theta)^{n/3} + 1]^2}{(n + 1)^2}
\]

(4.14)
### Table 4.1. MC results at $T^* = 8.0$ (good solvent system)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n$</th>
<th>$\langle S^2 \rangle / n$ (error %)</th>
<th>$A_2 M_b^2 / N_A$ (error %)</th>
<th>$\langle S^2 \rangle / n$ (error %)</th>
<th>$A_2 M_b^2 / N_A$ (error %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>109°</td>
<td>30</td>
<td>0.302 (0.1)</td>
<td>0.251 (0.1)</td>
<td>0.389 (0.1)</td>
<td>0.261 (0.2)</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.340 (0.1)</td>
<td>0.203 (0.1)</td>
<td>0.443 (0.1)</td>
<td>0.213 (0.2)</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>0.365 (0.1)</td>
<td>0.181 (0.1)</td>
<td>0.477 (0.1)</td>
<td>0.191 (0.1)</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>0.400 (0.1)</td>
<td>0.159 (0.1)</td>
<td>0.523 (0.1)</td>
<td>0.168 (0.1)</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.452 (0.1)</td>
<td>0.133 (0.1)</td>
<td>0.591 (0.1)</td>
<td>0.142 (0.1)</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>0.508 (0.1)</td>
<td>0.113 (0.1)</td>
<td>0.667 (0.1)</td>
<td>0.120 (0.2)</td>
</tr>
<tr>
<td></td>
<td>900</td>
<td>0.548 (0.1)</td>
<td>0.102 (0.0)</td>
<td>0.716 (0.1)</td>
<td>0.109 (0.1)</td>
</tr>
<tr>
<td>120°</td>
<td>30</td>
<td>0.379 (0.1)</td>
<td>0.285 (0.1)</td>
<td>0.500 (0.1)</td>
<td>0.301 (0.2)</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.431 (0.1)</td>
<td>0.241 (0.1)</td>
<td>0.565 (0.1)</td>
<td>0.252 (0.2)</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>0.462 (0.1)</td>
<td>0.218 (0.1)</td>
<td>0.604 (0.1)</td>
<td>0.229 (0.1)</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>0.502 (0.1)</td>
<td>0.194 (0.1)</td>
<td>0.656 (0.1)</td>
<td>0.205 (0.1)</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.561 (0.1)</td>
<td>0.165 (0.2)</td>
<td>0.732 (0.1)</td>
<td>0.175 (0.1)</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>0.626 (0.1)</td>
<td>0.141 (0.1)</td>
<td>0.817 (0.1)</td>
<td>0.150 (0.2)</td>
</tr>
<tr>
<td></td>
<td>900</td>
<td>0.667 (0.0)</td>
<td>0.129 (0.1)</td>
<td>0.872 (0.0)</td>
<td>0.137 (0.0)</td>
</tr>
<tr>
<td>135°</td>
<td>30</td>
<td>0.553 (0.1)</td>
<td>0.327 (0.2)</td>
<td>0.790 (0.1)</td>
<td>0.336 (0.2)</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.663 (0.1)</td>
<td>0.287 (0.3)</td>
<td>0.907 (0.2)</td>
<td>0.297 (0.1)</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>0.720 (0.1)</td>
<td>0.268 (0.1)</td>
<td>0.966 (0.2)</td>
<td>0.278 (0.2)</td>
</tr>
<tr>
<td></td>
<td>150</td>
<td>0.782 (0.1)</td>
<td>0.246 (0.2)</td>
<td>1.036 (0.1)</td>
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(continued)
\[ \theta = 175^\circ \]

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respectively, with the indicated values of $\theta$. We note that eq 4.14 is a special case of the theoretical expression obtained by Guenza et al.\(^{20}\) for the regular $f$-arm star freely rotating chain.

In the cases of $\theta = 109^\circ$, 120$^\circ$ and 135$^\circ$, the MC values for both the star and linear chains deviate upward from the corresponding ideal-chain ones because of the intramolecular excluded-volume effect and also of the effect of the interactions between beads on the unperturbed chain dimension through the short-range interference.\(^{12,21,22}\) In the cases of $\theta = 165^\circ$ and 175$^\circ$, on the other hand, the MC values for both the star and linear chains are almost identical with the corresponding ideal-chain values, because the effects of the intramolecular interactions between beads become negligibly small for a typical stiff chain if its total contour length is not very long. We note that the ring-closure or intramolecular-contact probability approaches zero in the rigid-rod limit and therefore the intramolecular excluded-volume effect vanishes in the limit.

Figure 4.2 shows plots of $g_S$ against log $n$ at $T^* = 8.0$ (good solvent system), where $g_S$ is the ratio of $\langle S^2 \rangle$ of the regular three-arm star chain to that of the corresponding linear one. The symbols represent the MC values of $g_S$ calculated from the values of $\langle S^2 \rangle/n$ given in the second and fourth columns in Table 4.1 for $\theta = 109^\circ$ (○), 120$^\circ$ (□), 135$^\circ$ (◇), 165$^\circ$ (△), and 175$^\circ$ (▽). The solid curves represent the theoretical values for the ideal chains calculated from eqs 4.14 and 4.15. The upper and lower dotted horizontal lines represent the asymptotic values $7/9$ in the ideal (unperturbed) random-coil limit\(^{8}\) and $4/9$ in the rigid-rod limit,\(^{7}\) respectively.

In the case of $\theta = 109^\circ$ (typical flexible chain), the MC value 0.76 of $g_S$ for $n = 900$ is somewhat (ca. 2%) smaller than the value 7/9 ($= 0.778$) in the ideal random-coil limit. The available experimental values of $g_S$ for regular $f$-arm star polystyrenes (PS) in good solvents obtained by Khasat et al.\(^{23}\) for $f = 3$ and by Okumoto et al.\(^{24,25}\) for $f = 4$ and 6 are 0.79, 0.61, and 0.45, respectively, and are in good agreement with the respective ideal random-coil limiting values $7/9$ ($= 0.778$), 5/8 ($= 0.625$), and 4/9 ($= 0.444$), which have been calculated from $g_S = (3f - 2)/f^2$.\(^{8}\) The available theoretical values of $g_S$ obtained by Douglas and Freed\(^9\) on the bases of the polymer RG theory for the regular three-, four-, and six-arm stars in the perturbed random-coil limit are 0.778, 0.631, and 0.453, respectively, and are consistent with the experimental ones. Then the present MC result for long flexible chains is consistent with the above-mentioned experimental and theoretical results.

As a natural consequence of the results shown in Figure 4.1, the MC value of $g_S$ becomes almost identical with the ideal chain value as the chain becomes stiffer. The ratio $g_S$, on the whole, remarkably depends on the chain stiffness but scarcely on the intramolecular excluded volume.
Figure 4.1. Double-logarithmic plots of \( \langle S^2 \rangle / n \) against \( n \) at \( T^* = 8.0 \) (good solvent system). The open and closed symbols represent the MC values for the regular three-arm star and linear freely rotating chains, respectively, of \( \theta = 109^\circ \) (△, ▪), 120° (□, ■), 135° (◇, ◆), 165° (△, ▲), and 175° (▽, ▼). The solid and dashed curves represent the theoretical values for the ideal regular three-arm star and linear freely rotating chains, respectively, with the indicated values of \( \theta \).
Figure 4.2. Plots of $g_S$ against $\log n$ at $T^* = 8.0$ (good solvent system) for the freely rotating chains of $\theta = 109^\circ$ (○), 120$^\circ$ (□), 135$^\circ$ (◊), 165$^\circ$ (△), and 175$^\circ$ (▽). The solid curves represent the theoretical values for the ideal freely rotating chains with the indicated values of $\theta$. The upper and lower dotted horizontal lines represent the asymptotic values $7/9$ in the ideal random-coil limit and $4/9$ in the rigid-rod limit, respectively.
4.3.2 Second Virial Coefficient

We have numerically evaluated $A_2 M^2 / N_A$ from eqs 4.11—4.13 on the basis of the sets of sample configurations generated in the above-mentioned MC evaluation of $\langle S^2 \rangle$. The MC values of $A_2 M^2_b / N_A$ with $M_b = M / n$ so obtained for the star and linear chains are given in the third and fifth columns, respectively, of Table 4.1, the number in the parentheses attached to each value indicating its statistical error. The $A_2 M^2_b / N_A$ value and its error for each chain are the mean and the standard deviation, respectively, of independent MC results.

Figure 4.3 shows double-logarithmic plots of $A_2 M^2_b / N_A$ against $n$ at $T^* = 8.0$ (good solvent system), where all the open and closed symbols have the same meaning as those in Figure 4.1. In the case of $\theta = 109^\circ$ (typical flexible chain), the value for the regular three-arm star is somewhat (6% at most) smaller than the corresponding value for the linear chain. For both the star and linear chains, $A_2$ decreases with increasing $n$ and the slopes of the plots become almost identical with the asymptotic value $-0.2$ for very large $n$. The difference in the value of $A_2$ between the two chains becomes small and the slopes become gentle, as $\theta$ (chain stiffness) is increased. In the case of $\theta = 175^\circ$ (typical semiflexible or stiff chain), $A_2$ becomes almost independent of $n$ for $n \geq 100$, as in the case of the rigid rod.

It is interesting to note that the value of $A_2$ for the chain of $\theta = 175^\circ$ is smaller than that of $\theta = 165^\circ$ over the whole range of $n$ examined. It may be regarded as arising from the fact that two chains with the LJ potential (having the attractive interaction), which are close to each other, prefer to be parallel rather than perpendicular to each other when their stiffness becomes large and their $A_2$ may therefore be suppressed (see the next Chapter), as discussed by Schoot and Odijk$^{26}$ in the case of long rigid rods with a van der Waals-type potential.

Figure 4.4 shows plots of $g_{A_2}$ against log $n$ at $T^* = 8.0$ (good solvent system). The symbols, which have the same meaning as those in Figure 4.2, represent the MC values of $g_{A_2}$ calculated from the values of $A_2 M^2_b / N_A$ given in the third and fifth columns in Table 4.1. We note that from the defining equation 4.1 for the effective intermolecular excluded volume $V_E$, $g_{A_2}$ may be written in terms of $V_E$ as follows,

$$g_{A_2} = \frac{A_2 \text{(star)}}{A_2 \text{(linear)}} = \frac{V_E \text{(star)}}{V_E \text{(linear)}} \quad (4.16)$$

In the case of $\theta = 109^\circ$ (typical flexible chain), $g_{A_2}$ slightly decreases with increasing $n$ and then seems to approach a constant independent of $n$. The asymptotic value of $g_{A_2}$ in the limit of $n \to \infty$ may be estimated to be 0.94 which is the mean of the three MC values 0.94$_0$, 0.94$_0$, and 0.93$_6$ for $n = 300, 600$, and 900, respectively. The $g_{A_2}$ value so...
obtained is rather in good agreement with the RG theory value\(^9\) 0.968 represented by the dotted horizontal line segment in Figure 4.4. In this connection, we refer to literature experimental and theoretical values of \(g_{A_2}\) for regular stars. The former values are 0.89 and 0.79 for the regular four- and six-arm star PSs\(^{24,25}\) respectively, and they are rather in good agreement with the corresponding RG theory values\(^9\) 0.923 and 0.808. Strictly speaking, the RG theory values are somewhat (ca. 3%) larger than the corresponding experimental and present MC values.

It is more important to see in Figure 4.4 that \(g_{A_2}\) slightly increases with increasing \(\theta\) (chain stiffness) in the range of \(\theta\) from 109° to 165° and it becomes almost identical with unity for \(\theta = 165°\) and 175° in the range of \(n \geq 100\). We note that \(g_{A_2}\) in the cases of \(\theta = 165°\) and 175° approaches, of course, the above-mentioned asymptotic value in the limit of \(n \to \infty\). In contrast to the cases of \(g_S\) and \(g_\eta\), \(g_{A_2}\) is rather insensitive to change in \(\theta\) (chain stiffness). Such characteristic behavior of \(g_{A_2}\) may be regarded as arising from the fact that in terms of the (classical) perturbation theory\(^4\) effects of the multiple intermolecular contact become negligibly small, if any, in the rigid-rod limit. The decreases in \(A_2\) of the star and linear chains due to the attractive interaction in the LJ potential may be considered to cancel out with each other in \(g_{A_2}\).

Finally, we consider the apparent interpenetration function \(\Psi_{ap}\) defined by\(^{27}\)

\[
A_2 = 4\pi^{3/2}N_A \frac{\langle S^2 \rangle^{3/2}}{M^2} \Psi_{ap}
\]  

(4.17)

from the whole \(A_2\) including effects of chain ends\(^{28,29}\). Figure 4.5 shows double-logarithmic plots of \(\Psi_{ap}\) against \(n\) at \(T^* = 8.0\) (good solvent system). The open and closed symbols, which have the same meaning as those in Figures 4.1 and 4.3, represent the MC values for the star and linear chains, respectively, calculated from eq 4.17 with the values of \(\langle S^2 \rangle/n\) and \(A_2M_b^2/N_A\) given in Table 4.1.

In the case of \(\theta = 109°\) (typical flexible chain), \(\Psi_{ap}\) for both the star and linear chains slightly decreases with increasing \(n\) and then seems to approach the asymptotic values 0.34 and 0.24, respectively, which are the means of the three \(\Psi_{ap}\) values for \(n = 300, 600,\) and 900, although \(\Psi_{ap}\) is appreciably larger for the star chain than for the linear one. The two asymptotic values are in good agreement with the literature MC values 0.35 for the star chain and 0.235 for the linear one obtained by Ohno \textit{et al.}\(^{30}\) and Barrett,\(^{31}\) respectively, by the use of self-avoiding chains on the simple cubic lattice. In Figure 4.5, the two literature values for the star and linear chains are represented by the solid and dashed horizontal line segments, respectively. We note that the RG theory values\(^9\) 0.384 and 0.269 of \(\Psi_{ap}\) for the perturbed flexible regular three-arm star and linear chains, respectively, are somewhat (ca. 10%) larger than the above-mentioned present and literature MC values. As \(\theta\) (chain stiffness) is increased from 109° to 175°, \(\Psi_{ap}\) decreases monotonically.
Figure 4.3. Double-logarithmic plots of $A_2 M_b^2 / N_A$ against $n$ at $T^* = 8.0$ (good solvent system). All the symbols have the same meaning as those in Figure 4.1.
Figure 4.4. Plots of $g_{A_2}$ against $\log n$ at $T^* = 8.0$ (good solvent system). All the symbols have the same meaning as those in Figure 4.2. The dotted horizontal line segment represents the RG theory value$^9$ 0.968.
Figure 4.5. Double-logarithmic plots of $\Psi_{ap}$ against $n$ at $T^* = 8.0$ (good solvent system). All the symbols have the same meaning as those in Figure 4.1. The solid horizontal line segment represents the asymptotic value 0.35 in the random-coil limit obtained by Ohno et al.$^{30}$ for the self-avoiding regular three-arm star chain on the simple cubic lattice and the dashed one the corresponding value 0.235 obtained by Barrett$^{31}$ for the self-avoiding linear chain on the same lattice.
Figure 4.6. Plots of $g_\Psi$ against $\log n$ at $T^* = 8.0$ (good solvent system). All the symbols have the same meaning as those in Figure 4.2.
From eqs 4.16 and 4.17, $g_{A_2}$ may be written in the form,

$$g_{A_2} = g_S^{3/2} \Phi$$  \hspace{1cm} (4.18)

where $\Phi$ is the ratio of $\Psi_{ap}$ of the regular three-arm star chain to that of the corresponding linear one. Figure 4.6 shows plots of $\Phi$ against log $n$ at $T^* = 8.0$ (good solvent system), where all the symbols have the same meaning as those in Figures 4.2 and 4.4. It is seen that $\Phi$ remarkably increases with increasing $\theta$ (chain stiffness). From eq 4.18 with the results shown in Figures 4.2, 4.4, and 4.6, it may be said that the increase in $\Phi$ compensates the decrease in $g_S^{3/2}$ and therefore $g_{A_2}$ is insensitive to change in $\theta$ (chain stiffness).

### 4.4 Conclusion

We have examined the effects of chain stiffness on the ratio $g_{A_2}$ of $A_2$ of the regular three-arm star chain to that of the corresponding linear one on the basis of the MC results for $A_2$ and $(S^2)$ of the freely rotating chains with the LJ 6-12 potential corresponding to a good solvent system. It is found that $g_{A_2}$ is rather insensitive to change in chain stiffness in contrast to the cases of the ratios $g_S$ and $g_\eta$ related to $(S^2)$ and $[\eta]$, respectively, which remarkably decrease with increasing chain stiffness. Such characteristic behavior of $g_{A_2}$ may be regarded as arising from the fact that in terms of the (classical) perturbation theory,\(^4\) the effects of the multiple intermolecular contact become negligibly small, if any, in the rigid-rod limit, or in other words, the increase in the ratio $\Phi$ related to $\Psi_{ap}$ compensates the decrease in $g_S^{3/2}$.

### References

4 Second Virial Coefficient
5 Some Comments on the Second Virial Coefficient of Semiflexible Polymers

5.1 Introduction

When we consider dilute solution behavior of nonionic flexible polymers in good solvents, interactions between segments constituting the polymer chains are all, without exception, treated in the binary cluster approximation, both in the frameworks of the (classical) perturbation theory\(^1\) and of the polymer renormalization group theory.\(^2\)–\(^4\) The approximation assumes that the effect of the interactions, i.e., the excluded-volume effect, does not depend on a detailed form of the pair potential \(u(R)\) of mean force between the segments as a function of their separation \(R\) but only on the binary-cluster integral \(\beta\) defined by

\[
\beta = 4\pi \int_0^\infty [1 - e^{-u(R)/k_BT}]R^2 dR
\]  

(5.1)

with \(k_B\) the Boltzmann constant and \(T\) the absolute temperature. The quantity \(\beta\) so defined in the perturbation theory represents the effective volume excluded to one segment by the presence of another,\(^1\) which, strictly speaking, might be distinguished from an analog defined in the polymer renormalization group theory. Since \(u(R)\) is in general short-ranged for nonionic polymers, the approximation is considered to be valid and has actually been working well as far as nonionic flexible polymers are concerned. In this chapter, we consider the question of whether or not the approximation is still valid when the polymer chain becomes stiff. We note that such a question is, of course, outside the scope of the polymer renormalization group theory which completely discards information about local structures of polymer chains.

For the present purpose, we carry out a Monte Carlo (MC) simulation study of the mean-square radius of gyration \(\langle S^2 \rangle\) and second virial coefficient \(A_2\) for the freely rotating chain\(^1,\)\(^5\) of bond angle \(\theta\) ranging from 109° to 175° and with the Lennard–Jones (LJ) 6-12 potential\(^6\) or the hard-sphere (HS) one. The chain of \(\theta = 109°\) (≃ tetrahedral bond angle) is usually used as a model for flexible polymer chains,\(^7\)–\(^9\) and the stiffness of the chain increases with increasing \(\theta\) from 109° to 175°. Considering the fact that the binary cluster approximation works well for flexible chains, it is expected that we can find an appropriate diameter of the HS model so that both its \(\langle S^2 \rangle\) and \(A_2\) agree with the corresponding ones
of the LJ model with given energy parameters. In anticipation of results, we note that such a diameter really exists. Then the above question becomes more concrete, i.e., how the values of \( \langle S^2 \rangle \) and \( A_2 \) for the two interaction potentials come to be, as \( \theta \) is increased from 109° to 175°.

5.2 Models and Methods

The model chain and the simulation algorithm used in this study is the same as those in the previous studies\(^7,8\) of \( \langle S^2 \rangle \) and \( A_2 \), except for the inclusion of the HS potential. Here we give brief descriptions of the model and algorithm.

The model chain is the freely rotating chain\(^1,5\) composed of \( n \) bonds, each of length unity, and of \( n + 1 \) identical beads, whose centers are located at the \( n - 1 \) junctions of two successive bonds and at the two terminal ends. The beads are numbered 0, 1, 2, \( \cdots \), \( n \) from one end to the other, and the \( i \)th bond vector connects the centers of the \( (i - 1) \)th and \( i \)th beads with its direction from the \( (i - 1) \)th to the \( i \)th bead. All the \( n - 1 \) bond angles (not supplements) are fixed at \( \theta \), so that the configuration of the entire chain may be specified by the set of \( n - 2 \) internal rotation angles \( \{ \phi_{n-2} \} = (\phi_2, \phi_3, \cdots, \phi_{n-1}) \) apart from its position and orientation in an external Cartesian coordinate system, where \( \phi_i \) is the internal rotation angle around the \( i \)th bond vector.

The total intramolecular (excluded-volume) potential energy \( U \) of the chain as a function of \( \{ \phi_{n-2} \} \) may be given by

\[
U(\{ \phi_{n-2} \}) = \sum_{i=0}^{n-4} \sum_{j=i+4}^{n} u(R_{ij})
\]

(5.2)

where \( u(R_{ij}) \) is the pair potential as a function of the distance \( R_{ij} \) between the centers of the \( i \)th and \( j \)th beads. We must note here that the pairwise decomposability of the potential energy has been assumed, as is usually done in the field.\(^1\) We also note that in eq 5.2 the interactions between the third-neighbor beads along the chain have been neglected, since they seem to make the chain locally take the cis conformation to excess in the case of the LJ 6-12 potential with an attractive interaction.

Similarly, the total intermolecular (excluded-volume) potential energy \( U_{12}(1,2) \) between two chains 1 and 2 as a function of all the coordinates of chains 1 and 2 may be given by

\[
U_{12}(1,2) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} u(R_{i_1i_2})
\]

(5.3)

where \( R_{i_1i_2} \) is the distance between the centers of the \( i_1 \)th bead of chain 1 and of the \( i_2 \)th one of chain 2. We note that we use the McMillan–Mayer symbolism\(^1,10\) to formulate \( A_2 \),
5.2 Models and Methods  

here and hereafter. Then the \(i\)th bead \((i = 0, 1, 2, \ldots, n)\) of chain \(\alpha\) \((\alpha = 1, 2)\) is labeled as \(i_\alpha\), and the symbol \((\alpha)\) \((\alpha = 1, 2)\) denotes all the coordinates (external and internal) of chain \(\alpha\).

As mentioned in Section 5.1, we adopt as the pair potentials \(u(R)\) the LJ 6-12 potential in a cutoff version and the HS one, the former being given by

\[
\begin{align*}
  u(R) &= \infty \quad \text{for } 0 \leq R < c\sigma \\
  &= u^{\text{LJ}}(R) \quad \text{for } c\sigma \leq R < 3\sigma \\
  &= 0 \quad \text{for } 3\sigma \leq R
\end{align*}
\]

where \(u^{\text{LJ}}(R)\) is the LJ 6-12 potential given by

\[
  u^{\text{LJ}}(R) = 4\epsilon \left[ \left( \frac{\sigma}{R} \right)^{12} - \left( \frac{\sigma}{R} \right)^6 \right]
\]

with \(\sigma\) and \(\epsilon\) the collision diameter and the depth of the potential well at the minimum of \(u^{\text{LJ}}(R)\), respectively. We note that \(u(R)\) given by eqs 5.4 is the LJ potential cut off at the upper bound \(3\sigma\). The lower bound \(c\sigma\) in eqs 5.4 has been introduced for numerical convenience; the factor \(c\) is properly chosen so that the Boltzmann factor \(e^{-u^{\text{LJ}}(c\sigma)/k_B T}\) may be regarded as numerically vanishing compared to unity. In practice, in double-precision numerical computation, we put

\[
c = \left[ \frac{2}{\sqrt{1 + 36T^*}} \right]^{1/6}
\]

so that \(e^{-u^{\text{LJ}}/k_B T} \lesssim 2 \times 10^{-16}\) for \(0 \leq R < c\sigma\), where \(T^*\) is the reduced temperature defined by \(T^* = k_B T/\epsilon\). On the other hand, the HS potential is simply written in the form,

\[
  u(R) = \infty \quad \text{for } 0 \leq R < \sigma_c \\
  = 0 \quad \text{for } \sigma_c \leq R
\]

where \(\sigma_c\) is the collision diameter.

The mean-square radius of gyration \(\langle S^2 \rangle\), i.e., the ensemble average of the square radius of gyration \(S^2\) may be evaluated from

\[
  \langle S^2 \rangle = N_s^{-1} \sum_{\{\phi_{n-2}\}} S^2(\{\phi_{n-2}\})
\]

where the sum is taken over \(N_s\) sample configurations \(\{\phi_{n-2}\}\) generated in an MC run by an application of the pivot algorithm\(^{11,12}\) and the Metropolis method of importance sampling,\(^{13}\) as done in the previous MC study\(^7\) of \(\langle S^2 \rangle\). For each sample configuration, \(S^2\) as a function of \(\{\phi_{n-2}\}\) may be calculated from

\[
  S^2 = \frac{1}{n + 1} \sum_{i=0}^{n} |\mathbf{r}_i - \mathbf{r}_{\text{c.m.}}|^2
\]
with $\mathbf{r}_i$ the vector position of the center of the $i$th bead and $\mathbf{r}_{c.m.}$ the vector position of the center of mass of the chain given by

$$
\mathbf{r}_{c.m.} = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{r}_i
$$

(5.10)

In every MC run, an initial configuration is generated by trial and error so that all the distances between the centers of beads are greater than or equal to $c\sigma$ in the case of the LJ potential or $\sigma_c$ in the case of the HS one. One configuration is sampled at every $M_{\text{nom}}$ (nominal) pivot steps, so that $N_s \times M_{\text{nom}}$ pivot steps are required to obtain a set of $N_s$ sample configurations.

As for the second virial coefficient $A_2$, it may be evaluated from

$$
A_2 = \frac{2\pi N_A}{M^2} \int_0^\infty \left\{ 1 - \exp \left[ -\frac{U_{12}(r)}{k_B T} \right] \right\} r^2 dr
$$

(5.11)

where $N_A$ is the Avogadro constant, $M$ is the molecular weight, and $U_{12}(r)$ is the averaged intermolecular potential as a function of the distance $r = |\mathbf{r}|$ between the centers of mass of chains 1 and 2 defined by

$$
U_{12}(r) = -k_B T \ln \left\langle \exp \left[ -\frac{U_{12}(1,2)}{k_B T} \right] \right\rangle_r
$$

(5.12)

In eq 5.12, $\langle \cdots \rangle_r$ indicates the conditional equilibrium average taken over the configurations of the two chains with $\mathbf{r}$ fixed by the use of the single-chain distribution function for each with the intramolecular excluded-volume potential given by eq 5.2. In practice, the conditional average may be calculated by the use of a set of $N_s$ sample configurations generated above, as follows. First, we randomly sample a pair of sample configurations (chains 1 and 2) from the set and calculate the intermolecular potential $U_{12}$ from eq 5.3 at given $r$ after randomizing the orientations of the two configurations in the external coordinate system. Numerical evaluation of $U_{12}$ may be carried out following the procedure used in the previous study of $A_2$ with the use of the "zippering" method. Then the average on the right-hand side of eq 5.12 may be evaluated from

$$
\left\langle \exp \left[ -\frac{U_{12}(1,2)}{k_B T} \right] \right\rangle_r = N_p^{-1} \sum'_{(1,2)} \exp \left[ -\frac{U_{12}(1,2)}{k_B T} \right]
$$

(5.13)

where $\sum'_{(1,2)}$ indicates summation over $N_p$ pairs of sample configurations (1,2) at a given $r$. With the values of $U_{12}(r)$ so evaluated for various values of $r$ in the case of the LJ potential at a given $T^*$ or of the HS potential with a given $\sigma_c$, the quantity $A_2M^2$ for a given $n$ may then be calculated from eq 5.11 by numerical integration with the use of the trapezoidal rule formula.
In the practical evaluation of $\langle S^2 \rangle$, we have generated 10 sets of $10^5$ ($= N_s$) sample configurations for $n = 6, 10, 20, 50, 100,$ and 200, 5 sets of those for $n = 500$, and 2 sets of those for $n = 1000$. In each case, $M_{\text{nom}}$ has been chosen to be ca. $2n$. In the evaluation of $U_{12}$ (and $A_2$), $10^6$ or $10^7$ ($= N_p$) sample pairs have been taken from each set. Then the total number of sample pairs is equal to the number $N_p$ of sample pairs in each set multiplied by the number of sets. In the both evaluations of $\langle S^2 \rangle$ and $A_2$, the parameter $\sigma$ in the LJ potential is set equal to unity.

All the numerical work has been done by the use of a personal computer with an AMD Athlon XP 2200+ CPU. A source program coded in C has been compiled by the GNU C compiler version 2.95.4 with real variables of double precision. In the program, the subroutine package MT19937 supplied by Matsumoto and Nishimura has been used instead of the subroutine RAND included in the standard C library.

### 5.3 Results and Discussion

#### 5.3.1 Mean-Square Radius of Gyration

In the second column of Table 5.1 are given the MC values of $\langle S^2 \rangle / n$ for the freely rotating chains with the LJ potential at $T^* = 8.0$ (good solvent condition) and with $\theta = 109^\circ$, $120^\circ$, $135^\circ$, $165^\circ$, and $175^\circ$ and $n = 6, 10, 20, 50, 100, 200, 500,$ and $1000$, the number in the parentheses attached to each value indicating its statistical error. The value and error for each chain are the mean and the standard deviation, respectively, of independent MC results. We note that the values for the chains with $\theta = 109^\circ$ have been reproduced from Table I of Ref. 7 and Table III of Ref. 8.

Figure 5.1 shows double-logarithmic plots of $\langle S^2 \rangle / n$ against $n$. The open symbols represent the above-mentioned MC values for the chains with the LJ potential for $\theta = 109^\circ$ ($\bigcirc$), $120^\circ$ ($\Box$), $135^\circ$ ($\Diamond$), $165^\circ$ ($\triangle$), and $175^\circ$ ($\triangledown$). The solid curves represent the theoretical values of $\langle S^2 \rangle / n$ for the ideal freely rotating chain without interactions between beads, which have been calculated from

$$
\langle S^2 \rangle = \frac{11 - \cos \theta}{6} n + \frac{11 + 6 \cos \theta - \cos^2 \theta}{(1 + \cos \theta)^2} + \frac{1 - 7 \cos \theta + 7 \cos^2 \theta + 3 \cos^3 \theta}{(1 + \cos \theta)^3} \frac{1}{n + 1} - \frac{2 \cos^2 \theta}{(1 + \cos \theta)^4} \frac{1 - (- \cos \theta)^n}{(n + 1)^2} \quad (5.14)
$$
Table 5.1. Results of Monte Carlo simulation.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\langle S^2 \rangle/n$ (error %)</th>
<th>$10A_2 M_b^2/N_A$ (error %)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LJ</td>
<td>HS</td>
</tr>
<tr>
<td>$\theta = 109^\circ$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.2892 (0.1)</td>
<td>0.2886 (0.1)</td>
</tr>
<tr>
<td>10</td>
<td>0.3158 (0.1)</td>
<td>0.3134 (0.1)</td>
</tr>
<tr>
<td>20</td>
<td>0.3599 (0.1)</td>
<td>0.3549 (0.1)</td>
</tr>
<tr>
<td>50</td>
<td>0.4287 (0.1)</td>
<td>0.4219 (0.1)</td>
</tr>
<tr>
<td>100</td>
<td>0.4867 (0.1)</td>
<td>0.4807 (0.1)</td>
</tr>
<tr>
<td>200</td>
<td>0.5511 (0.2)</td>
<td>0.5462 (0.1)</td>
</tr>
<tr>
<td>500</td>
<td>0.6472 (0.1)</td>
<td>0.6458 (0.1)</td>
</tr>
<tr>
<td>1000</td>
<td>0.7294 (0.1)</td>
<td>0.7298 (0.1)</td>
</tr>
<tr>
<td>$\theta = 120^\circ$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.3539 (0.1)</td>
<td>0.3544 (0.1)</td>
</tr>
<tr>
<td>10</td>
<td>0.3987 (0.1)</td>
<td>0.3991 (0.1)</td>
</tr>
<tr>
<td>20</td>
<td>0.4631 (0.1)</td>
<td>0.4634 (0.1)</td>
</tr>
<tr>
<td>50</td>
<td>0.5491 (0.1)</td>
<td>0.5491 (0.1)</td>
</tr>
<tr>
<td>100</td>
<td>0.6151 (0.1)</td>
<td>0.6170 (0.1)</td>
</tr>
<tr>
<td>200</td>
<td>0.6867 (0.1)</td>
<td>0.6911 (0.2)</td>
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<tr>
<td>500</td>
<td>0.7947 (0.1)</td>
<td>0.8005 (0.1)</td>
</tr>
<tr>
<td>1000</td>
<td>0.8868 (0.1)</td>
<td>0.8994 (0.0)</td>
</tr>
<tr>
<td>$\theta = 135^\circ$</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>0.4588 (0.0)</td>
<td>0.4590 (0.0)</td>
</tr>
<tr>
<td>10</td>
<td>0.5623 (0.0)</td>
<td>0.5625 (0.0)</td>
</tr>
<tr>
<td>20</td>
<td>0.7107 (0.1)</td>
<td>0.7118 (0.1)</td>
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<td>50</td>
<td>0.8800 (0.1)</td>
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<tr>
<td>100</td>
<td>0.9820 (0.1)</td>
<td>0.9898 (0.1)</td>
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<tr>
<td>200</td>
<td>1.074 (0.2)</td>
<td>1.086 (0.1)</td>
</tr>
<tr>
<td>500</td>
<td>1.198 (0.0)</td>
<td>1.218 (0.3)</td>
</tr>
<tr>
<td>1000</td>
<td>1.304 (0.1)</td>
<td>1.333 (0.2)</td>
</tr>
<tr>
<td>$\theta = 165^\circ$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.6381 (0.0)</td>
<td>0.6381 (0.0)</td>
</tr>
<tr>
<td>10</td>
<td>0.9307 (0.0)</td>
<td>0.9307 (0.0)</td>
</tr>
<tr>
<td>20</td>
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<td>1.598 (0.0)</td>
</tr>
<tr>
<td>50</td>
<td>3.161 (0.0)</td>
<td>3.161 (0.0)</td>
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<tr>
<td>100</td>
<td>4.820 (0.0)</td>
<td>4.828 (0.1)</td>
</tr>
<tr>
<td>200</td>
<td>6.534 (0.1)</td>
<td>6.534 (0.1)</td>
</tr>
<tr>
<td>500</td>
<td>8.199 (0.1)</td>
<td>8.204 (0.0)</td>
</tr>
<tr>
<td>1000</td>
<td>8.924 (0.3)</td>
<td>8.952 (0.1)</td>
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</table>

(continued)
<table>
<thead>
<tr>
<th>( \theta = 175^\circ )</th>
<th>6</th>
<th>0.663 &amp; (0.0)</th>
<th>0.663 &amp; (0.0)</th>
<th>5.319 &amp; (0.3)</th>
<th>5.570 &amp; (0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.991 &amp; (0.0)</td>
<td>0.991 &amp; (0.0)</td>
<td>4.350 &amp; (0.4)</td>
<td>4.833 &amp; (0.4)</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1.805 &amp; (0.0)</td>
<td>1.804 &amp; (0.0)</td>
<td>3.649 &amp; (0.4)</td>
<td>4.312 &amp; (0.5)</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>4.171 &amp; (0.0)</td>
<td>4.171 &amp; (0.0)</td>
<td>3.248 &amp; (1.1)</td>
<td>4.022 &amp; (0.3)</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>7.885 &amp; (0.0)</td>
<td>7.885 &amp; (0.0)</td>
<td>3.107 &amp; (0.8)</td>
<td>3.928 &amp; (0.6)</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>14.55 &amp; (0.0)</td>
<td>14.55 &amp; (0.0)</td>
<td>3.029 &amp; (1.0)</td>
<td>3.859 &amp; (0.5)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>29.82 &amp; (0.0)</td>
<td>29.83 &amp; (0.0)</td>
<td>2.986 &amp; (1.4)</td>
<td>3.836 &amp; (0.2)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>45.52 &amp; (0.0)</td>
<td>45.53 &amp; (0.0)</td>
<td>2.948 &amp; (0.7)</td>
<td>3.822 &amp; (0.1)</td>
<td></td>
</tr>
</tbody>
</table>
with the indicated values of $\theta$. For the chain of $\theta = 109^\circ$, which corresponds to a typical flexible chain, the MC data points deviate upward from the corresponding solid curves $n$ is increased because of the intramolecular excluded-volume effect. For the chains of $\theta = 120^\circ$ and $135^\circ$, the situations are the same as that in the case of $\theta = 109^\circ$ but the deviation of the MC value from the ideal one becomes small as $\theta$ is increased. As for the chains of $\theta = 165^\circ$ and $175^\circ$, which correspond to typical semiflexible or stiff chains, the MC values almost agree with the corresponding ideal ones over the whole range of $n$ examined. We note that the intramolecular excluded-volume effect may be neglected for typical semiflexible or stiff chains of short chain lengths.

In the third column of Table 5.1 are also given the MC values of $\langle S^2 \rangle/n$ for the freely rotating chains with the HS potential with $\sigma_c = 0.624$, the $\sigma_c$ value being chosen so that the $\langle S^2 \rangle/n$ values for the chains of $\theta = 109^\circ$ are as close as possible to the corresponding values given in the second column for the chains with the LJ potential at $T^* = 8.0$. It is seen that the values of $\langle S^2 \rangle/n$ for the two freely rotating chains of each $\theta$ and $n$ agree with each other within ca. 2%. In Figure 5.1 are also shows the HS values by the closed symbols, although they are almost hidden by the open symbols (LJ values). We note that the agreement between the $\langle S^2 \rangle/n$ values for the two chains of larger $\theta$ (165$^\circ$ and 175$^\circ$) is rather natural since the intramolecular excluded-volume effect becomes immaterial for typical semiflexible or stiff chains with short chain lengths, as mentioned above. The results imply that we may assign a proper value to $\sigma_c$ of the chain with the HS potential so that its $\langle S^2 \rangle$ agree well with that for the chain with the LJ potential at a given $T^*$ irrespective of the chain stiffness.

It is pertinent to make here a remark on the unperturbed value $\langle S^2 \rangle_0$ of $\langle S^2 \rangle$ without the excluded-volume effect. In a real polymer–solvent system, $\langle S^2 \rangle_0$ means the $\langle S^2 \rangle$ value in the $\Theta$ state which may be realized when repulsive and attractive interactions between repeat units (beads) cancel out each other but does not correspond to that of the ideal chain without any interactions between them. For the freely rotating chain with $\theta = 109^\circ$, the $\Theta$ state may be reproduced by the use of the LJ potential at $T^* = 3.72$ (and with $\sigma = 1$) where $\langle S^2 \rangle/n$ becomes a constant independent of $n$ for very large $n$, which is ca. 20% larger than the asymptotic value (ca. 1/3) of $\langle S^2 \rangle/n$ calculated from eq 5.14 for the ideal chain in the limit of $n \to \infty$. The implication is that the interactions between beads may appreciably affect $\langle S^2 \rangle_0$ through the short-range interference. The situation is somewhat different in the case of the HS potential only with a repulsive core, by the use of which we cannot reproduce the real unperturbed chain but only the ideal one with $\sigma_c = 0$. (Note that the chain with the LJ potential becomes the ideal one when
Figure 5.1. Double-logarithmic plots of $\langle S^2 \rangle / n$ against $n$. The open and closed symbols represent the MC values for the freely rotating chains with the LJ potential $T^* = 8.0$ and the HS one with $\sigma_c = 0.624$, respectively, for $\theta = 109^\circ$ (○, ●), 120° (□, ■), 135° (◇, ♦), 165° (△, ▲), and 175° (▽, ▼). The solid curves represent the theoretical values for the ideal freely rotating chain without interactions between beads calculated from eq 5.14 with the indicated values of $\theta$. 
Although the interaction between beads affects $\langle S^2 \rangle_0$ through the short-range interference also in the case of the HS potential, we cannot evaluate it separately from $\langle S^2 \rangle$ with the excluded-volume effect.

### 5.3.2 Second Virial Coefficient and Interpenetration Function

Now the problem is whether or not the values of $A_2$ for the two freely rotating chains considered above agree with each other irrespective of the chain stiffness. In the fourth and fifth columns of Table 5.1 are given the MC values of $A_2 M_b^2 / N_A$ for the chains with the LJ potential at $T^* = 8.0$ and with the HS one with $\sigma_c = 0.624$, respectively, where $M_b (= M/n)$ is the molecular weight per bond, the number in the parentheses attached to each value indicating its statistical error.

Figure 5.2 shows double-logarithmic plots of $A_2 M_b^2 / N_A$ against $n$, all the symbols having the same meaning as those in Figure 5.1. In the case of $\theta = 109^\circ$, which corresponds to a typical flexible chain, the values of $A_2$ for the chain with the LJ potential (○) are slightly (5% at most) smaller than the corresponding ones for the chain with the HS potential (●) for all $n$ except $n = 6$. For both the chains, $A_2$ decreases with increasing $n$ and the slopes of the plots become almost identical with the asymptotic value $-0.2$ for very large $n$. For a given $n$, $A_2$ increases with increasing $\theta$ in the range of large $n$. In the case of $\theta = 175^\circ$, which corresponds to a typical semiflexible or stiff chain, $A_2$ becomes almost independent of $n$ for $n \geq 100$, as in the case of the rigid rod.

It is more important to see in Figure 5.2 that as $\theta$ (chain stiffness) is increased, the value of $A_2$ for the chain with the LJ potential becomes remarkably smaller than the corresponding value for the chain with the HS one and that $A_2$ for the former chain with a given $n$ has a maximum at $\theta = 165^\circ$ while it increases monotonically with increasing $\theta$ for the latter. It may be regarded as arising from the fact that two chains with the LJ potential (having the attractive interaction) which are close to each other prefer to be parallel rather than perpendicular to each other when their stiffness becomes large and their $A_2$ may therefore be suppressed, as discussed by Schoot and Odijk in the case of long rigid rods with a van der Waals-type potential.

Finally, we examine the behavior of the apparent interpenetration function $\Psi_{ap}$ defined by

$$A_2 = 4 \pi^{3/2} N_A \frac{\langle S^2 \rangle^{3/2}}{M^2} \Psi_{ap}$$

from the whole $A_2$ including the effects of chain ends as a function of $n$. Figure 5.3 shows double-logarithmic plots of $\Psi_{ap}$ against $n$, where the values of $\Psi_{ap}$ have been calculated from eq 5.15 with the MC values of $\langle S^2 \rangle/n$ and $A_2 M_b^2 / N_A$ given in Table 5.1.
Figure 5.2. Double-logarithmic plots of $A_2 M_b^2 / N_A$ against $n$. All the symbols have the same meaning as those in Figure 5.1.
Figure 5.3. Double-logarithmic plots of $\Psi_{ap}$ against $n$. All the symbols have the same meaning as those in Figure 5.1. The dashed horizontal line represents the asymptotic value 0.235 of $\Psi$ obtained by Barrett.$^{22}$
All the symbols have the same meaning as those in Figure 5.1 and the dashed horizontal line represents the asymptotic value 0.235 of $\Psi_{ap}$ (or $\Psi$ without the effects of chain ends) obtained by Barrett.\(^{22}\)

In the case of $\theta = 109^\circ$, the values of $\Psi_{ap}$ for the chains with the LJ and HS potentials agree with each other within 7% at most, which is a natural consequence of the results for $\langle S^2 \rangle$ and $A_2$ for the two chains shown in Figures 5.1 and 5.2, respectively, and they decrease monotonically with increasing $n$ and seem to approach the above asymptotic value. In the cases of $\theta = 120^\circ$, $135^\circ$, and $165^\circ$, $\Psi_{ap}$ for the chains with a given $n$ decreases with increasing $\theta$ and $\Psi_{ap}$ as a function of $n$ exhibits a minimum and increases monotonically with increasing $n$ after passing through the minimum. In the case of $\theta = 175^\circ$, $\Psi_{ap}$ does not exhibit a minimum and decreases monotonically with increasing $n$ in the whole range of $n$ examined. We note that even in this case, $\Psi_{ap}$ approaches to the asymptotic value in the limit of $n \rightarrow \infty$ after passing through a minimum at some value of $n$ larger than 1000. It is seen that the larger downward deviation from the asymptotic value is, the larger difference between the values of the two chains with the LJ and HS potentials is.

From the MC results shown in Figures 5.2 and 5.3, it may be concluded that the intermolecular excluded volume of the chain with the LJ potential becomes smaller than that of the corresponding chain with the HS one as the chain stiffness is increased even if $\sigma_c$ properly determined so that $\langle S^2 \rangle$ of the chain with the HS potential agrees well with that of the chain with the LJ one at the reduced temperature $T^* = 8.0$ irrespective of the chain stiffness (i.e., the bond angle $\theta$ of the freely rotating chain in the range from $109^\circ$ to $175^\circ$). It is then found that $A_2$ of the latter chain becomes remarkably smaller than that of the former as the chain stiffness ($\theta$) is increased in contrast to the case of $\langle S^2 \rangle$. The result implies that the binary cluster approximation does not seem to work well for typical semiflexible and stiff polymers.

## Conclusion

We have examined the behavior of the mean-square radius of gyration $\langle S^2 \rangle$ and second virial coefficient $A_2$ for the two freely rotating chains with the LJ 6-12 and HS potentials under a good solvent condition by MC simulation, giving major attention to the effect of chain stiffness on the intra- and intermolecular excluded volumes. It is shown that we may satisfactorily assign the value 0.624 to the collision diameter $\sigma_c$ of the HS potential so that $\langle S^2 \rangle$ of the chain with the HS potential agrees well with that of the chain with the LJ one at the reduced temperature $T^* = 8.0$ irrespective of the chain stiffness (i.e., the bond angle $\theta$ of the freely rotating chain in the range from $109^\circ$ to $175^\circ$). It is then found that $A_2$ of the latter chain becomes remarkably smaller than that of the former as the chain stiffness ($\theta$) is increased in contrast to the case of $\langle S^2 \rangle$. The result implies that the binary cluster approximation does not seem to work well for typical semiflexible and stiff polymers.
References

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CHAPTER 2

CHAPTER 3

CHAPTER 4

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