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EXPLOSION OF STRANGE ATTRACTORS EXHIBITED BY DUFFING'S EQUATION

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INTRODUCTION

Various fascinating phenomena occur in nonlinear systems.[1] They are discussed generally by making use of steady solutions of a differential equation. The equation is usually derived from a real system by neglecting small uncertain factors—hence, it is deterministic. If a single solution of the equation is asymptotically stable and its basin is large compared with random noise, the corresponding phenomenon turns out to be deterministic. But if a bundle of solutions containing infinitely many unstable periodic solutions is asymptotically orbitally stable, a chaotic phenomenon appears, which results from the small uncertain factors in the real system. That is, the representative point of the physical state wanders chaotically in the bundle of solutions. Because of this characteristic, we have called the phenomenon a "chaotically transitional process."[2-5]*

We have long been studying this subject in connection with forced oscillatory phenomena in nonlinear electrical and electronic circuits. As analytical solutions of differential equations cannot be expected in these problems, we have been relying on analog and digital computers. Thus, both the global structure of solutions and the long-term movement of a representative point have been examined. So the question arises, Are computer solutions valid? However, we have not entered into details of this problem, though we have found consistency between the analog and digital results. Therefore, our results may lack mathematical rigor; nevertheless, they will have an important influence on many researchers in various fields.

This paper also describes the chaotically transitional processes exhibited by Duffing's equation. Special attention is directed toward the transition of the processes and the explosion of strange attractors is clarified.

PRELIMINARIES

This report deals with Duffing's equation,

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B_0 + B_1 \cos t \tag{1}$$

*At first, we used the term "random" instead of "chaotic." This revision is due to the advice of Professor Joseph Ford of the Georgia Institute of Technology.

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or

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -ky - x^3 + B_0 + B_1 \cos t. \tag{2}$$

In this and the following sections, we present a brief review concerned with the chaotically transitional processes.

Discrete Dynamical System

A discrete dynamical system on the xy plane is introduced by using the solutions of equation 2. In order to see this, first consider the solution $(x(t, x_0, y_0), y(t, x_0, y_0))$ of (2), which, when $t = 0$, is at the point $p_0 = (x_0, y_0)$ of the xy plane. Let $p_1 = (x_1, y_1)$ denote the point specified by $x_1 = x(2\pi, x_0, y_0)$, $y_1 = y(2\pi, x_0, y_0)$; then either a C^∞-diffeomorphism f_λ,

$$\begin{aligned} f_\lambda : R^2 &\longrightarrow R^2 \\ p_0 &\longmapsto p_1, \\ \lambda &= (k, B_0, B_1), \end{aligned} \tag{3}$$

of the xy plane into itself or a discrete dynamical system on R^2 is defined.

Bundle of Solutions and Strange Attractors

Let us consider the case in which chaotic motion takes place in the real system exhibited by (2). This chaotically transitional process, $\{X(T)\}$, is represented by a bundle of solutions in the txy space that is asymptotically orbitally stable and contains infinitely many unstable periodic solutions. The set of points on the xy plane consisting of the cross section of the bundle at $t = 2n\pi$ $(n \in Z)$ is called a strange attractor.

Average Power Spectrum

By assuming that the process $\{X(t)\}$ is the periodic random process $\{X_T(t)\}$ with a sufficiently long period T, where T is a multiple of 2π, we can estimate the mean value $m_X(t)$ and the average power spectrum $\Phi_X(\omega)$ of $\{X(t)\}$ as follows:

$$\begin{aligned} m_X(t) &= \langle X(t) \rangle \simeq \langle X_T(t) \rangle \\ &= \left\langle \frac{a_0}{2} \right\rangle + \sum_{m=1}^{\infty} \{\langle a_m \rangle \cos m\omega_0 t + \langle b_m \rangle \sin m\omega_0 t\}, \qquad \omega_0 = \frac{2\pi}{T}, \end{aligned} \tag{4}$$

$$\Phi_X(\omega) = \lim_{T\to\infty}\left\langle \frac{1}{T}\left| \int_{-T/2}^{T/2} x_T(t)e^{-i\omega t}\,\mathrm{d}t \right|^2 \right\rangle \simeq \Phi_X(m\omega_0)$$

$$= \frac{2\pi}{\omega_0}\left\langle \frac{1}{4}(a_m^2 + b_m^2)\right\rangle, \qquad \omega_0 = \frac{2\pi}{T}, \qquad m = 0, 1, 2, \ldots . \tag{5}$$

where a_m and b_m are Fourier coefficients of a realization $x_T(t)$ of the process $\{X_T(t)\}$, i.e.,

$$x_T(t) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}\{a_m \cos m\omega_0 t + b_m \sin m\omega_0 t\}, \qquad \omega_0 = \frac{2\pi}{T},$$

$$a_m = \frac{2}{T}\int_{-T/2}^{T/2} x_T(t)\cos m\omega_0 t\,\mathrm{d}t, \qquad m = 0, 1, 2, \ldots, \tag{6}$$

$$b_m = \frac{2}{T}\int_{-T/2}^{T/2} x_T(t)\sin m\omega_0 t\,\mathrm{d}t, \qquad m = 1, 2, \ldots .$$

Exponentlike Quantities

One of the stochastic properties of the strange attractors is estimated by the exponentlike quantities e_u and e_s. To make this clear, let us calculate the characteristic roots m_{1i}, m_{2i} of $Df_\lambda(p_i)$, the derivative of f_λ evaluated at $p = p_i$, where $p_i = f_\lambda^i(p_0)$, $(i \in Z^+)$. Then, by setting

$$e_{un} = \frac{1}{2n\pi}\sum_{i=0}^{n-1}\ln \rho_{ui}, \qquad e_{sn} = \frac{1}{2n\pi}\sum_{i=0}^{n-1}\ln \rho_{si}, \tag{7}$$

where $\rho_{ui} = \max\{|m_{1i}|, |m_{2i}|\}$, $\rho_{si} = \min\{|m_{1i}|, |m_{2i}|\}$, we can define the exponentlike quantities by taking the limits

$$e_u = \lim_{n\to\infty} e_{un}, \qquad e_s = \lim_{n\to\infty} e_{sn}. \tag{8}$$

From our experiments, we see that the limits seem to exist and to be independent of p_0. According to the above definition, one of them, e_u, indicates the rate of divergence of nearby points in the attractor and the other, e_s, indicates the rate of attraction of the strange attractor under f_λ.

Proceeding in the same manner as above, exponentlike quantities e_u^j, e_s^j under every jth iteration of the mapping f_λ can be defined. These quantities, e_u^j and e_s^j, indicate the rates of divergence and attraction under the mapping f_λ^j, respectively. The following relation can be easily proved.

$$e_u^j + e_s^j = -k. \tag{9}$$

Main Results and Remaining Unsolved Problems Relating to the Chaotically Transitional Processes of the Previous Investigations[2–5]

Introduction

Duffing's equation has no statistical parameters and every solution is uniquely determined by the initial condition. The appearance of statistical properties in the physical phenomena in spite of the perfectly deterministic nature of the equation is caused by the existence of noise in the real systems as well as in the global structure of the solutions. A bundle of solutions representing the chaotically transitional process appears in certain domains of the system parameters. The detalis of these stochastic regions have not been discussed as yet.

Strange Attractor

We have emphasized that the strange attractor is identical with a closure of unstable manifolds of a saddle of the diffeomorphism f_λ. Also, it is defined by the asymptotically stable, invariant, closed set of f_λ containing infinitely many unstable minimal sets connected to one another by the influence of noise in the real system. A decomposition of strange attractors is still open.

It seems that the structure of a strange attractor is unstable in the Andronov-Pontryagin sense. However, the experimental results show that the closure of unstable manifolds always comes out corresponding to the chaotically transitional process and is not affected by the small perturbations of the real system. Judging from these facts, the strange attractor will have structural stability in some sense.

Time Evolution Characteristics

The mean value $m_X(t)$ of the chaotically transitional process $\{X(t)\}$ is a periodic function of t with period 2π. The chaotically transitional process can be regarded as a periodic nonstationary process.

Roughly speaking, though stochastic properties arise from the noise in the real system, the average power spectrum of the process is characterized by the structure of the bundle of solutions alone. The other statistical characteristics of the processes may depend, of course, on the nature of noise in the real system. However, no attempt has been made to determine what that effect is.

Transition of the Processes

When the system parameters are varied between the deterministic and the stochastic regions, strange attractors usually develop from periodic points, producing periodic points of successive twice orders of the original order. Let us call this process

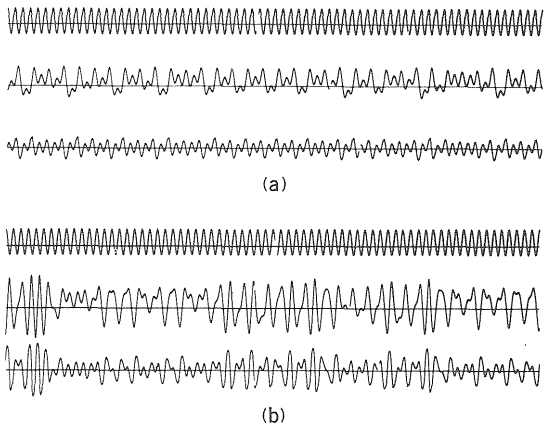

FIGURE 1. Waveforms of the external force and the chaotically transitional processes $\{X(t)\}$ and $\{Y(t)\}$. (a) $k = 0.05$, $B_0 = 0.030$, and $B_1 = 0.16$. (b) $k = 0.05$, $B_0 = 0.045$, and $B_1 = 0.16$.

SI branching because stable points change into inversely unstable points. It is frequently observed that stable periodic points disappear through coalescence with directly unstable periodic points (SD coalescence or SD extinction) and then strange attractors take their place. Strange attractors commonly change into periodic points by tracing SI branching inversely. It sometimes occurs that strange attractors disappear at the spot where the transition chains are formed. That is, as soon as the unstable manifolds composing strange attractors touch another stable manifold, strange attractors cannot generally exist. In any case, the onset and the limit of chaotically transitional processes are closely related to the global structure of the solutions and to the magnitude of uncertain factors in the real system.

EXPERIMENTAL RESULTS ON THE CHAOTICALLY TRANSITIONAL PROCESSES

In this section, we first show representative examples of the chaotically transitional processes exhibited by (2) with two specified values of the system parameters. We then study the transition of the processes by varying one of the system parameters. Finally, we discuss experimental results.

Representative Examples

As is well known, the subharmonic oscillation of order ½ is likely to occur in the system exhibited by Duffing's equation.[6] In this section, let us give two representative examples of the chaotically transitional processes developed from ½ harmonic oscillation. The system parameters used are as follows:

$$\begin{aligned} &\text{a. } k = 0.05, \quad B_0 = 0.030, \quad \text{and} \quad B_1 = 0.16, \\ &\text{b. } k = 0.05, \quad B_0 = 0.045, \quad \text{and} \quad B_1 = 0.16. \end{aligned} \tag{10}$$

Waveforms and Trajectories

FIGURE 1 shows the waveforms of the external periodic force and the resulting chaotically transitional processes $\{X(t)\}$ and $\{Y(t)\}$. The xy trajectories are given in FIGURE 2. These are the realizations of the chaotically transitional processes.

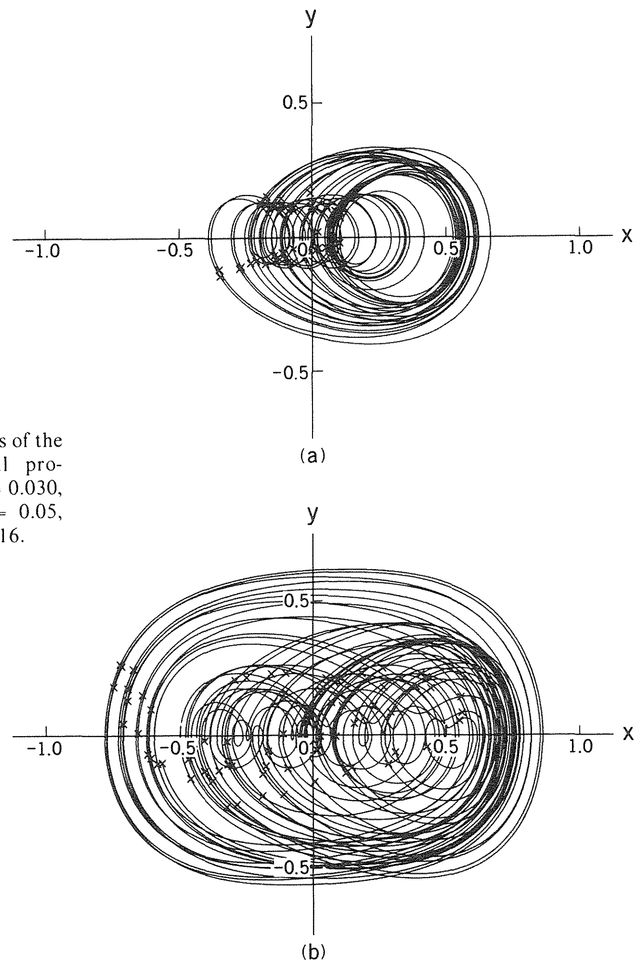

FIGURE 2. Trajectories of the chaotically transitional processes. (a) $k = 0.05$, $B_0 = 0.030$, and $B_1 = 0.16$. (b) $k = 0.05$, $B_0 = 0.045$, and $B_1 = 0.16$.

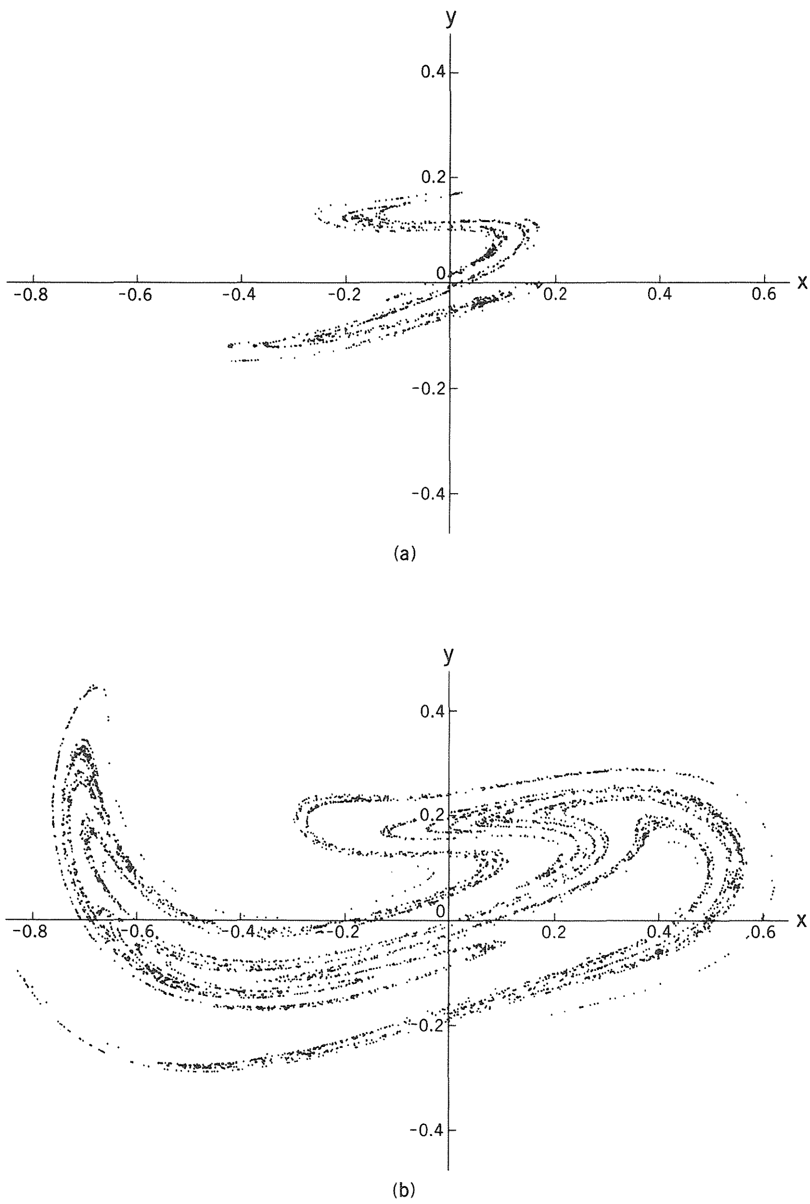

FIGURE 3. Strange attractors of the chaotically transitional processes. (a) $k = 0.05$, $B_0 = 0.030$, and $B_1 = 0.16$. (b) $k = 0.05$, $B_0 = 0.045$, and $B_1 = 0.16$.

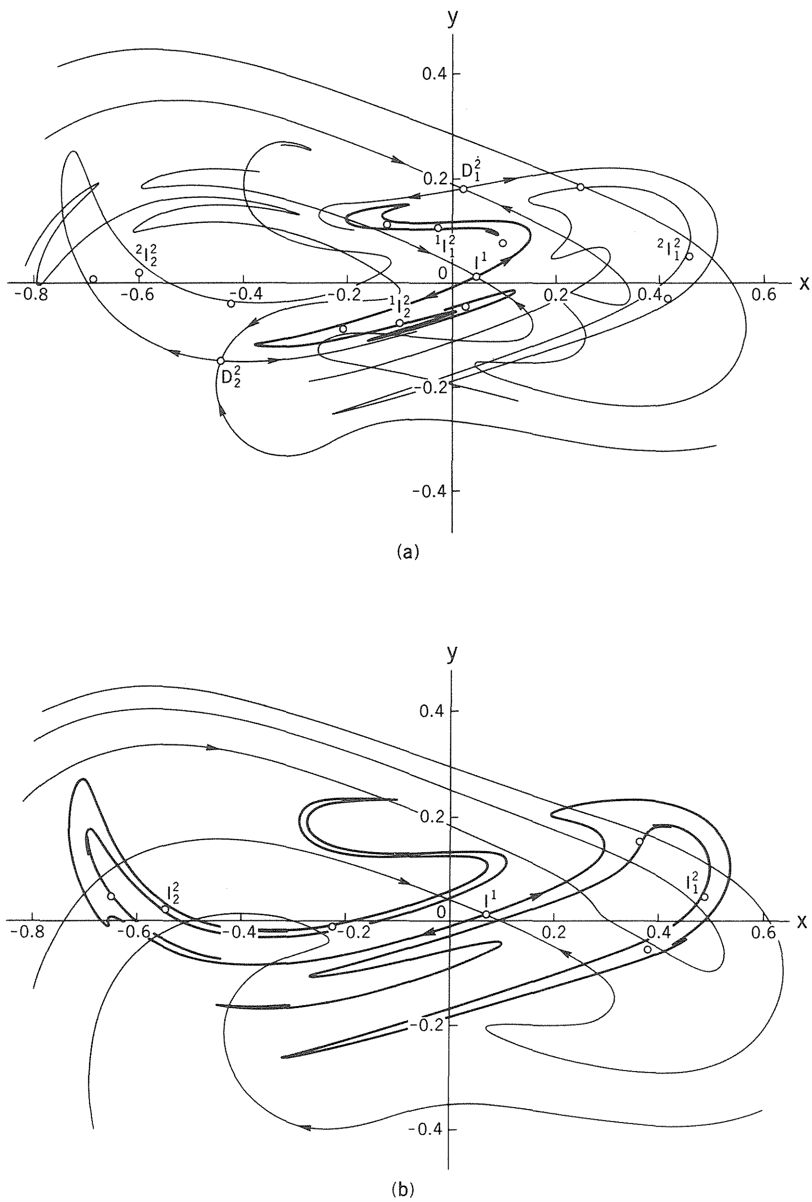

FIGURE 4. Global phase plane structures of the diffeomorphism f_λ. (a) λ = (0.05, 0.030, 0.16). (b) λ = (0.05, 0.045, 0.16).

Strange Attractors and Global Phase Plane Structures of f_λ

FIGURES 3 and 4 show the strange attractors and the global phase plane structures of the diffeomorphism f_λ. In FIGURE 4, the symbols ${}^iD_j^n$ and ${}^iI_j^n$ indicate the ith group of saddles of order n and the subscript j represents the order of the successive movements of the images under f_λ. The symbols D and I mean directly and inversely unstable types, respectively. The unlabeled circles in FIGURE 4 are all inversely unstable 4-periodic points.

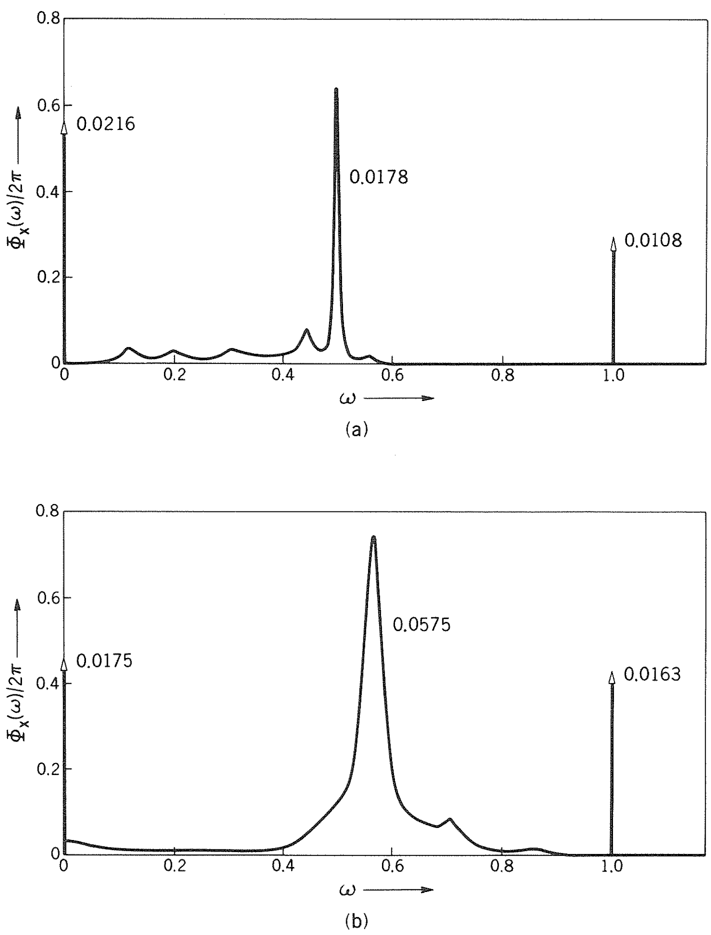

FIGURE 5. Average power spectra for the chaotically transitional processes $\{X(t)\}$. (a) $k = 0.05$, $B_0 = 0.030$, and $B_1 = 0.16$. (b) $k = 0.05$, $B_0 = 0.045$, and $B_1 = 0.16$.

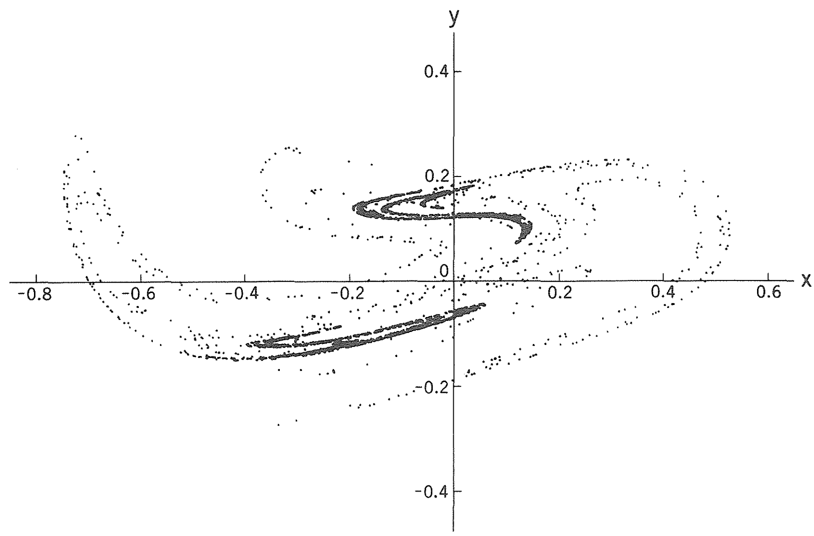

FIGURE 6. A strange attractor has just burst out, the system parameters being $k = 0.05$, $B_0 = 0.035$, and $B_1 = 0.16$.

Time Evolution Characteristics

The mean values of cases a and b are estimated as follows:

$$\begin{aligned} &\text{a. } m_X(t) = 0.147 - 0.207 \cos t + 0.018 \sin t, \\ &\text{b. } m_X(t) = 0.132 - 0.251 \cos t + 0.044 \sin t. \end{aligned} \tag{11}$$

The average power spectra are shown in FIGURE 5. In the figures, line spectra indicate the periodic components and continuous finitenesses show the statistical components. Numerical values in the figures represent the power focused and distributed on those frequencies. The peaks of the statistical components are at $\omega = 0.50$ and $\omega = 0.57$ in cases a and b, respectively.

Transition of the Chaotically Transitional Processes

Let us begin by examining the transition between the above two processes by varying the parameter B_0. When B_0 is increased from the value of case a, the unstable manifold of I^1, $W^u(I^1)$, comes into contact with the stable manifold of D_j^2, $W^s(D_j^2)$, and the explosion of the strange attractor occurs. FIGURE 6 shows the state just burst out for the system parameters $k = 0.05$, $B_0 = 0.035$, and $B_1 = 0.16$. In this strange attractor, the difference in image density is observed but, as B_0 increases, it immediately becomes uniform.

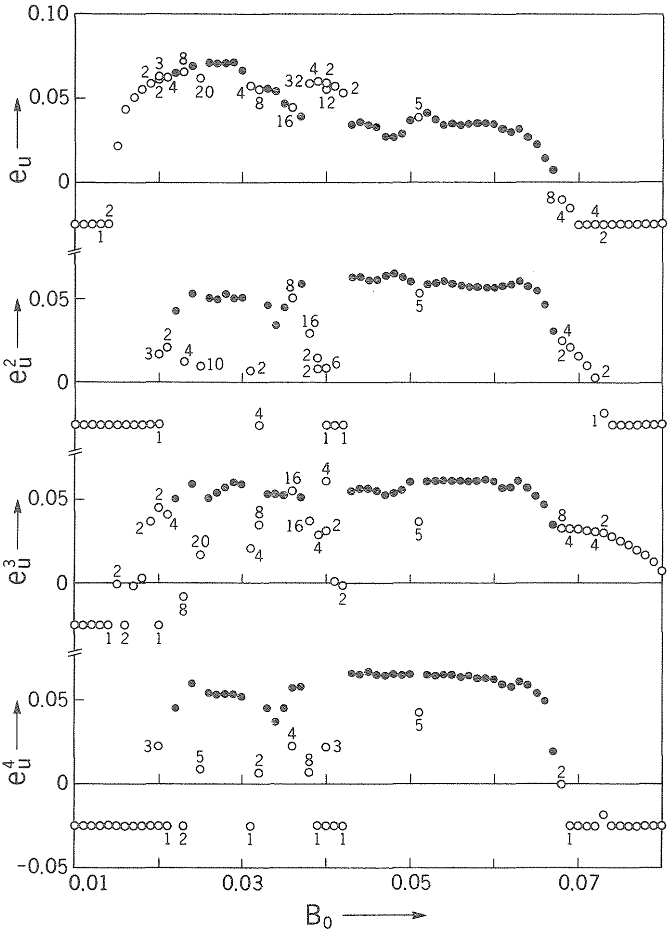

FIGURE 7. Exponentlike quantities e_u, e_u^2, e_u^3, and e_u^4 under the variation of the system parameter B_0.

In order to survey every aspect of the transition, the exponentlike quantities e_u, e_u^2, e_u^3, and e_u^4 are estimated and plotted in FIGURE 7. They are calculated both by increasing and by decreasing the parameter B_0 at an interval of 0.001. In the figure, the chaotically transitional process and the deterministic periodic process are differentiated. The former is marked ● and the latter ○. The number attached to the mark ○ indicates the order of the periodic point. The order of the periodic points without assigned numbers is the same as that of the neighboring one.

Independently of the observable phenomena in the real system, some of the exponentlike quantitites of the deterministic periodic processes take a positive value. This is due to the definition of the exponentlike quantities. However, if the order of the exponentlike quantities is the same as that of the periodic points, then the exponentlike quantities agree with the characteristic exponents of the periodic solution.

Discussion

In this section, we discuss the experimental results obtained in the preceding sections.

1. We see, in FIGURE 4a, that one of the branches of the unstable manifold $W^u(D_j^2)$ constitutes a homoclinic cycle, but the other does not. Let us call this structure partially homoclinic. On the other hand, both branches of $W^u(I^1)$ are homoclinic. Let us call this case totally homoclinic. Furthermore, the saddle D_j^2 is chained to I^1 through transition chain $W^u(D_j^2) \cup W^s(I^1)$, whereas the saddle I^1 is not chained to D_j^2. Therefore, the strange attractor of this case is composed only of the unstable manifold of I^1.

2. The explosion of a strange attractor occurs at the instant when I^1 is chained to D_j^2, that is, when $W^u(I^1)$ touches $W^s(D_j^2)$. The difference in image density in the strange attractor just burst out show the outline of the transition probability density distribution of the chaotically transitional process. Simultaneously with the explosion, continuous finite components of the power spectrum protrude at some frequencies.

3. As B_0 increases, the periodic points ${}^1I_j^2$ of FIGURE 4a change into completely stable periodic points ${}^1S_j^2$ at some value of B_0 between 0.039 and 0.040. The points ${}^1S_j^2$ and D_j^2 of FIGURE 4a then approach each other and, finally, cease to exist through coalescence (SD extinction) at some value of B_0 between 0.042 and 0.043. This gives the outline of the changes of the global phase plane structures from case a to case b.

The points ${}^1I_j^2$ of FIGURE 4a become stable below 0.021 and the points I_j^2 of FIGURE 4b become stable above 0.073. Needless to say, these transitions are the SI branching.

4. Subharmonic oscillations of various orders appear in the stochastic region. Among them, different from those of order 2^n, appearances of the subharmonic oscillations of orders 3, 5, 12, and 20 are interesting. Though we have not examined this in detail, it seems that the transition from strange attractors to periodic points is due to the transition chain. Let us call this type of transition "extinction of strange attractors." The transitions from periodic points to strange attractors are described above under the heading "Transition of the Processes."

5. We see in FIGURE 7 that, in the intervals of B_0, 0.026–0.030 and 0.043–0.065, the variations of the exponentlike quantities e_u^j for the chaotically transitional processes tend to be uniform as j increases. However, this situation is not present in the interval 0.033–0.037 and the explosion takes place in the interval.

Almost all exponentlike quantities, e_u^j, for the periodic points have an inclination to reduce as j increases. This tendency will be utilized for discriminating a deterministic process from a statistical process.

CONCLUSION

As in our previous reports, experimental results have been introduced relating to the chaotically transitional processes exhibited by Duffing's equation. In particular, the explosion of strange attractors and the appearances of the deterministic periodic points of various orders have been obtained. Furthermore, the whole aspect of the transition is clarified by estimating the exponentlike quantities.

Experimental study using computers produces valuable information about the solutions of Duffing's equation. However, the unsolved problems mentioned above are not yet settled.

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