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RANDOM PHENOMENA RESULTING FROM NON-LINEARITY IN THE SYSTEM DESCRIBED BY DUFFING'S EQUATION†

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1. INTRODUCTION

In physical phenomena, uncertainties lie between causes and effects. When uncertain factors are small, their effects may be neglected in most physical systems and the phenomena under consideration are treated as deterministic ones. Whereas in non-linear systems on some conditions, however small uncertain factors may be, they sometimes cause global changes of state variables even in the steady states. These kind of phenomena originate from global structures of the solutions for non-linear equations describing physical systems. In such systems, steady motions may be observed exhibiting stochastic properties.

This paper deals with random oscillations which occur in a series-resonance circuit containing a saturable inductor under the impression of a sinusoidal voltage. Firstly, a differential equation is derived from the electrical circuit under discussion. The discrete dynamical system is introduced by using the solutions of the differential equation. The terminology used in the following descriptions is explained briefly. Secondly, experimental results concerning random oscillations are obtained by using analog and digital computers. Finally, the experimental results are examined and the problems arising from them are summarized.

The phenomenon treated in this paper should be called turbulence in electric circuits. A series of results obtained in this paper disclose an important feature of non-linear phenomena not only in electrical circuits but also in general physical systems.

2. STEADY OSCILLATIONS AND ATTRACTORS

2.1. Differential equation

A series-resonance circuit containing a saturable inductor is shown in Fig. 1. With the notation of the figure, the equation for the circuit is written as

$$n\frac{d\phi}{dt} + Ri_{R} = E\sin\omega t$$

$$Ri_{R} = \frac{1}{C}\int i_{c}dt, \quad i = i_{R} + i_{c}$$
(1)

Fig. 1. Series-resonance circuit with non-linear inductance.

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where *n* is the number of turns of the inductor coil, and ϕ is the magnetic flux in the core. Let us consider the case in which the saturation curve of the core is expressed by

$$i = a\phi^3 \tag{2}$$

It is to be noted that the effect of hysteresis is neglected in equation (2). Here we introduce the dimensionless variable x, defined by

$$\phi = \Phi_n x \tag{3}$$

where Φ_n is an appropriate base quantity of the flux and is fixed by the relation

$$n\omega^2 C\Phi_n = a\Phi_n^3 \tag{4}$$

Then, eliminating i_R and i_C in equation (1) and using equations (2), (3) and (4), we obtain the well-known Duffing's equation

$$\frac{d^{2}x}{d\tau^{2}} + k\frac{dx}{d\tau} + x^{3} = B\cos\tau$$

$$\tau = \omega t - \tan^{-1}k, \qquad k = \frac{1}{\omega CR}$$

$$B = \frac{E}{n\omega\Phi_{n}}\sqrt{1 + k^{2}}.$$
(5)

where

2.2. Discrete dynamical system

Equation (5) is rewritten in simultaneous form as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -ky - x^3 + B\cos\tau \tag{6}$$

A discrete dynamical system on the xy plane is introduced by using the solutions of equation (6). In order to see this, first consider the solution $(x(\tau, x_0, y_0), y(\tau, x_0, y_0))$ of equation (6), which, when $\tau = 0$, is at the point $p_0 = (x_0, y_0)$ of the xy plane. Let $p_1 = (x_1, y_1)$ denote the point specified by $x_1 = x(2\pi, x_0, y_0), y_1 = y(2\pi, x_0, y_0)$; then either a C[∞]-diffeomorphism f_{λ}

where

 $\begin{cases}
f_{\lambda} : R^{2} \to R^{2} \\
p_{0} \mapsto p_{1} \\
\lambda = (k, B) \in \Lambda
\end{cases}$ (7)

of the xy plane into itself or a discrete dynamical system on R^2 is defined. For the theory of dynamical systems, see refs. [1–6].

For the circuit with dissipation (k > 0), f_{λ} is of class *D* or a dissipative system for large displacements and is a contractive mapping of the *xy* plane into itself. This implies that an orbit $Orb(p) = \{f_{\lambda}^{(n)}(p) | n \in Z\}$ of the discrete dynamical system (7) which starts from an arbitrary point $p \in R^2$ is positively stable in the sense of Lagrange and that there exists positively asymptotically stable, f_{λ} -invariant, maximum compact set $\Delta(f_{\lambda})$ with zero area. Therefore, investigation of the steady oscillations leads to examining the maximum compact set $\Delta(f_{\lambda})$ on R^2 and the behavior of its neighboring orbits.

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2.3. Steady oscillations and attractors

Since there exist uncertain factors, such as noise, in actual electric circuits, changes of voltages and/or currents are represented approximately by the solutions of the differential equations of the circuit. Accordingly, when the representative point of the circuit moves along an asymptotically stable solution, effects of noise may be neglected and deterministic phenomenon occurs. But stochastic phenomenon is caused when the representative point wanders, under the influence of noise, in the neighborhood of infinite solutions.

In the following, let us define the attractor as asymptotically stable, f_{λ} -invariant, compact set on the xy plane which exhibits a steady oscillation sustained in the actual circuit of Fig. 1. An attractor exhibiting a periodic oscillation is either a fixed point or a periodic group of the discrete dynamical system. An attractor exhibiting a random oscillation is considered to be an f_{λ} -invariant compact set containing infinite minimal sets.

3. EXPERIMENTS ON THE RANDOM OSCILLATIONS

In this section, experimental results obtained by using analog and digital computers are shown. Simulation and/or calculation errors are unavoidable in the computer solutions for the differential equation. Therefore, random quantities are not introduced intentionally but these errors are regarded as uncertainties acting on the system. These errors seem to be sufficiently small compared with noises in the actual circuit.

In the electric circuit as shown in Fig. 1, random oscillations can be observed within some intervals of the applied voltage and the circuit's constants. In fact, they occur in the ranges B = 9.9-13.3 for k = 0.1 and k = 0-0.31 for B = 12.0. In the following, experimental results are given for the steady oscillations, which occur in the computer-simulated systems for the representative values of the system parameters

$$\lambda = (k, B) = (0.1, 12.0) \tag{8}$$

3.1. Waveforms

Figure 2 shows two waveforms of the steady states sustained in the analog computersimulated system. Figure 2(a) shows a periodic oscillation containing remarkable higher harmonic components. The upper waveform is the applied voltage $B \cos \tau$ and the lower one is the normalized magnetic flux $x(\tau)$. Figure 2(b) shows a random oscillation. This waveform is not reproducible in every analog computer experiment. In the digital simulation, different waveforms are observed depending on the integration method and the step size. Therefore, the waveform of Fig. 2(b) is a realization of the random process $\{X(\tau)\}$.

3.2. Phase-plane analysis

Figure 3 shows a long-term orbit (a realization) of a random oscillation. The movement of images under iterations of f_{λ} is not uniquely determined even for the same initial point, but



Fig. 2. Waveforms of the steady oscillations in the system prescribed by equation (8).

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Fig. 3. Observed orbit corresponding to the random oscillation.

the general aspect (location, shape and size) of the orbit is reproducible, and further it seems stable in the Poisson sense. Therefore, a set of points as shown in Fig. 3 should be regarded as an outline of an attractor M representing the random oscillation.

Figure 4 shows fixed points ${}^{1}D^{1}$ (directly unstable), ${}^{1}I^{1}$ and ${}^{2}I^{1}$ (inversely unstable), and outlines of unstable $W^{u}({}^{1}D^{1})$ and stable $W^{s}({}^{1}D^{1})$ manifolds of ${}^{1}D^{1}$. If the unstable manifolds (thick lines in the figure) are prolonged, they tend to the orbit of Fig. 3. In other words, the closure of unstable manifolds of ${}^{1}D^{1}$, i.e. $ClW^{u}({}^{1}D^{1})$, is regarded as the attractor M for the random oscillation.

Figure 5 shows the global phase-plane structure of the diffeomorphism f_{λ} . The stable manifolds (thick lines) of the saddle ${}^{2}D^{1}$ (directly unstable fixed point) are the boundary of the two domains of attraction, and the sink S^{1} is the attractor corresponding to the periodic oscillation of Fig. 2(a).

3.3. Spectral analysis

In this section, spectral analysis of the random oscillation is shown. To this end, we regard the random process $\{X(\tau)\}$ as the periodic random process $\{X_T(\tau)\}$ with a sufficiently long period *T*, where *T* is a multiple of 2π . That is, let $x_T(\tau)$ be a periodic function with period *T*, which coincides with a realization $x(\tau)$ of $\{X(\tau)\}$ in the interval (-T/2, T/2]. Then a realization $x_T(\tau)$ is expanded into Fourier series as

$$x_T(\tau) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m \,\omega_0 \tau + b_m \sin m \,\omega_0 \tau), \qquad \omega_0 = \frac{2\pi}{T}$$
(9)



Fig. 4. Unstable and stable manifolds of the saddle point ${}^{1}D^{1}$.

where

$$a_{m} = \frac{2}{T} \int_{-T/2}^{T/2} x_{T}(\tau) \cos m \,\omega_{0} \tau \,\mathrm{d}\tau$$

$$b_{m} = \frac{2}{T} \int_{-T/2}^{T/2} x_{T}(\tau) \sin m \,\omega_{0} \tau \,\mathrm{d}\tau$$

$$m = 0, 1, 2, \dots$$
(10)

Fourier's coefficients a_m and b_m are random variables because $x_T(\tau)$ is a realization of the random process $\{X_T(\tau)\}$. From these coefficients, the mean value $m_X(\tau)$ and the average power spectrum $\Phi_X(\omega)$ of the random process $\{X(\tau)\}$ can be estimated as follows.

$$m_{X}(\tau) = \langle X(\tau) \rangle = \lim_{T \to \infty} \langle X_{T}(\tau) \rangle$$

$$= \langle X_{T}(\tau) \rangle = \left\langle \frac{a_{0}}{2} \right\rangle + \sum_{m=1}^{\infty} \left[\langle a_{m} \rangle \cos m \omega_{0} \tau + \langle b_{m} \rangle \sin m \omega_{0} \tau \right] \qquad (11)$$

$$\Phi_{X}(\omega) = \lim_{T \to \infty} \left\langle \frac{1}{T} \middle| \int_{-T/2}^{T/2} x_{T}(\tau) e^{-i\omega\tau} d\tau \middle|^{2} \right\rangle$$

$$= \Phi_{X}(m \omega_{0}) = \frac{2\pi}{\omega_{0}} \left\langle \frac{1}{4} (a_{m}^{2} + b_{m}^{2}) \right\rangle$$

$$\omega_{0} = \frac{2\pi}{T}$$

$$(12)$$



Fig. 5. Phase-plane structure of the diffeomorphism f_{λ} , $\lambda = (0.1, 12.0)$.

The ensemble average is calculated by regarding successive waveforms at the intervals ((n - 1/2)T, (n + 1/2)T] $(n = 0, 1, 2, ..., N_s)$ as sample processes of $\{X_T(\tau)\}$.

Let us give the results thus estimated for the specified values of the system parameters given by equation (8). The mean value of $\{X(\tau)\}$ is given by

$$m_{X}(\tau) = 1.72 \cos \tau + 0.22 \sin \tau + 1.21 \cos 3\tau - 0.26 \sin 3\tau + 0.25 \cos 5\tau - 0.06 \sin 5\tau + 0.07 \cos 7\tau - 0.02 \sin 7\tau + 0.02 \cos 9\tau - 0.01 \sin 9\tau$$
(13)

The mean value is found to be a periodic function. This indicates that the random process $\{X(\tau)\}$ is a non-stationary random process. Figure 6 shows the average power spectrum estimated by using equation (12). In the figure, line spectra at $\omega = 1, 3, 5, \ldots$ indicate the periodic components of the mean value as given by equation (13), and numerical values attached to line spectra represent the power concentrated on those frequencies. In every

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computer experiment, the general aspect (location, shape and size) of the average power spectrum is reproducible. The average power of the random process $\{X(\tau)\}$ is given by

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \langle X_T^2(\tau) \rangle \, \mathrm{d}\tau = \frac{1}{T} \int_{-T/2}^{T/2} \langle X_T^2(\tau) \rangle \, \mathrm{d}\tau = 3.08 \tag{14}$$

3.4. Spectral decomposition of the power

It is easily seen that, due to the non-linearity of the inductor, the results of Fig. 6 and equation (14) in the preceding section do not have dimension of the electric power. Therefore, let us here examine the situation of spectral dispersion of electric power supplied by the source of single frequency. The terminal voltage of the capacitor

$$v(\tau) = \frac{B}{\sqrt{1+k^2}}\sin(\tau + \tan^{-1}k) - y(\tau)$$
(15)

is also a random process $\{V(\tau)\}$, and the mean value is given by

$$m_{\nu}(\tau) = 0.97 \cos \tau + 13.60 \sin \tau + 0.77 \cos 3\tau + 3.63 \sin 3\tau + 0.28 \cos 5\tau + 1.27 \sin 5\tau + 0.16 \cos 7\tau + 0.51 \sin 7\tau$$
(16)
+ 0.06 \cos 9\tau + 0.15 \sin 9\tau
+ 0.02 \cos 11\tau + 0.04 \sin 11\tau
+ 0.00 \cos 13\tau + 0.01 \sin 13\tau

Figure 7 shows an average power spectrum of $\{V(\tau)\}$, † The average power of the random process is given by

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \langle V_T^2(\tau) \rangle \, \mathrm{d}\tau = \frac{1}{T} \int_{-T/2}^{T/2} \langle V_T^2(\tau) \rangle \, \mathrm{d}\tau = 104$$
(17)

From these results, spectral decomposition of the electric power dissipated in the shunt resistance with the capacitor $kv^2(\tau)$ is calculated. That is, the power supplied by the single frequency is decomposed and dissipated in the resistor at the following rate of frequencies:

fundamental component	89.3 %
third harmonic component	6.6%
random component	3.1 %
fifth harmonic component	0.8 %

The results of this section are obtained by using analog and digital computers. In the analog computer experiments, various types of non-linear elements and a number of operating speeds (time scales) are used. In the digital computer experiments, integration

$$\frac{B}{\sqrt{1+k^2}}\sin(\tau + \tan^{-1}k) = 1.19\cos\tau + 11.88\sin\tau$$

and the mean value of the current by

 $\langle X^3(\tau) \rangle = 13.70 \cos \tau + 0.39 \sin \tau + \cdots$

Therefore, the power supplied by the source turns out to be 10.4.

[†] In the present discussion, voltage, current and time are normalized by $n\omega\Phi_n$, $a\Phi_n^3$ and $1/\omega$, respectively. In this case, the source voltage is given by

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Fig. 6. Average power spectrum of the random process $\{X(\tau)\}$.

methods of Runge-Kutta-Gill and of Hamming with various step sizes are used, and further experiments have also been executed for both single and double precisions. It is confirmed that these results agree well with each other not only qualitatively but also quantitatively.

Fourier's transformation, equation (10), has been carried out by applying the discrete FFT algorithm

$$a_{m} = \frac{1}{N} \sum_{k=-(N-1)}^{N} x \left(k \frac{T}{2N} \right) \cos \left(mk \frac{\pi}{N} \right),$$

$$m = 0, 1, 2, \dots, N$$

$$b_{m} = \frac{1}{N} \sum_{k=-(N-1)}^{N} x \left(k \frac{T}{2N} \right) \sin \left(mk \frac{\pi}{N} \right),$$

$$m = 1, 2, \dots, N - 1$$

$$(18)$$

for 2N sampled values $x(\tau_k)$ at $\tau_k = kT/2N$ (k = -(N-1), ..., N). In the calculations, taking the errors related to the approximations of continuous variables by discrete variables and of infinite interval by finite interval into account, the values $T = 2\pi \times 2^{10}$, $2N = 2^{15}$ are used, and the ensemble average is estimated for 100 ($N_s = 99$) realizations.



Fig. 7. Average power spectrum of the random process $\{V(\tau)\}$.

4. DISCUSSION OF THE EXPERIMENTAL RESULTS

In the present section, the experimental results given in the preceding section are discussed and the problems arising from them are summarized.

(1) The random oscillation is not a special one which appears only for particular values of the system parameters, but can be observed in a rather wide range of values. From Figs. 3 and 4 the attractor M of the random oscillation specified by the parameter values of equation (8) is regarded as a closure of unstable manifolds of the directory unstable fixed point ${}^{1}D^{1}$ of f_{λ} , i.e. $M = ClW^{u}({}^{1}D^{1})$. An appearance of the attractor seems to change continuously when the parameters are varied in the neighborhood of $\lambda = (0.1, 12.0)$. The movement of images in the attractor M under iterations of f_{λ} is not reproducible, but seems to be stable in the Poisson sense.

(2) The stable manifolds $W^{s}({}^{1}D^{1})$ intersect the unstable manifolds $W^{u}({}^{1}D^{1})$ forming a homoclinic cycle. As we see in Fig. 4, most intersections (doubly asymptotic points) are transversal, but, as is seen in the neighborhood of the point (2.8, -2.0) of Fig. 4, it is expected that there exist some tangential points (doubly asymptotic points of the special type) on the prolonged manifolds. This fact suggests that the structure of the attractor may be unstable in the sense of Andronov-Pontryagin.

Existence of homoclinic points indicates that the attractor M contains infinite periodic groups. As shown by experiments, every periodic group is unstable. This fact implies that, even if sinks exist, their domains of attraction are so narrow that they are subject to the perturbations by the uncertainties acting on the system.

(3) The observed orbit $\{X(2n\pi), Y(2n\pi)\}$ $(n \in Z^+)$ of the discrete dynamical system, in other words, stroboscopic sequence of the computer solution with the same period as that of the periodic forcing seems to be a 2-dimensional stationary sequence taking values in the attractor.

(4) The random process $\{X(\tau)\}$ can be regarded as a sample process of the periodic nonstationary random process. Therefore, the mean value $m_X(\tau)$ is a periodic function with period 2π and the correlation function of the random component $\{R(\tau)\} = \{X(\tau) - m_X(\tau)\}$ is invariant under the periodic translations: $\tau \to \tau + 2n\pi$ ($n \in \mathbb{Z}$) [7]–[9].

In the computer experiments, an outline of the average power spectrum of the random process is reproducible. This indicates that the average power spectrum of the process is characterized by the structure of solutions regardless of the nature of uncertain factors. In other words, the simulation and/or calculation errors are not amplified into a random process but only bring about randomness in the phenomenon.

(5) From the above facts, the genesis and the properties of the random oscillation are summarized as follows. "The representative point of the actual system (which is not prescribed by the solution of the differential equation in the mathematical sense) continues to transit randomly among the infinitely many solutions due to the perturbations by uncertain factors of the system. The average power spectrum of the random oscillation depends practically not on the nature of uncertain factors but on the structure of the solutions emanating from the attractor".

In succession, the attractor representing random oscillations should be defined appropriately by "the asymptotically stable, compact, f_{λ} -invariant set which contains infinitely minimal sets connected to one another by the influence of uncertain factors in the actual system".

Because of those aforementioned, we have called this type of oscillation "the randomly transitional oscillation" [10]. The difference between randomly transitional oscillations and almost periodic oscillations is explained as follows. The attractor of the former is composed of infinite minimal sets, whereas that of the latter is made up of a single minimal set.

The problems arising from the above matters are summarized as follows.

(6) Let $\Omega(f_{\lambda})$ be a set of non-wandering points in the domain of attraction for the attractor $M = ClW^{u}({}^{1}D^{1})$. Is $\Omega(f_{\lambda})$ identical with M, or a proper subset of M? Does $\Omega(f_{\lambda})$ contain minimal sets different from periodic groups? Does $\Omega(f_{\lambda})$ contain minimal sets different from periodic groups? How is $\Omega(f_{\lambda})$ decomposed?

(7) From the above item (1), the attractor M seems to be structurally stable in some sense. What is the concept of structural stability? That is, what kinds of space (of differential equations) and topology are used for the discussion of structural stability?

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(8) How does the transition probability of the stroboscopic sequence $\{X(2n\pi), Y(2n\pi)\}\$ $(n \in \mathbb{Z}^+)$ and the stochastic properties of $\{X(\tau)\}$ depend on the nature of uncertain factors of the system?

As mentioned in item (4), an average power spectrum scarcely depends on the simulation and/or calculation errors but is determined from the structure of the solutions emanating from the attractor. This fact seems applicable to general electric circuits provided that the uncertain factors are sufficiently small and have no special characteristics. For the case in which this conjecture does not hold, namely, for the case in which some kind of resonance may be expected, what kind of relationship is expected between the nature of the noise and the structure of the solutions passing the attractor? Under the influence of random noise having the characteristics above, the phenomenon must be discussed by introducing random parameters into the differential equations describing the electric circuit.

5. CONCLUSION

In the present paper, random phenomena resulting from non-linearity have been studied in the series-resonance circuit containing a saturable inductor. As a result of this investigation, a part of the genesis and of the stochastic properties of the random oscillation has been first clarified. This phenomenon should be called turbulence in electric circuits.

Although an example of randomly transitional phenomena has been studied in detail for the system described by Duffing's equation, this kind of phenomena have been observed in another non-linear system. Hence they may be regarded as general steady phenomena in non-linear systems [10]. Further, it seems interesting to examine the phenomena in reference to the turbulence in fluid dynamics [11].

The unsolved problems (6)-(8) pointed out in the preceding section relate closely to the global structure of solutions of differential equations both in time and in space. They also relate to uncertain factors of actual systems. They are really fundamental and difficult problems. It is hoped that these problems will deserve attention as material for further study.

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POSTSCRIPT

I deem it a great honour to be given the opportunity to translate my article into English and I would like to express my thanks to the members of the editorial board. In the following I am writing down some comments and fond memories of days past when I was preparing the manuscript with tremendous difficulty.

It was on 27 November 1961 when I met with chaotic motions in an analog computer simulating a forced selfoscillatory system. Since then my interest has been held by the phenomenon, and I have been fascinated by the problem "What are steady states in non-linear systems?". After nearly ten years, I understood "randomly transitional phenomenon", I published my findings in the *Transactions of the Institute of Electronics and Communication Engineers of Japan*, Vol. 56, April 1973 [10]. My paper then received a number of unfavorable criticisms from some of my colleagues: such as, "Your results are of no importance because you have not examined the effects of simulation and/or calculation errors at all", "Your paper is of little importance because it is merely an experimental result", "Your result is no more than an almost periodic oscillation. Don't form a selfish concept of steady states", and so forth. Professor Hiromu Momota of the Institute of Plasma Physics was the first to appreciate the worth of my work. He said "Your results give an important feature relating to stochastic phenomena" on

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3 March 1974. Through his good offices I joined the Collaborating Research Program at the Institute of Plasma Physics of Nagoya University. These events gave me such unforgettable impressions that I continued the research with tenacity. At this moment I yearn for those days with great appreciation for their criticisms and encouragements.

By the middle of the 1970s, I had obtained many data of strange attractors for some systems of differential equations, but I had no idea to what journals and/or conferences I might submit these results. I was then luck y enough to meet with Professor David Ruelle who was visiting Japan in the early summer of 1978. He advised me to submit my results to the Journal of Statistical Physics [P1]. Further, he named the strange attractor of Fig. 3 "Japanese Attractor" and introduced it to the whole world (P2-P5). At that time chaotic behavior in deterministic systems began to come under the spotlight in various fields of natural sciences. I fortunately had several opportunities to present my accumulated results [P6-P11]. It is worth while mentioning that, due to the efforts of Professor David Ruelle and Professor Jean-Michel Kantor, the Japanese Attractor will be displayed at the National Museum of Sciences, Techniques and Industries which will open in Paris, 1986. In these circumstances this paper is a commemorative for me and I sincerely appreciate their kindness on these matters.

As the reader will notice in this translation and also in ref. [P1] I was rather nervous of using the term "strange attractor", because I had no understanding of its mathematical definition in those days. Although I do not think I fully understand the definition of it even today, I begin to use the term "strange attractor" without hesitation because it seems to agree with reality. However, it seems to me that the term "chaos", though it is short and simple, is a little bit exaggerated. In the universe one does have a lot more complicated, mysterious and incomprehensible phenomena! I should be interested in readers" views of my opinion.

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