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Selection 1
SOME PROBLEMS IN THE THEORY OF NONLINEAR OSCILLATIONS

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Abstract

This is the condensed version of the author's doctoral dissertation submitted to the Faculty of Engineering, Kyoto University, in February 1965. The text deals with two subjects: one is the generation of higher harmonics in simple nonlinear electrical circuits, and the other is concerned with the phenomenon of frequency entrainment which occurs in self-oscillatory systems under the impression of external periodic force.

1. HIGHER-HARMONIC OSCILLATIONS IN A SERIES-RESONANCE CIRCUIT

1.1 Introduction

Under the action of a sinusoidal external force, a nonlinear system may exhibit phenomena which are basically different from those found in linear systems. One of the salient features of such phenomena is the generation of higher harmonics and subharmonics. A considerable number of papers have been published concerning subharmonic oscillations in nonlinear systems [1-4]; however, very few investigations have been reported on the generation of higher harmonics.

This chapter deals with higher harmonic oscillations which occur in a series-resonance circuit containing a saturable inductor and a capacitor in series. The differential equation which describes the system takes the form of Duffing's equation. The amplitude characteristics of approximate periodic solutions are obtained by using the harmonic balance method, and the stability of these solutions is investigated by applying Hill's equation. These results are then tested by using analog and digital computers. An experimental result using a series-resonance circuit is cited at the end of this chapter [5]. This experiment was a motive for the present study.
1.2 Derivation of the Fundamental Equation

The schematic diagram illustrated in Fig. 1.1 shows an electrical circuit in which nonlinear oscillation takes place due to the saturable-core inductance $L(\phi)$ under the impression of the alternating voltage $E \sin \omega t$. As shown in the figure, the resistor $R$ is paralleled with the capacitor $C$, so that the circuit is dissipative. With the notation of the figure, the equations for the circuit are written as

$$
n \frac{d\phi}{dt} + R i_R = E \sin \omega t$$

$$R i_R = \frac{1}{C} \int i_C dt, \quad i = i_R + i_C$$

(1.1)

where $n$ is the number of turns of the inductor coil, and $\phi$ is the magnetic flux in the core. Then, neglecting hysteresis, we may assume the saturation curve of the form

$$i = a_1 \phi + a_3 \phi^3$$

(1.2)

where higher powers of $\phi$ than the third are neglected. We introduce dimensionless variables $u$ and $v$, defined by

$$i = I \cdot u, \quad \phi = \Phi \cdot v$$

(1.3)

where $I$ and $\Phi$ are appropriate base quantities of the current and the flux, respectively. Then Eq. (1.2) becomes

$$u = \frac{a_1 \Phi}{I} v + \frac{a_3 \Phi^3}{I} v^3 = c_1 v + c_3 v^3.$$  

(1.4)

Although the base quantities $I$ and $\Phi$ can be chosen quite arbitrarily, it is preferable, for brevity of expression, to fix them by the relations

---

![Fig. 1.1 Series-resonance circuit with nonlinear inductance.](image)
Some Problems in the Theory of Nonlinear Oscillations

\[ n \omega^2 C \Phi = I, \quad c_1 + c_3 = 1. \quad (1.5) \]

Then, after elimination of \( i_R \) and \( i_C \) in Eqs. (1.1) and use of Eqs. (1.3), (1.4) and (1.5), the result in terms of \( v \) is

\[ \frac{d^2 v}{d\tau^2} + k \frac{dv}{d\tau} + c_1 v + c_3 v^3 = B \cos \tau \quad (1.6) \]

where

\[ \tau = \omega t - \tan^{-1} k, \quad k = \frac{1}{\omega CR}, \quad B = \frac{E}{n\omega \Phi} \sqrt{1 + k^2}. \]

Equation (1.6) is a well-known equation in the theory of nonlinear oscillations and is known as Duffing’s equation [6].

1.3 Amplitude Characteristics of Approximate Periodic Solutions Using Harmonic Balance Method

(a) Periodic Solution Consisting of Odd-order Harmonics

As the amplitude \( B \) of the external force increases, an oscillation develops in which higher harmonics may not be ignored in comparison with the fundamental component. Since the system is symmetrical, we assume, for the time being, that these higher harmonics are of odd orders; hence a periodic solution for Eq. (1.6) may be written as

\[ v_0(\tau) = x_1 \sin \tau + y_1 \cos \tau + x_3 \sin 3\tau + y_3 \cos 3\tau. \quad (1.7) \]

Terms of harmonics higher than the third are certain to be present but are ignored to this order of approximation.

The coefficients on the right side of Eq. (1.7) may be found by the method of harmonic balance [5, 7-8]; that is, substituting Eq. (1.7) into (1.6) and equating the coefficients of the terms containing \( \sin \tau, \cos \tau, \sin 3\tau \) and \( \cos 3\tau \) separately to zero yields

\[ -A_1 x_1 - k y_1 - \frac{3}{4} c_3 [(x_1^2 - y_1^2)x_3 + 2x_1 y_1 y_3] \equiv X_1(x_1, y_1, x_3, y_3) = 0 \]
\[ k x_1 - A_1 y_1 + \frac{3}{4} c_3 [2x_1 y_1 x_3 - (x_1^2 - y_1^2)y_3] \equiv Y_1(x_1, y_1, x_3, y_3) = B \]
\[ -A_3 x_3 - 3k y_3 - \frac{1}{4} c_3 (x_1^2 - 3y_1^2)x_1 \equiv X_3(x_1, y_1, x_3, y_3) = 0 \]
\[ 3k x_3 - A_3 y_3 - \frac{1}{4} c_3 (3x_1^2 - y_1^2)y_1 \equiv Y_3(x_1, y_1, x_3, y_3) = 0 \]

\[ (1.8) \]
where

\[ A_1 = 1 - c_1 - \frac{3}{4} c_3 (r_1^2 + 2 r_3^2), \quad A_3 = 9 - c_1 - \frac{3}{4} c_3 (2 r_1^2 + r_3^2) \]

\[ r_1^2 = x_1^2 + y_1^2, \quad r_3^2 = x_3^2 + y_3^2. \]

Elimination of the \( x \) and \( y \) components in the above equations gives

\[
\left[ \left( A_1 - \frac{3 r_1^2}{r_1^2} A_3 \right)^2 + k^2 \left( 1 + \frac{9 r_3^2}{r_1^2} \right)^2 \right] r_1^2 = B^2
\]

\[
(A_3^2 + 9k^2)r_3^2 = \frac{1}{16} c_3^2 s_1^6.
\]

From these relations the components \( r_1 \) and \( r_3 \) of the approximate periodic solution are determined. By use of Eqs. (1.8) and (1.9) the coefficients of the periodic solution can be obtained from \( r_1 \) and \( r_3 \) by first computing

\[
x_1 = \frac{k(r_1^2 + 9r_3^2)}{B}, \quad y_1 = \frac{-(A_1r_1^2 - 3A_3r_3^2)}{B}
\]

and then subsequently

\[
x_3 = \frac{4r_3^2}{c_3 r_1^6} [-PA_3 + 3kQ], \quad y_3 = \frac{4r_3^2}{c_3 r_1^6} [-QA_3 - 3kP]
\]

where

\[ P = (x_1^2 - 3y_1^2)x_1, \quad Q = (3x_1^2 - y_1^2)y_1. \]

(b) Stability Investigation of the Periodic Solution

The periodic states of equilibrium determined by Eqs. (1.7), (1.10) and (1.11) are not always realized, but are sustained actually if they are stable. In this section the stability of the periodic solution will be investigated by considering the behavior of a small variation \( \xi(\tau) \) from the periodic solution \( v_0(\tau) \). If this variation \( \xi(\tau) \) tends to zero with increasing \( \tau \), the periodic solution is stable (asymptotically stable in the sense of Lyapunov [9-10]); if \( \xi(\tau) \) diverges, the periodic solution is unstable. Let \( \xi(\tau) \) be a small variation defined by

\[
v(\tau) = v_0(\tau) + \xi(\tau).
\]

Substituting Eq. (1.12) into (1.6) and neglecting terms of higher degree than the first in \( \xi \), we obtain the variational equation

\[
\frac{d^2 \xi}{d\tau^2} + k \frac{d\xi}{d\tau} + (c_1 + 3c_3 v_0^2)\xi = 0.
\]
Some Problems in the Theory of Nonlinear Oscillations

Introducing a new variable \( \eta(\tau) \) defined by

\[
\xi(\tau) = e^{-\delta \tau} \cdot \eta(\tau), \quad \delta = k/2
\]  

(1.14)

yields

\[
\frac{d^2 \eta}{d\tau^2} + (c_1 - \delta^2 + 3c_3\nu_0^2) \eta = 0,
\]

(1.15)

in which the first-derivative term has been eliminated. Inserting \( v_0(\tau) \) as given by Eq. (1.7) into (1.15) leads to a Hill’s equation of the form

\[
\frac{d^2 \eta}{d\tau^2} + \left( \theta_0 + 2 \sum_{n=1}^{3} \theta_{ns} \sin 2n\tau + 2 \sum_{n=1}^{3} \theta_{nc} \cos 2n\tau \right) \eta = 0
\]

(1.16)

where

\[
\theta_0 = c_1 - \delta^2 + \frac{3}{2} c_3 (r_1^2 + r_3^2)
\]

\[
\theta_{1s} = \frac{3}{2} c_3 (x_1y_1 - x_1y_3 + y_1x_3), \quad \theta_{1c} = -\frac{3}{4} c_3 (x_1^2 - y_1^2) + \frac{3}{2} c_3 (x_1x_3 + y_1y_3)
\]

\[
\theta_{2s} = \frac{3}{2} c_3 (x_1y_3 + y_1x_3), \quad \theta_{2c} = -\frac{3}{2} c_3 (x_1x_3 - y_1y_3)
\]

\[
\theta_{3s} = \frac{3}{2} c_3 x_3y_3, \quad \theta_{3c} = -\frac{3}{4} c_3 (x_3^2 - y_3^2).
\]

By Floquet’s theorem [11], the general solution of Eq. (1.16) takes the form

\[
\eta(\tau) = A e^{\mu \tau} \phi(\tau) + B e^{-\mu \tau} \psi(\tau)
\]

(1.17)

where \( A \) and \( B \) are arbitrary constants, \( \phi(\tau) \) and \( \psi(\tau) \) are periodic functions of \( \tau \) of period \( \pi \) or \( 2\pi \), and \( \mu \) is the characteristic exponent to be determined by the coefficients \( \theta_0, \theta_{ns} \) and \( \theta_{nc} \), and may be considered to be real or imaginary, but not complex. From the theory of Hill’s equation [12-14], we see that there are regions of coefficients in which the solution, Eq. (1.17), is either stable (\( \mu \): imaginary) or unstable (\( \mu \): real), and that these regions of stability and instability appear alternately as the coefficient \( \theta_0 \) increases. For convenience, we shall refer to the regions of instability as the first, the second, \( \cdots \) unstable regions as the coefficient \( \theta_0 \) increases from zero. It is known that the periodic functions \( \phi(\tau) \) and \( \psi(\tau) \) in Eq. (1.17) are composed of odd-order harmonics in the regions of odd orders and even-order harmonics in the regions of even orders and that, in the \( n \)-th unstable region, the \( n \)-th harmonic component predominates over other harmonics.
Since Eq. (1.7) is an approximate solution of Eq. (1.6), it is appropriate to consider a solution of Eq. (1.16) approximated to the same order. Therefore we assume that a particular solution in the first and the third unstable regions is given by

$$\eta(\tau) = e^{\mu\tau} \phi(\tau) = e^{\mu\tau} [b_1 \sin(\tau - \sigma_1) + b_3 \sin(3\tau - \sigma_3)].$$

(1.18)

We substitute this into Eq. (1.16) and apply the method of harmonic balance; the resulting homogeneous system has a nontrivial solution according to Cramer’s rule if

$$\Delta_1(\mu) = \begin{vmatrix} \theta_0 + \mu^2 - 1 - \theta_{1c} & \theta_{1s} - 2\mu & \theta_{1c} - \theta_{2c} & -\theta_{1s} + \theta_{2s} \\ \theta_{1s} + 2\mu & \theta_0 + \mu^2 - 1 + \theta_{1c} & \theta_{1s} + \theta_{2s} & \theta_{1c} + \theta_{2c} \\ \theta_{1c} - \theta_{2c} & \theta_{1s} + \theta_{2s} & \theta_0 + \mu^2 - 9 - \theta_{3c} & \theta_{3s} - 6\mu \\ -\theta_{1s} + \theta_{2s} & \theta_{1c} + \theta_{2c} & \theta_{3s} + 6\mu & \theta_0 + \mu^2 - 9 + \theta_{3c} \end{vmatrix} = 0.$$ 

(1.19)

From Eqs. (1.14) and (1.17) we see that the variation $\xi$ tends to zero with increasing $\tau$ provided that $|\mu| < \delta$. Clearly the stability threshold occurs when $\Delta_1(\delta) = 0$. Numerical experience has established that the stability condition for the first and the third unstable regions can be taken as

$$\Delta_1(\delta) > 0.$$  

(1.20)

By virtue of Eqs. (1.8) and the expressions for the coefficients $\theta_0$ and $\theta_{ns}, \theta_{nc}$, the stability condition (1.20) can be represented by

$$\Delta_1(\delta) \equiv \frac{\partial(X_1, Y_1, X_3, Y_3)}{\partial(x_1, y_1, x_3, y_3)} > 0.$$  

(1.21)

From this relation it is easily seen that the vertical tangency of the characteristic curves ($Br_1$- and $Br_3$-relations) occurs at the stability limit $\Delta_1(\delta) = 0$ of the first and the third unstable regions.

A particular solution of Eq. (1.16) in the second unstable region may appropriately be taken as

$$\eta(\tau) = e^{\mu\tau} \phi(\tau) = e^{\mu\tau} [b_0 + b_2 \sin(2\tau - \sigma_2)].$$

(1.22)
Some Problems in the Theory of Nonlinear Oscillations

Fig. 1.2 Amplitude characteristic curves of the approximate periodic solution given by Eq. (1.7).

Proceeding analogously as above, the characteristic exponent $\mu$ is determined by

$$\Delta_2(\mu) = \begin{vmatrix} \theta_0 + \mu^2 & \theta_{1s} & \theta_{1c} \\ 2\theta_{1s} & \theta_0 + \mu^2 - 4 - \theta_{2c} & \theta_{2s} - 4\mu \\ 2\theta_{1c} & \theta_{2s} + 4\mu & \theta_0 + \mu^2 - 4 + \theta_{2c} \end{vmatrix} = 0, \quad (1.23)$$

and the stability condition for the second unstable region, i.e., $|\mu| < \delta$, is given by

$$\Delta_2(\delta) > 0. \quad (1.24)$$

Figure 1.2 shows the amplitude characteristics for solutions of the form (1.7) calculated by numerical solution of Eqs. (1.9) with the system parameters $k = 0.4$, $c_1 = 0$ and $c_3 = 1$ in Eq. (1.6). The dashed portions of the characteristic curves represent unstable states, and the stability condition (1.20) or (1.24) is not satisfied in these intervals.
(c) Periodic Solution Including Even-order Harmonics

It has been pointed out above that, from the periodic state given by Eq. (1.7), even-order harmonics are excited in the second unstable region (see Fig. 1.2). In this region, the oscillation would gradually build up with increasing amplitude taking the form

\[ e^{(\mu-\delta)\tau} \left[ b_0 + b_2 \sin(2\tau - \sigma_2) \right] \quad \text{with} \quad \mu - \delta > 0 \]

and ultimately reach the steady state with a constant amplitude which is limited by the nonlinearity of the system. This implies that, for certain intervals of \( B \), such even-order harmonics must be considered in the periodic solution. Therefore we assume a periodic solution for Eq. (1.6) of the form

\[ v_0(\tau) = z + x_1 \sin \tau + y_1 \cos \tau + x_2 \sin 2\tau + y_2 \cos 2\tau. \quad (1.25) \]

Terms of harmonics higher than the second, especially the third harmonic, are certain to be present but are ignored to avoid unwieldy calculations. The unknown coefficients on the right side of Eq. (1.25) are determined in much the same manner as before; that is, substituting Eq. (1.25) into (1.6) and equating the coefficients of the non-oscillatory term and of the terms containing \( \sin \tau \), \( \cos \tau \), \( \sin 2\tau \) and \( \cos 2\tau \) separately to zero yields

\[
\begin{align*}
-A_0 z + \frac{3}{4} c_3 [2x_1 y_1 x_2 - (x_1^2 - y_1^2) y_2] &\equiv Z(z, x_1, y_1, x_2, y_2) = 0 \\
-A_1 x_1 - k y_1 + 3 c_3 z (y_1 x_2 - x_1 y_2) &\equiv X_1(z, x_1, y_1, x_2, y_2) = 0 \\
k x_1 - A_1 y_1 + 3 c_3 z (x_1 x_2 + y_1 y_2) &\equiv Y_1(z, x_1, y_1, x_2, y_2) = B \\
-A_2 x_2 - 2 k y_2 + 3 c_3 x_1 y_1 &\equiv X_2(z, x_1, y_1, x_2, y_2) = 0 \\
2 k x_2 - A_2 y_2 - \frac{3}{2} c_3 z (x_1^2 - y_1^2) &\equiv Y_2(z, x_1, y_1, x_2, y_2) = 0
\end{align*}
\]

where

\[
\begin{align*}
A_0 &= -c_1 - c_3 [z^2 + \frac{3}{2} (r_1^2 + r_2^2)] \\
A_1 &= 1 - c_1 - \frac{3}{4} c_3 (4z^2 + r_1^2 + 2r_2^2), \quad A_2 = 4 - c_1 - \frac{3}{4} c_3 (4z^2 + 2r_1^2 + r_2^2) \\
r_1^2 &= x_1^2 + y_1^2, \quad r_2^2 = x_2^2 + y_2^2.
\end{align*}
\]

Elimination of the \( x \) and \( y \) components in the above equations gives

\[
\left[ (A_1 - \frac{2r_2^2}{r_1^2} A_2)^2 + k^2 \left( 1 + \frac{4r_2^2}{r_1^2} \right)^2 \right] r_1^2 = B^2
\]
From these relations \( z, r_1 \) and \( r_2 \) are determined. By use of Eqs. (1.26) and (1.27) the coefficients of the periodic solution are found, once \( z, r_1 \) and \( r_2 \) are obtained, by first computing

\[
x_1 = \frac{k(r_1^2 + 4r_2^2)}{B}, \quad y_1 = \frac{-(A_1r_1^2 - 2A_2r_2^2)}{B}
\]

and then subsequently

\[
x_2 = \frac{4r_2^2}{3c_3zr_1^3}[A_2x_1y_1 + k(x_1^2 - y_1^2)]
\]

\[
y_2 = \frac{4r_2^2}{3c_3zr_1^3}[2kx_1y_1 - \frac{1}{2}A_2(x_1^2 - y_1^2)].
\]

Proceeding analogously as before, the condition for stability may also be derived; namely, inserting \( v_0(\tau) \) as given by Eq. (1.25) into (1.15) leads to a Hill's equation of the form

\[
\frac{d^2\eta}{d\tau^2} + (\theta_0 + 2 \sum_{n=1}^{4} \theta_{ns} \sin n\tau + 2 \sum_{n=1}^{4} \theta_{nc} \cos n\tau)\eta = 0.
\] (1.30)

A particular solution of Eq. (1.30) in the second unstable region may be assumed to have the form

\[
\eta(\tau) = e^{\mu\tau}\phi(\tau) = e^{\mu\tau}[b_0 + b_1 \sin(\tau - \sigma_1) + b_2 \sin(2\tau - \sigma_2)].
\] (1.31)

By use of Eqs. (1.26) the stability condition is obtained as

\[
\frac{\partial(Z, X_1, Y_1, X_2, Y_2)}{\partial(z, x_1, y_1, x_2, y_2)} > 0.
\] (1.32)

Figure 1.3 shows the amplitude characteristics of the approximate periodic solution given by Eq. (1.25). The system parameters are the same as in Fig. 1.2, i.e., \( k = 0.4, c_1 = 0 \), and \( c_3 = 1 \). The dashed portion in the first unstable region represents an unstable state for a solution of the form (1.25) with \( z = x_2 = y_2 = 0 \). The dashed portion in the second unstable region represents an unstable state for a solution (1.25) with even-order harmonics suppressed;
indeed the term second unstable region should be understood here to refer to an approximate solution involving only the fundamental frequency. It is to be noted that the second unstable region of Fig. 1.3 is narrower than that of Fig. 1.2 because the third harmonic in Eq. (1.25) was neglected. It may be expected that if an approximate solution including constant, fundamental, second- and third-order harmonics were computed, it would represent a steady state across the entire second unstable region of Fig. 1.2; this is consistent with the results of analog computer analysis presented in the next section.

It is worth mentioning that the second harmonic is sustained in the second unstable region even though the system is symmetrical.

1.4 Analog Computer Analysis

The approximate periodic solutions obtained in the preceding section are

*As the coefficient of η in Eq. (1.30) contains even and odd harmonics, there are regions of coefficients θ₀, θₙₓ and θₙᵥ in which the 1/2, 3/2, ..., harmonics are excited. This implies that in the second unstable region of Fig. 1.3 there may exist intervals of B such that 1/2, 3/2, ..., harmonics develop. A detailed investigation of such a case is, however, omitted here (cf. Secs. 1.4, 1.5 and 1.6).
compared with the solutions obtained by using an analog computer. The block diagram of Fig. 1.4 shows an analog computer setup for the solution of Eq. (1.6), in which the system parameters \( k, c_1 \) and \( c_3 \) are set equal to the values as given in the preceding section; i.e.,

\[
\frac{d^2 v}{d\tau^2} + 0.4 \frac{dv}{d\tau} + v^3 = B \cos \tau.
\]  

(1.33)

The symbols in the figure follow the conventional notation; that is, the integrating amplifiers in the block diagram integrate their inputs with respect to
Fig. 1.5 Amplitude characteristics for the periodic solutions of Eq. (1.33)
obtained by analog computer analysis.

the machine time (in seconds), which is, in this particular case, five times the
independent variable $\tau$. The solutions of Eq. (1.33) are sought for various values
of $B$, i.e., the amplitude of the external force. From the solutions obtained in
this way, each harmonic component is calculated and plotted against $B$ in Fig.
1.5. The first unstable region ranges from $B = 0.45$ to $0.53$; jump phenomena
take place in the direction of arrows. The second unstable region extends from
$B = 2.7$ to $12.6$. In this region the occurrence of the subharmonics of order
$1/2, 3/2, \cdots$ is confirmed in the interval of $B$ approximately from $7$ to $11$.
However, since the solutions accompanied by such subharmonics are extremely
sensitive to external disturbances, the result obtained by computer analysis was
not very accurate. Therefore, this region is indicated by dashed lines in Fig. 1.5.
The third unstable region occurs between $B = 12.6$ to $14.9$, and the oscillation
jumps into another stable state on the borders of this region. These results
show qualitative agreement with the results obtained in the preceding section
using harmonic balance methods.
1.5 Digital Computer Analysis

In the preceding sections we investigated the approximate solutions of Eq. (1.6) both by using the harmonic balance method and by using an analog computer. The results thus obtained were that there are several regions of $B$; in the first and the third unstable regions with respect to odd harmonics there exist two stable states (see Fig. 1.2) and in the second unstable region there is only one stable state (see Fig. 1.3). In this section we shall examine the periodic solutions in each unstable region by using the KDC-I digital computer.

The periodic solutions of Eq. (1.6), that is,

$$\frac{d^2v}{d\tau^2} + k\frac{dv}{d\tau} + c_1v + c_3v^3 = B\cos\tau$$

are determined by the following procedure.

The second-order differential equation (1.6) can be rewritten as simultaneous equations of the first order

$$\frac{dv}{d\tau} = \dot{v}$$

$$\frac{d\dot{v}}{d\tau} = -k\dot{v} - c_1v - c_3v^3 + B\cos\tau. \quad (1.34)$$

We consider the location of the points whose coordinates are given by $v(\tau)$ and $\dot{v}(\tau)$ at the instants $\tau = 0, 2\pi, 4\pi, \cdots$ in the $v\dot{v}$ plane, since the right sides of Eqs. (1.34) are periodic functions in $\tau$ of period $2\pi$. Mathematically, these points $P_n(v(2n\pi), \dot{v}(2n\pi))$ are defined as the successive images of the initial point $P_0(v(0), \dot{v}(0))$ under iterations of the mapping $T$ from $\tau = 0$ to $2n\pi$; we denote this by [15]

$$P_n = T^n(P_0), \quad n = 1, 2, 3, \cdots. \quad (1.35)$$

Actually, these points can be obtained approximately by performing the numerical integration of Eqs. (1.34) from $\tau = 0$ to $2n\pi$. Special attention is directed toward the location of the fixed points and the periodic points of Eq. (1.35). When an initial point $P_0$, the initial condition $(v(0), \dot{v}(0))$, is chosen sufficiently

\footnote{In the second unstable region, there are two oscillations differing in sign and in phase by $\pi$ radians, but their amplitudes are the same.}

\footnote{A point whose location is invariant under the mapping is called a fixed point; i.e.,

$$P_0 = P_1 (= T(P_0))$$

and the corresponding solution $v(\tau)$ is periodic in $\tau$ with the period $2\pi$. Periodic points are}
Table 1.1 Completely stable fixed and periodic points for Eq. (1.34) with $k=0.4$, $c_1=0$ and $c_3=1$

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<th>Point</th>
<th>$v$</th>
<th>$\dot{v}$</th>
<th>$h$</th>
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<td>0.5</td>
<td>1</td>
<td>0.2526</td>
<td>1.0398</td>
<td>$2\pi/60$</td>
<td>Fixed point</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>-0.5290</td>
<td>0.3134</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second</td>
<td>4.0</td>
<td>1</td>
<td>1.5220</td>
<td>3.1810</td>
<td>$2\pi/60$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>1.8626</td>
<td>-1.1065</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second</td>
<td>9.0</td>
<td>1</td>
<td>2.9857</td>
<td>3.2769</td>
<td>$2\pi/120$</td>
<td>2-periodic point</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>3.1460</td>
<td>2.2806</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>2.8192</td>
<td>-0.7005</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>2.9310</td>
<td>0.2684</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third</td>
<td>13.0</td>
<td>1</td>
<td>1.7823</td>
<td>-3.7474</td>
<td>$2\pi/120$</td>
<td>Fixed point</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>3.5927</td>
<td>2.0913</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

near the fixed (periodic) point, the point sequence $\{P_n\}$ converges to the fixed (periodic) point as $n \to \infty$ provided the fixed (periodic) point is completely stable. In order to determine the location of a stable fixed (periodic) point, we estimate the initial condition by making use of the values obtained in the preceding sections.

Then numerical integration of Eqs. (1.34) is performed from the above initial condition until the following condition is reached:

$$
| P_n - P_{n+1} | < \epsilon \quad \text{for a fixed point}
$$

$$
| P_n - P_{n+m} | < \epsilon \quad \text{for a periodic point}
$$

(1.36)

where $\epsilon$ is a small positive constant. In the numerical examples afterwards, $\epsilon = 10^{-5}$ is used. Because of this procedure, only the stable solutions are obtained.

Once the stable fixed (periodic) points are determined, we can easily obtain the time responses (waveforms) of $v(\tau)$ and/or $\dot{v}(\tau)$. Table 1.1 lists the values of the coordinates of the completely stable fixed and periodic points for the representative values of $B$ given in each unstable region. The same values of the system parameters are used as in the preceding examples; i.e., $k = 0.4$, defined by the following relations,

$$
P_0 \neq P_i \quad (1 \leq i \leq m-1), \quad P_0 = P_m = T^m(P_0),
$$

namely, periodic points are invariant under every $m$-th iterate of the mapping. The corresponding solution $v(\tau)$, in this case, is also periodic in $\tau$ but its least period is equal to $2m\pi$. 

16
Some Problems in the Theory of Nonlinear Oscillations

Fig. 1.6 Periodic points and associated phase plane trajectories in the second unstable region \((B = 9.0)\).

\[ c_1 = 0 \text{ and } c_3 = 1. \]

Numerical integration of Eqs. (1.34) was carried out by using the Runge-Kutta-Gill method with the time increment \(h\) equal to \(2\pi/60\) or \(2\pi/120\).

There are two stable periodic solutions in each unstable region. For the first unstable region \((B = 0.5)\), the fixed point 1 represents the resonant state, while the point 2 is non-resonant. For the second unstable region \((B = 4)\), the fixed points 1 and 2 are symmetrically related; that is, if we indicate the periodic solution associated with the fixed point 1 by \(v_{01}(\tau)\), then the fixed point 2 corresponds to the periodic solution \(v_{02}(\tau) = -v_{01}(\tau - \pi)\). For the third unstable region \((B = 13)\), by inspecting the magnitudes of the fundamental components of the periodic solutions represented by the fixed points 1 and 2, the point 1 corresponds to the upper branch of the \(r_1\) characteristic curve in the third unstable region (see Figs. 1.2 and 1.5).
Figure 1.6 shows the phase plane trajectories of the periodic solutions corresponding to the periodic points in the second unstable region ($B = 9$). The small circles in the figure indicate the location of the periodic points which are correlated with the subharmonic oscillations of order $1/2$. The periodic points $1$ and $2$ (or $3$ and $4$) lie on the same trajectory and, under iterations of the mapping, these points are transformed successively to the points that follow in the direction of the arrows. In order to distinguish clearly the trajectory of the point $1$ to $2$ (or $3$ to $4$) from that of the point $2$ to $1$ (or $4$ to $3$), we show the former by solid lines and the latter by dashed lines. The waveforms corresponding to the trajectories $1 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 4 \rightarrow 3$ are shown in Fig. 1.7. If we denote the periodic solution associated with the point $i$ by $v_{0i}(\tau)$, then the periodic solutions correlated with these periodic points are given by

$$v_{01}(\tau) = v_{02}(\tau - 2\pi) = -v_{03}(\tau - \pi) = -v_{04}(\tau - 3\pi)$$
Some Problems in the Theory of Nonlinear Oscillations

\[ -0.31 - 0.06 \sin \frac{1}{2} \tau + 0.01 \cos \frac{1}{2} \tau + 0.58 \sin \tau + 1.84 \cos \tau \\
- 0.00 \sin 3/2\tau + 0.01 \cos 3/2\tau + 0.26 \sin 2\tau + 0.34 \cos 2\tau \\
+ 0.10 \sin 5/2\tau - 0.07 \cos 5/2\tau + 0.04 \sin 3\tau + 0.89 \cos 3\tau \\
+ 0.01 \sin 7/2\tau + 0.02 \cos 7/2\tau + 0.11 \sin 4\tau + 0.05 \cos 4\tau \\
+ 0.03 \sin 9/2\tau - 0.03 \cos 9/2\tau + 0.06 \sin 5\tau + 0.18 \cos 5\tau \\
+ 0.00 \sin 11/2\tau + 0.00 \cos 11/2\tau + 0.05 \sin 6\tau + 0.02 \cos 6\tau \\
+ 0.01 \sin 13/2\tau - 0.01 \cos 13/2\tau + 0.02 \sin 7\tau + 0.05 \cos 7\tau \\
+ 0.00 \sin 15/2\tau - 0.00 \cos 15/2\tau + 0.02 \sin 8\tau + 0.00 \cos 8\tau \\
+ \cdots. \]

(1.37)

In summary, there is complete qualitative agreement between analog and digital simulations regarding the number of stable solutions, and the agreement is also quantitative to the expected accuracy.

1.6 Experimental Result

An experiment using the series-resonance circuit as illustrated in Fig. 1.1 has been performed [5, pp. 39-41]. The result was as follows.

Since \( B \) is proportional to the amplitude \( E \) of the applied voltage, varying \( E \) will bring about the excitation of higher harmonic oscillations. This is observed in Fig. 1.8, in which the effective value of the oscillating current is plotted (thick line) for a wide range of the applied voltage. By making use of a heterodyne harmonic analyzer, this current is analyzed into harmonic components. These are shown by fine lines, the numbers on which indicate the order of the harmonics. The first unstable region ranges between 24 and 40 volts of the applied voltage; the jump phenomenon in this region has been called ferro-resonance. The second unstable region extends from 180 to 580 volts. The third unstable region occurs between 660 and 670 volts, exhibiting another jump in amplitude.

Comparison of Fig. 1.8 with Fig. 1.5 shows full qualitative agreement concerning the number of stable solutions, and the amplitude variations of the harmonics up to order three. Quantitative agreement should not be expected, because of the approximation made in Eq. (1.2) assuming the saturation curve to have only linear and cubic terms.
The diagram shows the oscillating current and its harmonics as a function of the applied voltage. The graph is labeled with several curves, each representing a different series-resonance circuit.

**Fig. 1.8 Experimental result using a series-resonance circuit.**
(Reproduced with the courtesy of Nippon Printing and Publishing Company [5].)
2. HIGHER-HARMONIC OSCILLATIONS IN A PARALLEL-RESONANCE CIRCUIT

2.1 Introduction

In the preceding chapter, we investigated the higher-harmonic oscillations in a series-resonance circuit. Since the series condenser limits the current which magnetizes the reactor core, the applied voltage must be exceedingly high in order to bring the oscillation into the unstable regions of higher order. On the other hand, we may expect that a higher-harmonic oscillation is likely to occur in a parallel-resonance circuit because the reactor core is readily saturated under the impression of a comparatively low voltage; this will be investigated in the present chapter. The differential equation which describes the system takes the form of Mathieu's equation with additional damping and nonlinear restoring terms. An experimental result is also cited at the end of this chapter.

2.2 Derivation of the Fundamental Equation

Figure 2.1 shows the schematic diagram of a parallel-resonance circuit, in which two oscillation circuits are connected in series, each having equal values of $L$, $R$ and $C$, respectively. With the notation of the figure, the equations for the circuit are written as

\[
\begin{align*}
\frac{d\phi_1}{dt} + n \frac{d\phi_2}{dt} &= E \sin \omega t \\
R i_{R1} &= \frac{1}{C} \int i_{C1} dt \\
R i_{R2} &= \frac{1}{C} \int i_{C2} dt \\
\frac{d\phi_1}{dt} &= a_1 \phi_1 + a_3 \phi_1^3 \\
\frac{d\phi_2}{dt} &= a_1 \phi_2 + a_3 \phi_2^3.
\end{align*}
\]  

(2.1)

If the two oscillation circuits behave identically, we have, from the third of Eqs. (2.1)

\[
\phi_1 = \phi_2 = \frac{E}{2n\omega} \cos \omega t.
\]  

(2.2)
Fig. 2.1 Parallel-resonance circuit with nonlinear inductances.

An increase of the flux $\phi_1$ by $\phi$ results in the decrease of $\phi_2$ by the same amount

$$\phi_1 = -\frac{E}{2n\omega} \cos \omega t + \phi, \quad \phi_2 = -\frac{E}{2n\omega} \cos \omega t - \phi.$$  \hspace{1cm} (2.3)

After elimination of $i_{R1}$, $i_{R2}$, $i_{C1}$ and $i_{C2}$ in Eqs. (2.1) and by using Eqs. (2.3), we obtain

$$\frac{d^2 \phi}{dt^2} + \frac{1}{C R} \frac{d \phi}{dt} + \frac{1}{2nC} (i_{L1} - i_{L2}) = 0.$$  \hspace{1cm} (2.4)

Proceeding in the same manner as in Sec. 1.2, we introduce dimensionless variables defined by

$$i_{L1} = I \cdot u_{L1}, \quad i_{L2} = I \cdot u_{L2}, \quad \phi = \Phi \cdot v$$  \hspace{1cm} (2.5)

and fix the base quantities $I$ and $\Phi$ by the following relations

$$n\omega^2 C \Phi = I, \quad c_1 + c_3 = 1$$  \hspace{1cm} (2.6)

where

$$c_1 = \frac{a_1 \Phi}{I}, \quad c_3 = \frac{a_3 \Phi^3}{I}.$$  

Then, by use of Eqs. (2.2), (2.3), (2.5) and (2.6), Eq. (2.4) may be written in normalized form as

$$\frac{d^2 v}{d\tau^2} + k \frac{dv}{d\tau} + \left( c_1 + \frac{3}{2} c_3 B^2 + \frac{3}{2} c_3 B^2 \cos 2\tau \right) v + c_3 v^3 = 0$$  \hspace{1cm} (2.7)
Problems in the Theory of Nonlinear Oscillations

\[ \tau = \omega t, \quad k = \frac{1}{\omega CR}, \quad B = \frac{E}{2n\omega \Phi}. \]

2.3 Amplitude Characteristics of Approximate Periodic Solutions Using Harmonic Balance Method

We assume for the moment that \( k = 0 \) and that \( v \) is so small that we may neglect the nonlinear term in Eq. (2.7). Then equation (2.7) reduces to the Mathieu’s equation

\[ \frac{d^2 v}{d\tau^2} + (\theta_0 + 2\theta_1 \cos 2\tau)v = 0 \quad (2.8) \]

where

\[ \theta_0 = c_1 + \frac{3}{2}c_3B^2, \quad \theta_1 = \frac{3}{4}c_3B^2. \]

From the theory of Mathieu’s equation [13-14, 16] we see that there are regions of the coefficients, \( \theta_0 \) and \( \theta_1 \), in which the solution for Eq. (2.8) is either stable (remains bounded as \( \tau \) increases) or unstable (diverges unboundedly), and that these regions of stability and instability appear alternately as the coefficient \( \theta_0 \) increases. We shall call these regions of instability the first, the second, \( \cdots \) unstable regions as the coefficient \( \theta_0 \) increases from zero. When the coefficients \( \theta_0 \) and \( \theta_1 \) lie in the \( n \)-th unstable region, a higher harmonic of the \( n \)-th order is predominantly excited. Once the oscillation builds up, the nonlinear term \( c_3v^3 \) in Eq. (2.7) may not be ignored. It is this term that finally prevents the amplitude of the oscillation from growing unboundedly.

(a) Periodic Solutions

After these preliminary remarks, we now proceed to investigate the periodic solution of Eq. (2.7) and assume the following form of the solution.

Harmonic: \( v_0(\tau) = x_1 \sin \tau + y_1 \cos \tau \quad (2.9) \)

Second-harmonic: \( v_0(\tau) = z + x_2 \sin 2\tau + y_2 \cos 2\tau \quad (2.10) \)

Third-harmonic: \( v_0(\tau) = x_1 \sin \tau + y_1 \cos \tau + x_3 \sin 3\tau + y_3 \cos 3\tau \quad (2.11) \)

(i) Harmonic Oscillation

In order to determine the coefficients on the right side of Eq. (2.9), we use the method of harmonic balance; namely, substituting Eq. (2.9) into (2.7) and equating the coefficients of the terms containing \( \sin \tau \) and \( \cos \tau \) separately to zero yields
\[
-A_1 + \frac{3}{4}c_3B^2)x_1 - ky_1 \equiv X_1(x_1, y_1) = 0 \\
kx_1 - \left(A_1 - \frac{3}{4}c_3B^2\right)y_1 \equiv Y_1(x_1, y_1) = 0
\]
where
\[
A_1 = 1 - c_1 - \frac{3}{4}c_3(2B^2 + r_1^2), \quad r_1^2 = x_1^2 + y_1^2.
\]

Elimination of the \(x\) and \(y\) components in the above equations gives
\[
\left[A_1^2 + k^2 - \left(\frac{3}{4}c_3B^2\right)^2\right]r_1^2 = 0,
\]
from which the amplitude \(r_1\) is found to be either
\[
r_1^2 = 0
\]
or
\[
r_1^2 = \left(\frac{4}{3} - 2B^2\right)\pm \sqrt{B^4 - \left(\frac{4k}{3c_3}\right)^2}.\]

(ii) Second-harmonic Oscillation

Substituting Eq. (2.10) into (2.7) and equating the coefficients of the non-oscillatory term and of the terms containing \(\sin 2\tau\) and \(\cos 2\tau\) separately to zero, we obtain
\[
-A_0z + \frac{3}{4}c_3B^2y_2 \equiv Z(z, x_2, y_2) = 0 \\
-A_2x_2 - 2ky_2 \equiv X_2(z, x_2, y_2) = 0
\]
where
\[
2kx_2 - A_2y_2 + \frac{3}{2}c_3B^2z \equiv Y_2(z, x_2, y_2) = 0
\]
where
\[
A_0 = -c_1 - c_3[z^2 + \frac{3}{2}(B^2 + r_2^2)], \quad A_2 = 4 - c_1 - \frac{3}{4}c_3(2B^2 + 4z^2 + r_2^2)
\]
\[
r_2^2 = x_2^2 + y_2^2.
\]

Elimination of the \(x\) and \(y\) components in the above equations gives
\[
-A_0z^2 + \frac{1}{2}A_2r_2^2 = 0 \\
(A_2^2 + 4k^2)r_2^2 = \left(\frac{3}{2}c_3B^2\right)^2z^2
\]
Some Problems in the Theory of Nonlinear Oscillations

from which the unknown quantities \( z \) and \( r_2 \) are determined.

(iii) Third-harmonic Oscillation Substituting Eq. (2.11) into (2.7) and equating the terms containing \( \sin \tau \), \( \cos \tau \), \( \sin 3\tau \) and \( \cos 3\tau \) separately to zero, we obtain

\[
-\left( A_1 + \frac{3}{4} c_3 B^2 \right) x_1 - k y_1 - \frac{3}{4} c_3 \left[ (x_1^2 - y_1^2 - B^2) x_3 + 2 x_1 y_1 y_3 \right] = X_1(x_1, y_1, x_3, y_3) = 0
\]

\[
k x_1 - \left( A_1 - \frac{3}{4} c_3 B^2 \right) y_1 + \frac{3}{4} c_3 \left[ 2 x_1 y_1 x_3 - (x_1^2 - y_1^2 - B^2) y_3 \right] = Y_1(x_1, y_1, x_3, y_3) = 0
\]

\[
-A_3 x_3 - 3 k y_3 + \frac{1}{4} c_3 \left[ 3 B^2 - (x_1^2 - 3 y_1^2) \right] x_1 = X_3(x_1, y_1, x_3, y_3) = 0
\]

\[
3 k x_3 - A_3 y_3 + \frac{1}{4} c_3 \left[ 3 B^2 - (3 x_1^2 - y_1^2) \right] y_1 = Y_3(x_1, y_1, x_3, y_3) = 0
\]

where

\[
A_1 = 1 - c_1 - \frac{3}{4} c_3 (2 B^2 + r_1^2 + 2 r_3^2), \quad A_3 = 9 - c_1 - \frac{3}{4} c_3 (2 B^2 + 2 r_1^2 + r_3^2)
\]

\[
r_1^2 = x_1^2 + y_1^2, \quad r_3^2 = x_3^2 + y_3^2
\]

from which the unknown quantities \( x_1, y_1, x_3 \) and \( y_3 \), and consequently the amplitudes, \( r_1 \) and \( r_3 \), are determined.

(b) Stability Investigation of the Periodic Solutions

The periodic solutions given above are sustained actually only when they are stable. Here the stability of the periodic solutions will be investigated in the same manner as we have done in Sec. 1.3. We consider a small variation \( \xi(\tau) \) from the periodic solution \( \nu_0(\tau) \). Then the behavior of \( \xi(\tau) \) is described by the following variational equation

\[
\frac{d^2 \xi}{d\tau^2} + k \frac{d\xi}{d\tau} + \left( c_1 + \frac{3}{2} c_3 B^2 + \frac{3}{2} c_3 B^2 \cos 2\tau + 3 c_3 v_0^2 \right) \xi = 0.
\]  

(2.19)

Furthermore we introduce a new variable \( \eta(\tau) \) defined by

\[
\xi(\tau) = e^{-\delta \tau} \eta(\tau), \quad \delta = k/2
\]

(2.20)

to remove the first-derivative term. Then we obtain

\[
\frac{d^2 \eta}{d\tau^2} + \left( c_1 - \delta^2 + \frac{3}{2} c_3 B^2 + \frac{3}{2} c_3 B^2 \cos 2\tau + 3 c_3 v_0^2 \right) \eta = 0.
\]

(2.21)
Fig. 2.2 Amplitude characteristic curves of the approximate periodic solutions given by Eqs. (2.9), (2.10) and (2.11).

(i) Stability Condition for the Harmonic Oscillation

Inserting \(v_0(\tau)\) as given by Eq. (2.9) into (2.21) leads to

\[
\frac{d^2 \eta}{d\tau^2} + (\theta_0 + 2\theta_{1s} \sin 2\tau + 2\theta_{1c} \cos 2\tau) \eta = 0
\]  

(2.22)

where

\[
\theta_0 = c_1 - \delta^2 + \frac{3}{2} c_3 (B^2 + r_1^2)
\]

\[
\theta_{1s} = \frac{3}{2} c_3 x_1 y_1, \quad \theta_{1c} = \frac{3}{4} c_3 [B^2 - (x_1^2 - y_1^2)].
\]

We assume that a particular solution of Eq. (2.22) in the first unstable region is given by

\[
\eta(\tau) = e^{\mu \tau} \phi(\tau) = e^{\mu \tau} \sin(\tau - \sigma_1).
\]  

(2.23)
Some Problems in the Theory of Nonlinear Oscillations

Proceeding analogously as in Sec. 1.3, the stability condition $|\mu| < \delta$ leads to

$$\Delta_1(\delta) = \left| \begin{array}{cc} \theta_0 + \delta^2 - 1 - \theta_{1c} & \theta_1s - 2\delta \\ \theta_1s + 2\delta & \theta_0 + \delta^2 - 1 + \theta_{1c} \end{array} \right| \equiv \frac{\partial(X_1, Y_1)}{\partial(x_1, y_1)} > 0. \quad (2.24)$$

(ii) Stability Conditions for the Higher-harmonic Oscillations

The conditions for stability of the solutions given by Eqs. (2.10) and Eq. (2.11) may also be derived by the same procedure as above. The results are as follows.

Stability condition for the solution (2.10):

$$\Delta_2(\delta) \equiv \frac{\partial(Z, X_2, Y_2)}{\partial(z, x_2, y_2)} > 0. \quad (2.25)$$

Stability condition for the solution (2.11):

$$\Delta_3(\delta) \equiv \frac{\partial(X_1, Y_1, X_3, Y_3)}{\partial(x_1, y_1, x_3, y_3)} > 0. \quad (2.26)$$

The vertical tangency of the characteristic curves ($Bz$, $Br_1$, $Br_2$ and $Br_3$-relations) also occurs at the stability limit $\Delta_n(\delta) = 0 \ (n = 1, 2, 3)$.

Figure 2.2 shows amplitude characteristics of Eqs. (2.9), (2.10) and (2.11) for the choices of system parameters $k = 0$ and 0.4, $c_1 = 0$ and $c_3 = 1$ in Eq. (2.7). The dashed portions of the characteristic curves represent unstable states. The case $k = 0$ corresponds to the pairs of curves extending in thin tongues to the vertical axis at $B = 0$, while the case $k = 0.4$ has the stable and unstable branches joining smoothly in vertical tangency at the small circles. It is to be mentioned that the portions of the $B$ axis interposed between the end points of the paired characteristic curves are unstable. We see in the figure that

<table>
<thead>
<tr>
<th>Unstable Region</th>
<th>$B$</th>
<th>$\nu$</th>
<th>$\dot{\nu}$</th>
<th>$\dot{h}$</th>
<th>Classification*</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>0.8</td>
<td>1</td>
<td>0.2925</td>
<td>-0.6621</td>
<td>$\pi/30$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>-0.2925</td>
<td>0.6621</td>
<td>””</td>
</tr>
<tr>
<td>Second</td>
<td>1.8</td>
<td>1</td>
<td>0.4430</td>
<td>0.6727</td>
<td>$\pi/30$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>-0.4430</td>
<td>-0.6727</td>
<td>””</td>
</tr>
<tr>
<td>Third</td>
<td>2.8</td>
<td>1</td>
<td>0.1495</td>
<td>-1.7041</td>
<td>$\pi/60$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>-0.1495</td>
<td>1.7041</td>
<td>””</td>
</tr>
</tbody>
</table>

*The mapping $T$ is defined at the instants $\tau = 0, \pi, 2\pi, \cdots$.
increasing $B$ will bring about the excitation of higher-harmonic oscillations and that once the oscillation is started, it may only be stopped by decreasing $B$ to a value which is lower than before, thus exhibiting the phenomenon of hysteresis.

2.4 Analog and Digital Computer Analyses

The approximate periodic solutions obtained in the preceding section have been compared with solutions obtained by using analog and digital computers. The methods of analysis are the same as in Secs. 1.4 and 1.5, and results show good agreement with the results given in Fig. 2.2. Table 2.1 lists the values of the coordinates of the completely stable fixed and periodic points for representative values of $B$ in each unstable region. The same values of the system parameters are used as in Fig. 2.2; i.e., $k = 0.4$, $c_1 = 0$ and $c_3 = 1$. Figure 2.3
Some Problems in the Theory of Nonlinear Oscillations

Fig. 2.4 Experimental result using a parallel-resonance circuit. (Reproduced with the courtesy of Nippon Printing and Publishing Company [5].)

shows the phase plane trajectory of the periodic solution corresponding to the periodic points for the third unstable region \( B = 2.8 \).

2.5 Experimental Result

An experiment on the circuit of Fig. 2.1 has been performed [5, pp. 44-48]. The result was as follows.

The excitation of the fundamental and higher-harmonic oscillations was observed under varying \( E \). As a result of this excitation, the potential of the junction point of the two resonance circuits oscillates with respect to the neutral point of the applied voltage with the frequency of the harmonic. In Fig. 2.4, the
anomalous neutral voltage $V_N$ (which is related to the flux $\phi$) is shown against the applied voltage. 

*The excited oscillation in the first unstable region (marked by 1) has the same frequency as that of the applied voltage. See the waveform (a) in the figure. This phenomenon is known as neutral inversion in electric transmission lines.
3. PHENOMENON OF FREQUENCY ENTRAINMENT

3.1 Introduction

When a periodic force is applied to a self-oscillatory system, the frequency of the self-excited oscillation, that is, the natural frequency of the system, falls in synchronism with the driving frequency, provided these two frequencies are not far different [17-21]. This phenomenon of frequency entrainment may also occur when the ratio of the two frequencies in either order is in the neighborhood of an integer (different from unity) [22]. Thus, if the amplitude and frequency of the external force are appropriately chosen, the natural frequency of the system is entrained by a frequency which is an integral multiple or submultiple of the driving frequency. If the ratio of these two frequencies is not in the neighborhood of an integer, one may expect the occurrence of an almost periodic oscillation [23-24]. It is a salient feature of an almost periodic oscillation that the amplitude and phase of the oscillation vary slowly but periodically even in the steady state. However, the period of the amplitude variation is not an integral multiple of the period of the external force; the ratio of these two periods is in general incommensurable. Therefore, as a whole, there is no periodicity in the almost periodic oscillation.

In this chapter, response curves of the harmonic entrainment, the regions of entrainment at various frequencies, and almost periodic oscillations are studied by using the averaging method. These results are checked by using analog and digital computers.

3.2 Van der Pol’s Equation with Forcing Term

In the preceding chapters we treated the cases in which the restoring force of the system was nonlinear. In this chapter we consider a system in which the nonlinearity appears in the damping. The system considered is governed by

\[
\frac{d^2u}{dt^2} - \epsilon(1 - u^2)\frac{du}{dt} + u = B \cos \nu t + B_0
\]

(3.1)

where \( \epsilon \) is a small positive constant and the right side represents an external force containing a non-oscillatory component. The left side of this equation takes the form of van der Pol’s equation [25-26]. Introduction of a new variable defined by \( v = u - B_0 \) yields an alternative form of Eq. (3.1)

\[
\frac{d^2v}{dt^2} - \mu(1 - \beta v - \gamma v^2)\frac{dv}{dt} + v = B \cos \nu t
\]

(3.2)
Revised Chapter 3 for the Monograph, 1968

where
\[
\mu = (1 - B_0^2) \epsilon, \quad \beta = \frac{2B_0}{1 - B_0^2}, \quad \gamma = \frac{1}{1 - B_0^2}.
\]

Here let us introduce approximate expressions for the entrained oscillations. Since \( \mu \) is small, we see that when \( B = 0 \) the natural frequency of the system (3.2) is nearly equal to unity. Hence, when the driving frequency \( \nu \) is in the neighborhood of unity, we may expect an entrained oscillation at the driving frequency \( \nu \), that is, an occurrence of harmonic entrainment. The entrained harmonic oscillation \( v_0(t) \) may be expressed approximately by

\[
v_0(t) = b_1 \sin \nu t + b_2 \cos \nu t.
\]

On the other hand, when the driving frequency \( \nu \) is far different from unity, we may expect an occurrence of higher-harmonic or subharmonic entrainment. In this case, the entrained oscillation has a frequency which is an integral multiple or a submultiple of the driving frequency \( \nu \). An approximate solution for Eq. (3.2) may be expressed by

\[
v_0(t) = \frac{B}{1 - \nu^2} \cos \nu t + b_1 \sin n\nu t + b_2 \cos n\nu t
\]

where

\[
\begin{align*}
n &= 2 \text{ or } 3: \quad \text{for higher-harmonic oscillations,} \\
n &= 1/2 \text{ or } 1/3: \quad \text{for subharmonic oscillations.}
\end{align*}
\]

The first term on the right side represents the forced oscillation at the driving frequency \( \nu \). The second and the third terms represent the entrained oscillation at the frequency \( n\nu \), which is close to unity.

3.3 Analysis of the Equation Using the Averaging Method

(a) Derivation of an Autonomous System [10, 20, 27-29]

We now write the differential equation (3.2) in a simultaneous form

\[
\frac{dv}{dt} = \dot{v}
\]

\[
\frac{d\dot{v}}{dt} = \mu(1 - \beta \nu - \gamma \nu^2)\dot{v} - \nu + B \cos \nu t.
\]

The behavior of the system is described by the movement of a representative point \((v(t), \dot{v}(t))\) along the solution curves of Eqs. (3.5) in the \( v\dot{v} \) plane. These
solution curves are called trajectories of the representative point. Let us first consider the case in which the driving frequency $\nu$ of the external force is in the neighborhood of unity.* According to the form of the solution (3.3) considered in Sec. 3.2, we introduce a new coordinate system $(b_1(t), b_2(t))$ defined by

$$
\begin{align*}
\nu(t) &= b_1(t) \sin \nu t + b_2(t) \cos \nu t \\
\dot{\nu}(t) &= \nu b_1(t) \cos \nu t - \nu b_2(t) \sin \nu t
\end{align*}
$$

(3.6)

which rotates with angular frequency $\nu$. It may therefore be conjectured that the coordinates $(b_1(t), b_2(t))$ of the representative point vary rather slowly in comparison with $(\nu(t), \dot{\nu}(t))$. To see this let us transform Eqs. (3.5) by using Eqs. (3.6). Hence

$$
\frac{dx_1}{dt} = \frac{\mu}{2} \left\{ \left( 1 - r_1^2 \right) x_1 - \sigma_1 y_1 + \frac{B}{\nu a_0} \right\} \\
- \frac{1}{2} \beta a_0 (x_1^2 - y_1^2) \sin \nu t - \beta a_0 x_1 y_1 \cos \nu t \\
+ \left[ -\sigma_1 x_1 - (1 + 2x_1^2 - 2y_1^2) y_1 \right] \sin 2\nu t \\
+ \left[ (1 - 4y_1^2) x_1 - \sigma_1 y_1 + \frac{B}{\nu a_0} \right] \cos 2\nu t \\
- \frac{1}{2} \beta a_0 (x_1^2 - y_1^2) \sin 3\nu t - \beta a_0 x_1 y_1 \cos 3\nu t \\
- (3x_1^2 - y_1^2) y_1 \sin 4\nu t + (x_1^2 - 3y_1^2) x_1 \cos 4\nu t \right\}
$$

(3.7)

$$
\frac{dy_1}{dt} = \frac{\mu}{2} \left\{ \left[ \sigma_1 x_1 + (1 - r_1^2) y_1 \right] \\
- \beta a_0 x_1 y_1 \sin \nu t + \frac{1}{2} \beta a_0 (x_1^2 - y_1^2) \cos \nu t \\
+ \left[ -(1 - 2x_1^2 + 2y_1^2) x_1 + \sigma_1 y_1 - \frac{B}{\nu a_0} \right] \sin 2\nu t \\
+ \left[ -\sigma_1 x_1 - (1 - 4x_1^2) y_1 \right] \cos 2\nu t \\
+ \beta a_0 x_1 y_1 \sin 3\nu t - \frac{1}{2} \beta a_0 (x_1^2 - y_1^2) \cos 3\nu t \\
- (x_1^2 - 3y_1^2) x_1 \sin 4\nu t - (3x_1^2 - y_1^2) y_1 \cos 4\nu t \right\}
$$

*It is here assumed that $\nu - 1 = O(\mu)$ and $B = O(\mu)$. 

33
Revised Chapter 3 for the Monograph, 1968

where

\[ x_1 = \frac{b_1}{a_0}, \quad y_1 = \frac{b_2}{a_0}, \quad r_1^2 = x_1^2 + y_1^2 \]

\[ a_0 = \sqrt{\frac{4}{\gamma}}, \quad \sigma_1 = \frac{1 - \nu^2}{\mu \nu} \text{ (detuning)}. \]

From the form of the right sides of Eqs. (3.7), it is seen that both \( \frac{dx_1}{dt} \) and \( \frac{dy_1}{dt} \) are proportional to the small parameter \( \mu \), so that \( x_1 \) and \( y_1 \) will be slowly varying functions of \( t \) as we expected. Moreover \( \frac{dx_1}{dt} \) and \( \frac{dy_1}{dt} \) are periodic functions of \( t \) with period \( 2 \pi / \nu \). It may therefore be considered that \( x_1(t) \) and \( y_1(t) \) remain approximately constant during one period \( 2 \pi / \nu \). Hence averaging the right sides of Eqs. (3.7) over the period \( 2 \pi / \nu \), we obtain the relations to determine \( \frac{dx_1}{dt} \) and \( \frac{dy_1}{dt} \) to a first approximation as

\[
\frac{dx_1}{dt} = \frac{\mu}{2} \left[ (1 - r_1^2)x_1 - \sigma_1 y_1 + \frac{B}{\mu \nu a_0} \right] \equiv X_1(x_1, y_1) \\
\frac{dy_1}{dt} = \frac{\mu}{2} \left[ \sigma_1 x_1 + (1 - r_1^2)y_1 \right] \equiv Y_1(x_1, y_1).
\]

Equations (3.8) play an important role in the present investigation, since the singular points of this system correspond to the entrained harmonic oscillations and the limit cycles, if they exist, correspond to the asynchronized almost periodic oscillations. It is to be noted that \( x_1 \) and \( y_1 \) in Eqs. (3.8) denote the normalized amplitudes of the entrained oscillation since the constant \( a_0 \) represents the amplitude of the self-excited oscillation to a first approximation.

By the same procedure as above, we proceed next to derive an autonomous system for the cases in which the frequency \( \nu \) of the external force or its inverse is in the neighborhood of an integer (different from unity). In this case we make use of the transformation defined by

\[
v(t) = \frac{B}{1 - \nu^2} \cos \nu t + b_1(t) \sin \nu t + b_2(t) \cos \nu t \\
\dot{v}(t) = -\frac{\nu B}{1 - \nu^2} \sin \nu t + n \nu b_1(t) \cos \nu t - n \nu b_2(t) \sin \nu t.
\]

Then the derived autonomous systems are as follows.

\[
\begin{align*}
\frac{dx_2}{dt} &= \frac{\mu}{2} \left[ (D - r_2^2)x_2 - \sigma_2 y_2 \right] \equiv X_2(x_2, y_2) \\
\frac{dy_2}{dt} &= \frac{\mu}{2} \left[ \sigma_2 x_2 + (D - r_2^2)y_2 - \frac{\beta}{4a_0} A^2 \right] \equiv Y_2(x_2, y_2)
\end{align*}
\]

34
Some Problems in the Theory of Nonlinear Oscillations

\( n = 3: \)

\[
\begin{align*}
\frac{dx_3}{dt} &= \frac{\mu}{2} \left[ (D - r^3_3)x_3 - \sigma_3 y_3 \right] \equiv X_3(x_3, y_3) \\
\frac{dy_3}{dt} &= \frac{\mu}{2} \left[ \sigma_3 x_3 + (D - r^3_3)y_3 - \frac{\gamma}{12a_0} A^3 \right] \equiv Y_3(x_3, y_3)
\end{align*}
\]

(3.11)

\( n = 1/2: \)

\[
\begin{align*}
\frac{dx_{1/2}}{dt} &= \frac{\mu}{2} \left[ \left( D - r^2_{1/2} + \frac{1}{2} \beta A \right)x_{1/2} - \sigma_{1/2} y_{1/2} \right] \equiv X_{1/2}(x_{1/2}, y_{1/2}) \\
\frac{dy_{1/2}}{dt} &= \frac{\mu}{2} \left[ \sigma_{1/2} x_{1/2} + \left( D - r^2_{1/2} - \frac{1}{2} \beta A \right)y_{1/2} \right] \equiv Y_{1/2}(x_{1/2}, y_{1/2})
\end{align*}
\]

(3.12)

\( n = 1/3: \)

\[
\begin{align*}
\frac{dx_{1/3}}{dt} &= \frac{\mu}{2} \left[ (D - r^2_{1/3})x_{1/3} - \sigma_{1/3} y_{1/3} + \frac{2 A}{a_0} x_{1/3} y_{1/3} \right] \equiv X_{1/3}(x_{1/3}, y_{1/3}) \\
\frac{dy_{1/3}}{dt} &= \frac{\mu}{2} \left[ \sigma_{1/3} x_{1/3} + (D - r^2_{1/3})y_{1/3} + \frac{A}{a_0} (x_{1/3}^2 - y_{1/3}^2) \right] \equiv Y_{1/3}(x_{1/3}, y_{1/3})
\end{align*}
\]

(3.13)

where

\[
\begin{align*}
x_n &= \frac{b_1}{a_0}, \quad y_n = \frac{b_2}{a_0}, \quad r^2_n = x_n^2 + y_n^2, \quad a_0 = \sqrt{\frac{4}{\gamma}} \\
A &= \frac{B}{1 - \nu^2}, \quad D = 1 - \frac{2 A^2}{a_0^2}, \quad \sigma_n = \frac{1 - (nv)^2}{\mu n \nu} \quad \text{(detuning)}
\end{align*}
\]

(b) Singular Points Correlated with Periodic Oscillations

Let \( x_{10} \) and \( y_{10} \) be the coordinates of the singular point of Eqs. (3.8). They are obtained by putting \( dx_1/dt = 0 \) and \( dy_1/dt = 0 \); i.e.,

\[
X_1(x_{10}, y_{10}) = 0, \quad Y_1(x_{10}, y_{10}) = 0
\]

(3.14)

and represent the particular solutions corresponding to the equilibrium states of this system. The variational equation for these solutions is of the form

\[
\frac{d\xi}{dt} = a_1 \xi + a_2 \eta, \quad \frac{d\eta}{dt} = b_1 \xi + b_2 \eta
\]

(3.15)

with \( a_1 = (\partial X_1/\partial x_1)_0, a_2 = (\partial X_1/\partial y_1)_0, b_1 = (\partial Y_1/\partial x_1)_0, \) and \( b_2 = (\partial Y_1/\partial y_1)_0 \), where \( (\partial X_1/\partial x_1)_0, (\partial Y_1/\partial y_1)_0 \) denote the values of \( \partial X_1/\partial x_1, \cdots, \partial Y_1/\partial y_1 \) at \( x_1 = x_{10} \) and \( y_1 = y_{10} \), respectively, and are constants.
Let us assume that the characteristic equation of this system has no root the real part of which is equal to zero. It is known that in this case the system (3.7) has for sufficiently small $\mu$ one and only one periodic solution which reduces to the solution $x_1 = x_{10}$ and $y_1 = y_{10}$ for $\mu = 0$. Moreover the stability of this solution is decided by the signs of the real parts of the corresponding characteristic roots. That is, if the real parts of the roots of the characteristic equation of the system (3.15) are negative, the corresponding periodic solution is stable; if at least one root has a positive real part, the periodic solution is unstable [27].

The coordinates of the singular point are given by

\[ x_{10} = \frac{-\mu \nu a_0}{B} (1 - r_{10}^2) r_{10}^2, \quad y_{10} = \frac{\mu \nu a_0}{B} \sigma_1 r_{10}^2 \]  

(3.16)

where $r_{10}^2$ is determined by the equation

\[ [(1 - r_{10}^2)^2 + \sigma_1^2] r_{10}^2 = \left( \frac{B}{\mu \nu a_0} \right)^2. \]  

(3.17)

Equation (3.17) yields what we call the amplitude characteristics (response curves) for the harmonic oscillation and is obtained by eliminating $x_{10}$ and $y_{10}$ from Eqs. (3.14). Figure 3.1 is obtained by plotting Eq. (3.17) in the $\sigma_1 r_{10}^2$ plane for various values of the magnitude $(B/\mu \nu a_0)^2$. Evidently the curves are symmetrical with respect to the $r_{10}^2$ axis. Each point on these curves yields the amplitude $r_{10}$, which is correlated with the frequency $\nu$ of a possible harmonic oscillation for a given value of the amplitude $B$.

The singular points and the relations representing the amplitude characteristics of the entrained oscillations for the derived autonomous systems (3.10), (3.11), (3.12) and (3.13) are easily obtained in the same manner as above.

(c) Conditions for Stability of Singular Points

The periodic states of equilibrium of the initial system (3.2) are not always realized, but are actually able to exist only so long as they are stable. We have already seen that the stability of the harmonic solution of Eq. (3.2) is to be decided in accordance with the characteristic roots of the corresponding singular point (3.16). Here we will therefore consider the stability condition for the singular point.

Let $\xi$ and $\eta$ be small variations from the singular point defined by

\[ x_1 = x_{10} + \xi, \quad y_1 = y_{10} + \eta \]  

(3.18)

and let us determine whether these variations approach zero or not with the increase of time $t$. We again write the variational equations (3.15) which are
obtained by substituting Eqs. (3.18) into (3.8) and neglecting terms of higher order than the first in $\xi$ and $\eta$

$$\frac{d\xi}{dt} = a_1 \xi + a_2 \eta, \quad \frac{d\eta}{dt} = b_1 \xi + b_2 \eta$$

(3.19)

where

$$a_1 = \frac{\mu}{2} (1 - r_{10}^2 - 2x_{10}^2), \quad a_2 = \frac{\mu}{2} (-\sigma_1 - 2x_{10}y_{10})$$

$$b_1 = \frac{\mu}{2} (\sigma_1 - 2x_{10}y_{10}), \quad b_2 = \frac{\mu}{2} (1 - r_{10}^2 - 2y_{10}^2).$$

The characteristic equation of the system (3.19) is given by

$$\begin{vmatrix} a_1 - \lambda & a_2 \\ b_1 & b_2 - \lambda \end{vmatrix} = \lambda^2 + p\lambda + q = 0$$

(3.20)

where

$$p = -(a_1 + b_2) = \mu(2r_{10}^2 - 1)$$

$$q = a_1b_2 - a_2b_1 = \frac{\mu^2}{4} [(1 - r_{10}^2)(1 - 3r_{10}^2) + \sigma_1^2].$$
The variations $\xi$ and $\eta$ approach zero with the time $t$, provided that the real parts of $\lambda$ are negative. This stability condition is given by the Routh-Hurwitz criterion [30], that is

$$p > 0 \quad \text{and} \quad q > 0.$$  \hspace{1cm} (3.21)

On Fig. 3.1 the stability limits $p = 0$ and $q = 0$ are also drawn; the curve $p = 0$ is a horizontal line $r_{10}^2 = 1/2$, and the curve $q = 0$ is an ellipse which is the locus of the vertical tangents of the response curves. The unstable portions of the response curves are shown dashed in the figure. We can obtain the region of harmonic entrainment in the $B\nu$ plane by referring to the stability limit (drawn as a thick line in the figure) of Fig. 3.1. The portion of the ellipse $q = 0$ applies in the case where the amplitude $B$ and consequently the detuning $\sigma_1$ are comparatively small, while if $B$ and $\sigma_1$ are large the stability limit $p = 0$ applies. For intermediate values of $B$ and $\sigma_1$ some complicated phenomena may occur, but we will not enter this problem here. A detailed investigation of such cases is reported by Cartwright [18]. See also [19].

The stability conditions for the singular points of the autonomous systems (3.10) to (3.13) are derived in the same manner as above.

(d) Regions of Frequency Entrainment

Thus far, the singular points of the derived autonomous systems (3.8), (3.10), (3.11), (3.12) and (3.13) and the relations representing the amplitude characteristic of the entrained oscillations have been investigated. The stability for these singular points has also been investigated by making use of the Routh-Hurwitz criterion. From these results we can obtain the regions of frequency entrainment on the $B\nu$ plane; namely, if the amplitude $B$ and the frequency $\nu$ of the external force are given in these regions, the corresponding autonomous system possesses at least one stable singularity. Consequently, entrainment occurs at the corresponding harmonic, higher-harmonic, or subharmonic frequency of the external force. Figure 3.2 shows an example of the regions of frequency entrainment, computed by numerical evaluation of the stability conditions (3.21) for the derived autonomous system (3.8) and the analogous conditions for (3.10), (3.11), (3.12) and (3.13). The system parameters under consideration are

$$\epsilon = 0.2 \quad \text{and} \quad B_0 = 0.5$$

as in Eq. (3.1). Consequently, the parameters in Eq. (3.2) would be

$$\mu = 0.15, \quad \beta = 4/3 \quad \text{and} \quad \gamma = 4/3.$$  

We see that higher-harmonic or subharmonic entrainment occurs within a narrow range of the driving frequency $\nu$. On the other hand, harmonic entrainment
occurs at any driving frequency $\nu$ provided the amplitude $B$ of the external force is sufficiently large.

In Fig. 3.2(a) the boundary curves of the higher-harmonic entrainment tend asymptotically to the curve $D = 0$ as the detuning $\sigma_n \ (n = 2, 3)$ increases. In the figure the curve $D = 0$ is plotted as a dashed line. We see that the inequality

$$D = 1 - \frac{2A^2}{\omega_0^2} < 0$$

occurs at any driving frequency $\nu$. 

Fig. 3.2 Regions of frequency entrainment.
(a) Harmonic and higher-harmonic entrainments.
(b) Harmonic and subharmonic entrainments.
is equivalent to the first condition of (3.21) which gives the boundary of harmonic entrainment for large detuning $\sigma_1$.\(^1\) It is easily ascertained that if $D < 0$, the higher-harmonic oscillations are stable. Furthermore since there are no abrupt changes in the amplitudes of the higher-harmonic components of an oscillation at the curve $D = 0$, the boundary curve $D = 0$ of the harmonic entrainment has practically no significance for the higher-harmonic entrainment.

In Fig. 3.2(b), the boundary of harmonic entrainment in the neighborhood of $\nu = 1 (\sigma_1 = 0)$ is given by the second condition of (3.21) for small detuning $\sigma_1$ and by the first condition for large detuning $\sigma_1$.\(^1\)

It is also mentioned that the regions of harmonic and 1/3-harmonic entrainments have an overlapping area. In this area common to the two regions, both the harmonic and the 1/3-harmonic oscillations are sustained. On the other hand, such a situation does not occur for the 1/2-harmonic entrainment.

(e) Limit Cycles Correlated with Almost Periodic Oscillations

The oscillations governed by van der Pol’s equation with forcing term are characterized by the behavior of the representative point of the derived autonomous systems within the accuracy of the approximation made in averaging. Now suppose that we fix a point $(x_n(0), y_n(0))$ in the $x_n y_n$ plane as an initial condition. Then the representative point moves, with the increase of time $t$, along the integral curve which emanates from the initial point and may lead ultimately to a stable singular point. Thus the transient solutions are correlated with the integral curves, and the stationary periodic solutions, with the singular points in the $x_n y_n$ plane. The representative point may not always lead to a singular point, but may tend to a closed trajectory along which it moves permanently. An isolated closed trajectory such that no trajectory sufficiently near it is also closed is called a limit cycle [31]. In such a case we see that $x_n(t)$ and $y_n(t)$ tend to periodic functions having the same period in $t$ and hence the solution of the original differential equation (3.2) will be one in which the amplitude and the phase after the lapse of sufficient time

\[a_0^2 r_{10}^2 \approx \left( \frac{B}{1 - \nu^2} \right)^2 = A^2.\]

With the above result the first condition of (3.21) becomes the inequality (3.22). It is to be noted that the assumptions made in the derivation of Eqs. (3.8) become inappropriate as the detuning $\sigma_1$ increases.

\(^1\)As we already noted, for intermediate values of $\sigma_1$, the boundary becomes complicated, but we overlook these situations in Fig. 3.2(b), since such ranges of the external force are extremely limited.
Some Problems in the Theory of Nonlinear Oscillations

vary slowly but periodically. In the same way that a singular point represents a periodic solution of the initial system, a limit cycle represents a stationary oscillation which is affected by amplitude and phase modulation.

The closed trajectory \( C \) is said to be orbitally stable if, given \( \epsilon > 0 \), there is \( \eta > 0 \) such that, if \( R \) is a representative point of another trajectory which is within a distance \( \eta \) from \( C \) at time \( t_0 \), then \( R \) remains within a distance \( \epsilon \) from \( C \) for \( t > t_0 \). If no such \( \eta \) exists, \( C \) is orbitally unstable. Moreover if \( C \) is orbitally stable and, in addition, if the distance between \( R \) and \( C \) tends to zero as \( t \) increases, \( C \) is said to be asymptotically orbitally stable \([19, 32]\).

The stability of a limit cycle can be tested by making use of Poincaré’s criterion for orbital stability. This stability criterion is the following inequality \([19]\);

\[
\oint_C \left( \frac{\partial X_n}{\partial x_n} + \frac{\partial Y_n}{\partial y_n} \right) dt < 0. \tag{3.23}
\]

We proceed to establish the existence of a limit cycle for the derived autonomous systems when the external force is given outside the regions of frequency entrainment. In such a case it follows from a careful consideration that there is only one singular point in the \( x_ny_n \) plane. Furthermore, this singular point is identified as an unstable focus. This means that any representative point starting near this singularity moves away from it with increasing \( t \); in fact there is an ellipse containing the focus in its interior with the property that all representative points cross it moving from its interior to its exterior as \( t \) increases. On the other hand, all integral curves of the autonomous systems remain, as \( t \) increases, within a circle of sufficiently large radius. This follows at once from the form of the autonomous systems \((3.8), (3.10), (3.11), (3.12) \) and \((3.13)\), since we have approximately for large \( x_n \) and \( y_n \):

\[
dx_n/dt = -(\mu/2)r_n^2x_n\]

and

\[
dy_n/dt = -(\mu/2)r_n^2y_n,\]

so that \( dy_n/dx_n = y_n/x_n \). This means that the integral curves are approximately rays through the origin and that a representative point on one of them moves toward the origin as \( t \) increases. Thus there is a ring-shaped domain bounded on the outside by this circle and on the inside by a small ellipse which is free from singular points and has the property that any solution curve which starts inside it remains there as \( t \) increases. The theorem of Poincaré and Bendixson can therefore be applied to establish the existence of at least one limit cycle \([10, 32-33]\).

Thus far, the existence of a limit cycle for the derived autonomous systems is established. However, it is in general not easy to obtain any further information (number, location, size and shape) about such a limit cycle. In order to determine the limit cycle precisely, we are compelled to resort to numerical or graphical means.
Let us give an example of a limit cycle when the amplitude $B$ and the frequency $\nu$ of the external force are prescribed close to the regions of entrainment. The system parameters considered are the same as those in Fig. 3.2.

By putting the external force just outside the region of harmonic entrainment, i.e.,

$$B = 0.2 \quad \text{and} \quad \nu = 1.1,$$

we obtain an example of a limit cycle of Eqs. (3.8). Figure 3.3 shows the limit cycle of this case, which is obtained by carrying out numerical integration both from the inside and from the outside of the limit cycle. The result establishes that the limit cycle is asymptotically orbitally stable. The periodic solution of Eqs. (3.8) represented by the limit cycle of Fig. 3.3 is given by

$$x_1(t) = \varphi_1(t) = 0.09 + 0.19 \sin \theta - 0.81 \cos \theta$$
$$- 0.19 \sin 2\theta + 0.06 \cos 2\theta + 0.04 \sin 3\theta + 0.04 \cos 3\theta + \cdots$$

$$y_1(t) = \psi_1(t) = -0.42 + 0.74 \sin \theta + 0.17 \cos \theta$$
$$- 0.06 \sin 2\theta - 0.17 \cos 2\theta - 0.03 \sin 3\theta + 0.04 \cos 3\theta + \cdots$$

\begin{equation}
(3.24)
\end{equation}

where

$$\theta = 0.0841 \cdots \times (t - h) \quad \text{and} \quad h: \text{an arbitrary constant.}$$
Some Problems in the Theory of Nonlinear Oscillations

The period $T$ of this solution is approximately $74.68(= 2\pi/\nu \times 13.07 \cdots)$. The variational equation for the periodic state (3.24) is a linear differential equation with periodic coefficients,

$$
\begin{align*}
\frac{d\xi}{dt} &= p_{11}(t)\xi + p_{12}(t)\eta, \\
\frac{d\eta}{dt} &= p_{21}(t)\xi + p_{22}(t)\eta
\end{align*}
$$

(3.25)

where

$$
\begin{align*}
p_{11}(t) &= \frac{\mu}{2} \{1 - 3\varphi_1^2(t) - \psi_1^2(t)\}, \\
p_{12}(t) &= \frac{\mu}{2} \{-\sigma_1 - 2\varphi_1(t)\psi_1(t)\} \\
p_{21}(t) &= \frac{\mu}{2} \{\sigma_1 - 2\varphi_1(t)\psi_1(t)\}, \\
p_{22}(t) &= \frac{\mu}{2} \{1 - \varphi_1^2(t) - 3\psi_1^2(t)\}.
\end{align*}
$$

As the Eq. (3.8) is autonomous and its periodic solution $(\varphi_1(t), \psi_1(t))$ is not a state of equilibrium, the linear system (3.25) necessarily has one characteristic multiplier equal to unity, or equivalently one characteristic exponent equal to zero. The other characteristic multiplier $\rho$ is given by [9, 19]

$$
\rho = \exp\left[\mu \int_0^T \{1 - 2\varphi_1^2(t) - 2\psi_1^2(t)\} dt\right] = \exp \left[\mu T (1 - 2\bar{r}_1^2)\right] = \exp (-8.20) = 0.000275
$$

(3.26)

where

$$
\bar{r}_1^2 = \frac{1}{T} \int_0^T \{\varphi_1^2(t) + \psi_1^2(t)\} dt = 0.866.
$$

This confirms that the limit cycle is strongly stable.§ The magnitude $\bar{r}_1$ is sometimes called the normalized r.m.s (root mean square) amplitude of an almost periodic oscillation.

(f) Transition between entrained oscillations and almost periodic oscillations

Let us here briefly explain the transition between entrained oscillations and almost periodic oscillations when the external force is varied across the boundary of harmonic entrainment. As stated above, the boundary of harmonic entrainment is given by the conditions (3.21). The second condition $q > 0$ applies in the case where the detuning $\sigma_1$ and consequently the amplitude $B$ are comparatively small. On the other hand, the first condition $p > 0$ applies in the case where $\sigma_1$ and $B$ are large.

First, let us suppose the case in which almost periodic oscillations are sustained for comparatively small detuning $\sigma_1$, and from this state the frequency

---

§The strong stability of limit cycles is defined by the condition $\rho < 1$; or one of the characteristic exponents of the Eqs. (3.25) is less than zero [34].
\( \nu \) and/or the amplitude \( B \) are varied across the boundary \( q = 0 \) into the region of harmonic entrainment. When the external force reaches the boundary \( q = 0 \), a higher order singularity appears suddenly on the limit cycle representing the almost periodic oscillation. This singularity is a coalesced node-saddle point, and as the external force enters into the region of harmonic entrainment, the singular points separate from one another; i.e., stable node and saddle. Thus the synchronized oscillation is established.

Second, let us consider the case in which almost periodic oscillations are sustained for large detuning \( \sigma_1 \), and from this state the external force is brought inside the region of harmonic entrainment through the boundary \( p = 0 \). In this situation limit cycles representing almost periodic oscillations become small in size, and when the external force is prescribed on the boundary \( p = 0 \) the limit cycle shrinks to a point; at the same time the unstable focus inside the limit cycle turns into a stable focus. Thus the synchronized periodic oscillation appears.

The difference in the above descriptions can be explained as follows. An almost periodic oscillation may be considered as a combination of two components, i.e., the free oscillation with the natural frequency of the system and the forced oscillation with the driving frequency. When the detuning \( \sigma_1 \) is small, the forced oscillation is not predominant, and the free oscillation is entrained by the driving frequency. On the other hand, for large detuning \( \sigma_1 \), the forced oscillation is predominant and the free oscillation is suppressed by the forced one. It should be added that no hysteresis phenomena are observed while varying the external force across the boundary of entrainment for the above two situations, both for small and for large detuning. However, for intermediate values of the detuning \( \sigma_1 \); i.e., \( 1/2 \leq |\sigma_1| \leq 1/\sqrt{3} \) (see Fig. 3.1), complicated phenomena occur. As the ranges of the frequency \( \nu \) and the amplitude \( B \) are extremely narrow for this case and consequently the original non-autonomous equation (3.2) cannot be treated by analog computer, we do not discuss this problem here. Quite similar phenomena are observed for rather wide ranges of the external force in forced self-oscillatory systems with nonlinear restoring term, and the phenomena will be discussed in the following chapters 4 and 5.

3.4 Analog and Digital Computer Analyses

The results obtained in the preceding section are compared with the solutions of the original Eq. (3.1) or (3.2) obtained by using analog and digital computer.

The block diagram of Fig. 3.4 shows an analog computer setup for the solution of Eq. (3.2), in which the system parameters \( \mu, \beta \) and \( \gamma \) are set equal
Some Problems in the Theory of Nonlinear Oscillations

Fig. 3.4 Block diagram of an analog computer setup for the solution of Eq. (3.2), the system parameters being 
\[ \mu = 0.15, \beta = 4/3 \text{ and } \gamma = 4/3. \]
to the values as given above; i.e., \( \mu = 0.15, \beta = 4/3 \) and \( \gamma = 4/3 \). It is confirmed that the regions of frequency entrainment thus obtained agree well with that of Fig. 3.2.

Here, let us investigate the behavior of the images under the mapping \( T \) of the \( \nu \dot{\nu} \) plane into itself, which is defined by Eqs. (3.5), or (3.7), in much the same way as described in Sec. 1.5.

As we have seen in Sec. 3.3(e), when the amplitude \( B \) and the frequency \( \nu \) of the external force are given outside the regions of entrainment, the corresponding autonomous system possesses at least one stable limit cycle. However, it
is tacitly assumed that to the limit cycle there actually corresponds an almost periodic solution of the initial system (3.7). It is known that, when the limit cycle is strongly stable, there exists a unique correspondence between the limit cycle of the autonomous system (3.8) and the simple closed curve which is invariant under the mapping $T$ defined by Eqs. (3.7) and differs little from the limit cycle provided $\mu$ is sufficiently small [34]. The closed curve invariant under the mapping $T$ is called an invariant closed curve. The movement of images on the invariant closed curve is characterized by the rotation number, which measures the average advance of an image under the mapping $T$. It is ascertained that, when the rotation number is irrational and the invariant closed curve is sufficiently smooth, there exists almost periodic solutions in Eqs. (3.7), or in Eqs. (3.5) [32].

Figure 3.5 shows an example of an invariant closed curve of the mapping $T$ defined by Eqs. (3.5). The system parameters and the external force are taken equal to the values as used in Fig. 3.3; i.e., $\mu = 0.15$, $\beta = 4/3$, $\gamma = 4/3$, $B = 0.2$ and $\nu = 1.1$.  

\[\begin{align*}
\text{Closed invariant curve} \\
\text{Unstable fixed point}
\end{align*}\]
Some Problems in the Theory of Nonlinear Oscillations

$B = 0.2$ and $\nu = 1.1$. The successive images $P_n(v(2n\pi/\nu), \dot{v}(2n\pi/\nu))$ $(n = 1, 2, 3, \cdots)$ of the initial point $P_0(v(0), \dot{v}(0))$ under the mapping $T$ are obtained both from the inside and from the outside of the invariant closed curve. The small circles on the curves represent examples of images under the mapping $T$, and they are transformed successively to the points that follow in the direction of the arrows. (It should be noted that the solid curves connecting the dots approaching the invariant closed curve do not represent solutions of Eq. (3.2), but are merely intended to guide the eye.) The period required for the image to complete one revolution along the invariant closed curve of Fig. 3.5 is $11.37 \cdots$ times the period of the external force. This implies that the rotation number associated with the invariant closed curve is $0.912 \cdots$. The closed curve drawn as a dashed line indicates the limit cycle of Fig. 3.3 converted through the relation (3.6).

---

The approximate solution (3.3) or the first equation of (3.6) may be represented as

$$v(t) = a_0r_1(t)\sin[\nu t + \theta(t)]$$

where $\theta(t) = \tan^{-1} y_1(t)/x_1(t)$.

As the limit cycle of Fig. 3.3 contains the origin $x_1 = y_1 = 0$ in its interior and the representative point moves in the clockwise direction, the phase angle of the oscillation lags $2\pi$ radians when the representative point makes one revolution along the limit cycle. Therefore the rotation number associated with the invariant closed curve of Fig. 3.5 is approximated by $1 - 1/11.37 \simeq 0.912$. 

47
4. HYSTERETIC TRANSITION FOR INTERMEDIATE VALUES OF THE DETUNING

4.1 Van der Pol/Duffing Mixed Type Equation

In this chapter we consider a problem in which both the damping and the restoring force are nonlinear; that is, a system governed by the van der Pol/Duffing mixed type equation

$$\frac{d^2 v}{dt^2} - \mu (1 - \gamma v^2) \frac{dv}{dt} + \gamma v^3 = B \cos \nu t$$

where $\mu$ is a small positive constant and $\gamma$ is positive also, and $B \cos \nu t$ represents a forcing term.

In the preceding chapter we overlooked the case where both entrained oscillations and almost periodic oscillations occur even for the same value of the external force, since such a range of the external force is extremely narrow in Eq. (3.1). On the other hand in the system described by Eq. (4.1) the response curves are skewed by the nonlinear restoring force as will be seen later. From this it may be conjectured that such a range of the external forces becomes broader than before. Therefore we consider a particular case in which two types of steady-state responses occur depending only on different value of the initial conditions. Special attention is directed toward the transition between entrained oscillations and beat oscillations which occurs under such circumstances.* The method of analysis is much the same as we have used in the foregoing chapter.

4.2 Response Curves and the Region of Harmonic Entrainment

When no external force is applied to the system, a free oscillation is self-excited. To find the ultimate amplitude and frequency of the oscillation, we put, to a first approximation,

$$v(t) = a_0 \cos \omega_0 t$$

*In the original version of the text, the term almost periodic oscillation is used for beat oscillation, which occurs when the external force is given outside the regions of entrainment. In the present version, however, we use the term beat oscillations instead of almost periodic oscillations.
and substitute this into Eq. (4.1) with $B = 0$. Then, equating the coefficients of the terms containing $\sin \omega_0 t$ and $\cos \omega_0 t$ separately to zero, we obtain

\begin{align*}
  a_0^2 &= \frac{4}{\gamma}, \\
  \omega_0^2 &= \frac{3}{4} a_0^2 = \frac{3}{\gamma}.
\end{align*}

We see that the natural frequency $\omega_0$ of the system is proportional to the amplitude $a_0$ of the free oscillation. This results from the presence of the nonlinear restoring term.

When the external force with frequency $\nu$ nearly equal to $\omega_0$ is present, either an entrained harmonic oscillation or a beat oscillation which develops from harmonic entrainment may occur depending on the value of $B$. Hence we assume an approximate solution of the form

\begin{equation}
  v(t) = b_1(t) \sin \nu t + b_2(t) \cos \nu t
\end{equation}

where $b_1(t)$, $b_2(t)$ become constants for entrained oscillations and slowly varying function of the time $t$ for beat oscillations.

Here let us derive an autonomous equation which determines the amplitudes, $b_1(t)$ and $b_2(t)$ in Eq. (4.4). Substituting Eq. (4.4) into (4.1) and equating the coefficients of the terms containing $\cos \nu t$ and $\sin \nu t$ separately to zero leads to

\begin{align*}
  \frac{dx_1}{dt} &= \frac{\mu}{2} \left[ (1 - r_1^2)x_1 - \sigma_1 y_1 + \frac{B}{\mu a_0} \right] \equiv X_1(x_1, y_1) \\
  \frac{dy_1}{dt} &= \frac{\mu}{2} \left[ \sigma_1 x_1 + (1 - r_1^2)y_1 \right] \equiv Y_1(x_1, y_1)
\end{align*}

where

\begin{align*}
  x_1 &= \frac{b_1}{a_0}, \\
  y_1 &= \frac{b_2}{a_0}, \\
  r_1^2 &= x_1^2 + y_1^2 \\
  a_0 &= \sqrt{\frac{4}{\gamma}}, \\
  \omega_0 &= \sqrt{\frac{3}{\gamma}}, \\
  \sigma_1 &= \frac{\omega_0^2 r_1^2 - \nu^2}{\mu \nu} \text{(detuning)}.
\end{align*}

In deriving the autonomous equations (4.5), the following assumptions are used:

1. The amplitudes $b_1(t)$ and $b_2(t)$ are slowly varying functions of $t$; therefore, $\frac{d^2 b_1}{dt^2}$ and $\frac{d^2 b_2}{dt^2}$ are neglected.
2. Since $\mu$ is a small quantity, $\mu db_1/dt$ and $\mu db_2/dt$ are also discarded.

The equations have the same form as Eqs. (3.8), but it should be noted that the definition of the detuning $\sigma_1$ differs from that in Sec. 3.3(a). In the present case, the detuning is a function of both the amplitude $r_1$ and the driving frequency $\nu$. 

49
This is a reasonable result since the natural frequency of the system is modified
to $\omega_0 r_1$. Equations (4.5) serve as the fundamental equations in studying beat
oscillations as well as entrained oscillations. The entrained harmonic oscillation
is obtained by putting $dx_1/dt = 0$ and $dy_1/dt = 0$, i.e.,

$$X_1(x_1, y_1) = 0, \quad Y_1(x_1, y_1) = 0.$$  \hspace{1cm} (4.6)

The above equations can be combined to give a single equation for $r_1$,

$$[(1 - r_1^2)^2 + \sigma_1^2]r_1^2 = \left( \frac{B}{\mu v a_0} \right)^2.$$ \hspace{1cm} (4.7)

Once $r_1$ is determined, $x_1$ and $y_1$, that is, the coordinates of the singular point,
are found to be

$$x_1 = -\frac{\mu v a_0}{B}(1 - r_1^2)r_1^2, \quad y_1 = \frac{\mu v a_0}{B}\sigma_1 r_1^2.$$  \hspace{1cm} (4.8)

Figure 4.1 shows the response curves as given by Eq. (4.7) for the following
values of the system parameters;

$$\mu = 0.2 \quad \text{and} \quad \gamma = 8.$$

One sees that the response curves in the case of nonlinear restoring force could
be thought of as arising from those for the linear case (see Fig. 3.1) by bending
the latter to the right. As a consequence of this bending, it can be seen in Fig.
4.1 that for many values of $B$ and $\nu$ three different response amplitudes are
possible.

Proceeding analogously as in Sec. 3.3(c), let us investigate the stability of
the equilibrium state by considering the behavior of small variations $\xi$ and $\eta$
from the singular point and determine whether these deviations approach zero
or not with increase of $t$. The variational equation is

$$\frac{d\xi}{dt} = a_1 \xi + a_2 \eta, \quad \frac{d\eta}{dt} = b_1 \xi + b_2 \eta$$ \hspace{1cm} (4.9)

with

$$a_1 = (\partial X_1/\partial x_1)_0, \quad a_2 = (\partial X_1/\partial y_1)_0, \quad b_1 = (\partial Y_1/\partial x_1)_0, \quad b_2 = (\partial Y_1/\partial y_1)_0$$

where $(\partial X_1/\partial x_1)_0, \cdots, (\partial Y_1/\partial y_1)_0$ stand for $\partial X_1/\partial x_1, \cdots, \partial Y_1/\partial y_1$ at the
singular point. By making use of the Routh-Hurwitz criterion, the conditions
Fig. 4.1 Response curves for the harmonic oscillation in the system described by Eq. (4.1), the system parameters being $\mu = 0.2$ and $\gamma = 8$.

for stability of the singular point are given by

$$p = -(a_1 + b_2) = \mu(2r_1^2 - 1) > 0$$

$$q = a_1 b_2 - a_2 b_1 = \frac{\mu^2}{4} \left[ (1 - r_1^2)(1 - 3r_1^2) + \sigma_1^2 + 2 \frac{\omega_0^2}{\mu \nu} \sigma_1 r_1^2 \right] > 0. \quad (4.10)$$

The stability limits $p = 0$ and $q = 0$ are also shown in Fig. 4.1. Hence the dashed portions of the response curves are unstable. It is readily verified that the vertical tangencies of the response curves lie on the stability limit $q = 0$.

From the above results, the region of harmonic entrainment is obtained in the $BV$ plane as illustrated in Fig. 4.2. In this plane each point $(\nu, B)$ cor-
Fig. 4.2 Region of harmonic entrainment in the system described by Eq. (4.1), the system parameters being $\mu = 0.2$ and $\gamma = 8$.

responds to a particular pattern of integral curves; i.e., a phase portrait of Eqs. (4.5) in the $x_1y_1$ plane. The number and character of the singularities are prescribed by the location of the point $(\nu, B)$. The boundary curve $BD\omega_0EA$ corresponds to the upper portion of $q = 0$ in Fig. 4.1, and $BGCA$ to the lower portion of $q = 0$. For points on these boundaries, phase portraits have a higher-order singularity corresponding to a point on the stability limit $q = 0$. The boundary curve $HDCF$ corresponds to the boundary $p = 0$. It touches the curve $BGA$ at the point $C$ and $B\omega_0$ at $D$. The points $C$ and $D$ correspond to the intersections of $q = 0$ with $p = 0$ in Fig. 4.1. For points on the curve $CD$, the singularity on $p = 0$ is a saddle point within the unstable region $q < 0$, and hence, this curve has no practical significance. On the other hand, for points on the curves $HD$ and $CF$, the singularity on $p = 0$ is a higher-order singularity representing the transition between a stable and unstable focus.

From the above considerations, we see that when $B$ and $\nu$ are in the region represented by the curvilinear triangle $\omega_0AB$, Eqs. (4.5) have three singularities, one of which is a saddle point. Outside this region, there is only one stable or unstable singularity, according to whether the point in question is above or
Fig. 4.3 Example of a phase portrait in which singularities and limit cycle coexist, the parameters being $\mu = 0.2$, $\gamma = 8$, $B = 0.35$ and $\nu = 1.0$.

below the boundaries $HG$ and $EF$. In the regions represented by the curvilinear triangles $ACE$ and $BDG$ there are three singularities, one of which is a saddle point and the remaining two are stable singularities in the region $ACE$, and unstable singularities in $BDG$. The region above the line $HD\omega_0EF$ (drawn as a thick line in the figure), therefore, becomes the region of harmonic entrainment since there is at least one stable singularity in the phase portrait.

4.3 Hysteretic Transition between Entrained Oscillations and Beat Oscillations

From the results above, we see that if the amplitude $B$ and frequency $\nu$ of the external force are prescribed to the right of the region of harmonic entrainment ($\nu > \omega_0$), Eqs. (4.5) have only an unstable singularity. It will also be seen that, for large values of $x_1$ and $y_1$, the representative point on the integral curve will cross the family of concentric circles centered at the origin from the outside to
Fig. 4.4 Example of amplitude characteristics when the driving frequency $\nu$ is kept constant at $\nu = 0.9$.

the inside as $t$ increases. Hence the existence of a stable limit cycle in the $x_1y_1$ plane may be concluded, which corresponds to a beat oscillation. However, even if $B$ and $\nu$ are given inside the region of harmonic entrainment, Eqs. (4.5) can possess a stable limit cycle in addition to a stable singular point for certain values of $B$ and $\nu$.

Figure 4.3 shows an example of a phase portrait of such a case. The amplitude $B$ and the frequency $\nu$ of the external force are given by

<table>
<thead>
<tr>
<th>Interval</th>
<th>$p$</th>
<th>$q$</th>
<th>$p^2 - 4q$</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>oa</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>Unstable focus</td>
</tr>
<tr>
<td>ab</td>
<td>$+$</td>
<td>$+$</td>
<td>$-$</td>
<td>Stable focus</td>
</tr>
<tr>
<td>bc</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>Stable node</td>
</tr>
<tr>
<td>cd</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>Saddle</td>
</tr>
<tr>
<td>de</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>Stable node</td>
</tr>
<tr>
<td>ef</td>
<td>$+$</td>
<td>$+$</td>
<td>$-$</td>
<td>Stable focus</td>
</tr>
</tbody>
</table>
which are located in the region of harmonic entrainment (see Fig. 4.2). The other system parameters are the same as in Fig. 4.1. We see in the figure a stable limit cycle which encircles the unstable focus 1. The integral curves (drawn as a thick line in the figure) that approach the saddle point 2 form a separatrix which divides the whole plane into two regions; in one all integral curves tend to the limit cycle, while in the other they tend to the nodal point 3. Thus both beat oscillation and entrained oscillation may occur in this system, which one it will be depending on the initial condition. Let us here consider the transitions between entrained oscillations and beat oscillations which occur under such circumstances.

Figure 4.4 shows an example of the amplitude characteristic ($Br^2$-relation) when $\nu$ is kept constant (\(= 0.9\)). By making use of the stability conditions (4.10) we classified the singular points along the amplitude characteristic illustrated in Fig. 4.4. The results are listed in Table 4.1.

The occurrence and extinction of a limit cycle, as $B$ varies along this characteristic, will most easily be understood by plotting the phase portraits of Eqs. (4.5) for various values of $B$. The results are shown in Fig. 4.5. When $B$ is given below $B_1$ of Fig. 4.4, a limit cycle exists which encircles an unstable focal point. This state is shown in Fig. 4.5(a). A coalesced singularity appears at $B = B_1$. Further increase in $B$ results in the separation of this higher-order singular point into two simple singular points, resulting in the coexistence of a stable limit cycle and a stable node. When $B = B_2$, there exists a closed integral curve starting from the saddle point and coming back to the same point, as shown in Fig. 4.5(d). The limit cycle disappears when $B$ is increased beyond $B_2$. However, when $B$ reaches $B_3$ the integral curves once again show the same behavior as in Fig. 4.5(d). A limit cycle appears once more for $B_3 < B < B_4$ and shrinks up to a stable focus at $B = B_4$. For values of $B$ between $B_4$ and $B_5$, there exist two distinct stable singularities, corresponding to resonant and non-resonant oscillations, but there is no limit cycle. The coalescence of singularities occurs at $B = B_5$ and then this higher-order singularity disappears. Hence it is concluded that a beat oscillation occurs for $B < B_2$ and $B_3 < B < B_4$, a resonant oscillation for $B_1 < B$, and a non-resonant oscillation for $B_4 < B < B_5$.

Here we introduce the mean-square amplitude of the beat oscillation defined by

$$\bar{r}_1^2 = \frac{1}{\tau} \int_C r_1^2 dt$$  \hspace{1cm} (4.11)

where $C$ is an integration path taken around the limit cycle and $\tau$ is the period.
Fig. 4.5 Various phase portraits of Eqs. (4.5) showing the transition between entrained oscillations and beat oscillations.

for the representative point \((x_1(t), y_1(t))\) to complete one revolution along the limit cycle. This mean square amplitude \(\tilde{r}_1^2\) of the beat oscillation is plotted by
uneven dashed lines in Fig. 4.6. The arrows in the figure show the transitions between these oscillations. Thus we see that between $B_1$ and $B_2$ there exists hysteresis and at $B_3$ and $B_5$ jump phenomena take place from the lower to the higher branch.

It should also be mentioned that, for other values of the external force parameters, the situation may somewhat be different from that mentioned above. Under certain conditions (for example, for different values of $\nu$), a beat oscillation occurs for any value of $B$ below $B_4$, that is, the two uneven dashed lines become one; or a beat oscillation may occur only for values below $B_1$ or between $B_3$ and $B_4$. In the latter case no hysteresis between $B_1$ and $B_2$ results, since the coalescence of singularities appears on the limit cycle.

In the above we investigated the right hand transitional region as given by $\nu > \omega_0$, but for the left hand region $\nu < \omega_0$, particularly in the neighborhood of the curvilinear triangle $BDG$ in Fig. 4.2, some complicated phenomena may occur [35-36]. However, we shall not enter into this problem here, because such a region of the external force parameters is extremely narrow. An interesting example of a transition for such a situation will be given in the last chapter of this text.
4.4 Analog Computer Analysis

In the preceding section two types of steady-state responses and the transition between them were investigated on the basis of the derived autonomous system (4.5). Here let us examine these results by using an analog computer for the original non-autonomous system.

The solutions of the original differential equation (4.1), i.e.,

\[
\frac{d^2v}{dt^2} - \mu(1 - \gamma v^2)\frac{dv}{dt} + v^3 = B \cos \nu t
\]

are sought for rather restricted ranges of \( B \) and \( \nu \), concentrating our attention on the hysteretic transition for the case in which the driving frequency \( \nu \) is greater than the natural frequency \( \omega_0 \) of the system. The system parameters are the same as in Secs. 4.2 and 4.3, i.e., \( \mu = 0.2 \) and \( \gamma = 8 \). The regions of different steady states thus obtained are shown in Fig. 4.7. The lines in the figure show the boundaries of entrainment, or boundaries of different types of
Some Problems in the Theory of Nonlinear Oscillations

oscillations. The hatched area indicates the presence of a beat oscillation in addition to an entrained oscillation. The other domains of the different types of oscillations are the same as already explained in Fig. 4.2 and should be self-explanatory. The result concerning hysteresis is also confirmed.
5. TRANSITIONS OF SINGULARITIES AND LIMIT CYCLES

5.1 Autonomous System Derived from Rayleigh/Duffing Mixed Type Equation

In the preceding chapter, we investigated the phenomenon of hysteretic transition which occurs in the self-oscillatory system with external force described by a second-order non-autonomous equation, the van der Pol/Duffing mixed type equation. The procedure was: first, the non-autonomous equation was approximated by an autonomous equation by applying the averaging method. Then, the singular points and the limit cycles of the derived autonomous system were investigated. Finally, by using an analog computer the results for the autonomous system were compared with the solutions of the original non-autonomous system.

In the present chapter, however, ignoring the applicability of the results for an autonomous system to a non-autonomous one, we consider the transitions of singularities and limit cycles of an autonomous system which determines the amplitudes of approximate steady states of a self-oscillatory system with external force. The reason is as follows: the parameter ranges treated in the following analysis are rather restricted and narrow; moreover the phenomenon itself in the circumstances is so complicated that it cannot be expected that analog-computer experiments can bring reliable solutions of the original non-autonomous system and that the phenomenon can be approximated by a simple form of solutions as was used in the preceding chapter.

Let us consider the autonomous system

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\mu}{2} \left( (1 - \gamma \nu^2 r^2)x - \sigma y + \frac{B}{\mu \nu a_0} \right) \equiv X(x, y) \\
\frac{dy}{dt} &= \frac{\mu}{2} [\sigma x + (1 - \gamma \nu^2 r^2)y] \equiv Y(x, y)
\end{align*}
\]

(5.1)

where

\[ r^2 = x^2 + y^2, \quad a_0^2 = \frac{4}{3\gamma}, \quad \omega_0^2 = \frac{1}{\gamma}, \quad \sigma = \frac{\omega_0^2 r^2 - \nu^2}{\mu \nu} \text{ (detuning)}. \]

The equations (5.1) arise as a derived autonomous system for the components in a phase plane rotating with frequency \( \nu \) of the forced oscillations in a self-oscillatory system with external force governed by the Rayleigh/Duffing mixed
Some Problems in the Theory of Nonlinear Oscillations

type equation

\[ \frac{d^2 v}{dt^2} - \mu \left[ 1 - \gamma^2 \left( \frac{dv}{dt} \right)^2 \right] \frac{dv}{dt} + v^3 = B \cos \nu t, \quad 0 < \mu \ll 1. \] \tag{5.2}

By assuming an approximate solution for the above equation (5.2) of the form

\[ v(t) = b_1(t) \sin \nu t + b_2(t) \cos \nu t \] \tag{5.3}

and by applying the harmonic balance method under the same assumptions made on \( b_1(t) \) and \( b_2(t) \) as in the preceding chapter, the equations (5.1) are derived with the relations \( x = b_1(t)/a_0 \) and \( y = b_2(t)/a_0 \). The constants \( a_0 \) and \( \omega_0 \) represent the values of the first approximation to the amplitude and frequency of the free oscillation, respectively. In the systems (5.1) and (5.2), \( \mu \) and \( \gamma \) are system parameters; \( B \) and \( \nu \) represent the amplitude and frequency of the external force; that is, the same notations as in the preceding chapter are used.

The system (5.2) also describes the phenomenon of frequency entrainment. Before giving an example of transition of singularities and limit cycles, let us show an outline of the region of harmonic entrainment.

5.2 Response Curves and the Region of Harmonic Entrainment

The region of harmonic entrainment is the region in the \( B \nu \) plane such that, if the amplitude \( B \) and the frequency \( \nu \) of the external force are given in the region, the autonomous system (5.1) possesses at least one stable singularity.

The coordinates of the singular point are given by

\[ x = -\frac{\mu \nu a_0}{B} (1 - \gamma \nu^2 r^2) r^2, \quad y = \frac{\mu \nu a_0}{B} \sigma r^2 \] \tag{5.4}

where \( r^2 \) is determined by the relation

\[ [(1 - \gamma \nu^2 r^2)^2 + \sigma^2] r^2 = \left( \frac{B}{\mu \nu a_0} \right)^2. \] \tag{5.5}

The conditions for stability of the singular point (5.4) may also be derived by the same procedure as used in the preceding chapters. The results are,

\[ p = \mu (2 \gamma \nu^2 r^2 - 1) > 0 \]

\[ q = \frac{\mu^2}{4} \left[ (1 - \gamma \nu^2 r^2)(1 - 3 \gamma \nu^2 r^2) + \sigma^2 + 2 \frac{\omega_0^2}{\mu \nu} \sigma r^2 \right] > 0. \] \tag{5.6}
Fig. 5.1 Response curves for the system described by Eqs. (5.1), the system parameters being $\mu = 0.2$ and $\gamma = 2$.

Figure 5.1 shows the response curves given by the relation (5.5) for the following values of the system parameters;

$$\mu = 0.2 \quad \text{and} \quad \gamma = 2.$$  

The stability limits $p = 0$ and $q = 0$ given by Eqs. (5.6) are also shown in the figure. The dashed portions of the resonance curves are unstable. One sees that Fig. 5.1 is similar to Fig. 4.1 except for the shape of the stability limit $p = 0$. From this result the region of harmonic entrainment is reproduced in the $B\nu$ plane as illustrated in Fig. 5.2, which is also analogous to Fig. 4.2. In the figure the curvilinear triangle $BDG$ corresponding to that of Fig. 4.2 is broader than before. Therefore, we can more readily deal with the behavior of integral curves
Some Problems in the Theory of Nonlinear Oscillations

Fig. 5.2 Region of harmonic entrainment in the system described by Eqs. (5.1), the system parameters being $\mu = 0.2$ and $\gamma = 2$.

when the external force is given in the vicinity of this region, or to the left of the frequency $\omega_0$ of the free oscillation.

5.3 Transition of Singularities and Limit Cycles

Let us first show an example of a phase portrait having three singularities and two limit cycles. The special case we have in mind is that specified by the following values of the external force parameters

$$B = 0.065 \quad \text{and} \quad \nu = 0.57$$

which is located in the curvilinear triangle $BDG$ of Fig. 5.2. The other system parameters are the same as in Figs. 5.1 and 5.2. The phase portrait of Eqs. (5.1) for the above parameters is plotted in Fig. 5.3. As illustrated in the figure, there are three singularities in this case. We see in the figure an unstable limit cycle which encircles the stable focus 3, and outside this a stable limit cycle enclosing three singular points. The unstable limit cycle divides the whole plane into two regions; inside this limit cycle all integral curves tend to the stable focus 3,
Fig. 5.3 Example of a phase portrait in which singularities and limit cycles coexist, the parameters being \( \mu = 0.2, \gamma = 2, \ B = 0.065 \) and \( \nu = 0.57 \).

while outside they tend to the stable limit cycle. Thus both the stable singular point and the limit cycle coexist in this system.

Here, let us proceed to the main subject of the transitions of singularities and limit cycles of Fig. 5.3 when \( B \) is varied. By substituting \( \nu = 0.57 \) into Eq. (5.5), the \( Br^2 \) relation is calculated and plotted in Fig. 5.4. As in the case of Sec. 4.3, the singularities are classified and the results are listed in Table 5.1.

Figure 5.5 shows the phase portraits of Eqs. (5.1) for various values of \( B \). The phase portrait of Fig. 5.3 appears again as Fig. 5.5(d). When \( B \) is decreased the unstable limit cycle diminishes in size and shrinks up to the unstable focus when \( B_2 \) of Fig. 5.4 is reached. This state is shown in Fig. 5.5(c). Further decrease in \( B \) brings about the coalescence of singular points at \( B = B_1 \) and then this higher-order singularity disappears (see Figs. 5.5(a) and (b)). When \( B \) is increased from Fig. 5.5(d) the unstable limit cycle grows large, and at \( B = B_3 \) a higher-order singularity is formed by the coalescence of two singular points at a location lying between two limit cycles. Two limit cycles persist
for values of $B$ between $B_3$ and $B_4$ coexisting with one stable singular point; when $B$ reaches $B_4$ the two limit cycles coincide and form a higher-order limit cycle which is semi-stable (Fig. 5.5(h)), which then disappears leaving only one stable focus.

Hence it is concluded that a stable limit cycle appears for $B < B_4$, an unstable limit cycle for $B_2 < B < B_4$ and a stable singular point for $B_2 < B$ in this particular case.

Table 5.1 Classification of singular points along the amplitude characteristic of Fig. 5.4

<table>
<thead>
<tr>
<th>Interval</th>
<th>$p$</th>
<th>$q$</th>
<th>$p^2 - 4q$</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>oa</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>Unstable focus</td>
</tr>
<tr>
<td>ab</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>Unstable node</td>
</tr>
<tr>
<td>bc</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>Saddle</td>
</tr>
<tr>
<td>cd</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>Unstable node</td>
</tr>
<tr>
<td>de</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>Unstable focus</td>
</tr>
<tr>
<td>ef</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>Stable focus</td>
</tr>
</tbody>
</table>
Fig. 5.5 Various phase portraits of Eqs. (5.1) showing the transition of
singularities and limit cycles.

As previously in Sec. 4.3, we again introduce the mean-square amplitude of
the limit cycle defined by
Some Problems in the Theory of Nonlinear Oscillations

Fig. 5.6 Amplitude characteristics with mean-square amplitude of the limit cycles.

\[ \bar{r}^2 = \frac{1}{\tau} \oint_C r^2 \, dt \]  

(5.7)

where \( C \) is an integration path taken around the limit cycle and \( \tau \) is the period for the representative point \((x(t), y(t))\) to complete one revolution along the limit cycle.

By using Eqs. (5.1) and (5.7), we have

\[ \oint_C \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \, dt = \mu \int_0^\tau (1 - 2\gamma\nu^2 r^2) \, dt = 2\mu\tau\gamma\nu^2 \left( \frac{1}{2\gamma\nu^2} - \bar{r}^2 \right). \]  

(5.8)

From Poincaré's criterion for orbital stability [19], we see that a stable limit cycle has mean-square amplitude greater than \(1/2\gamma\nu^2\), an unstable limit cycle less than \(1/2\gamma\nu^2\), and a limit cycle of higher order has mean-square amplitude equal to \(1/2\gamma\nu^2\).

The results obtained above are summarized and sketched in Fig. 5.6. In the figure the mean-square amplitude \(\bar{r}^2\) of the limit cycle is plotted by uneven
dashed lines, and the arrows show jump phenomena between stable steady states when B is varied. That is to say, in physical term, an entrained oscillation is sustained for \(B_2 < B\), a beat oscillation for \(B < B_4\), and hysteresis occurs between \(B_2\) and \(B_4\).

Thus far singularities and limit cycles of the autonomous system (5.1) have been investigated. Particular attention has been directed to the transitions of the phase portraits in the left hand region \((\nu < \omega_0)\) on the \(B\nu\) plane, and the example illustrating such transitions has been given. However, the application of these results to the original non-autonomous Rayleigh/Duffing mixed type equation requires further examination, since the representative point moves along the limit cycle rather quickly. It should be added that a region such as that represented by the curvilinear triangle \(BDG\) in Fig. 5.2 and in Fig. 4.2 does not appear in the self-oscillatory system having linear restoring force represented by the Eq. (3.1).

---

*When \(B = 0\), Eqs. (5.1) are easily integrated; that is

\[
r^2 = \frac{\omega_0^2 r_0^2}{(\omega_0^2 - \nu^2 r_0^2)e^{-\mu t} + \nu^2 r_0^2}
\]

where

\[
r_0^2 = r^2|_{t=0}.
\]

We see that the limit cycle in this particular case is a circle centered at the origin with radius \(r^2 = \omega_0^2/\nu^2 = 1/\gamma \nu^2\).
Some Problems in the Theory of Nonlinear Oscillations

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Some Problems in the Theory of Nonlinear Oscillations


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