Structure of Invariants in the Wave Equation of n-Dimensional Harmonic Oscillator

Shigeru MATSUO*
(Yukawa Laboratory)

Received August 25, 1966

The constants of motion of n-dimensional harmonic oscillator are proved to satisfy the same commutation relations as those of infinitesimal generators of an n-dimensional unitary unimodular group. An extention of our considerations to the relativistic case results in the isomorphism of symmetry group of our system to the special complex Lorents group, and this fact will be used in the non-local theory of fields to connect internal and external quantum numbers of elementary particles.

I. INTRODUCTION

We shall show in this article that the wave equation of n-dimensional harmonic oscillator has the so called unitary unimodular group $SU(n)$ as its own symmetry transformation group. As a frist step, the Lie theory of continuous transformation groups will be stated, and then the isomorphism of the group which leaves invariant the wave equation of n-dimensional harmonic oscillator to the unitary unimodular group in an n-dimensional unitary space will be proved. Finally, an extension of our arguments to the theory of non-local fields will be suggested.

II. PRELIMINARIES FROM THE LIE THEORY OF CONTINUOUS TRANSFORMATION GROUPS

An infinitesimal transformation of r-parameter continuous group in a certain n-dimensional space is given by

$$dx^i = \sum_{\alpha,\beta=1}^{r} \xi^i{}_\alpha(x^1, x^2, \cdots\cdots, x^n)\lambda_\beta{}^\alpha(a^1, a^2, \cdots\cdots, a^n)da^\beta, \quad \mathrm{i} = 1, 2, \cdots\cdots, \mathrm{n}, \tag{1}$$

where $a^1, a^2, \cdots\cdots, a^r$ are r independent parametrs of the Lie group, $\xi_\alpha{}^i$ are "velocity field" of the transformation, and $\lambda_\beta{}^\alpha$ are "velocity field" of the adjoint group. According to Lie's second fundamental theorem, if operators of infinitesimal transformation defined by

$$X_\alpha = \sum_{i=1}^{n} \xi_\alpha{}^i(x^1, x^2, \cdots\cdots, x^n)\frac{\partial}{\partial x^i} \tag{2}$$

satisfy the following commutation relations

---

* 松尾　茂

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma \tag{3}$$

then operators X_α generate the r-parameter continuous transformation group, and they are called the infinitesimal operators of the group considered. The constants $c_{\alpha\beta}{}^\gamma$ are called "structure constants" of the Lie group.

Let $F_1, F_2, \cdots\cdots, F_r$ be linearly independent classical integrals of a dynamical system described by a Hamiltonian H $(p^1, p^2, \cdots\cdots, p^n; q^1, q^2, \cdots\cdots, q^n)$ with the generalized coordinates p^i, q^i. Then the infinitesimal transformation near the identity transformation in the phase spces defined by

$$dp^k = -\sum_{\alpha=1}^{r} \frac{\partial F_\alpha}{\partial q^k} da^\alpha \quad , \qquad dq^k = \sum_{\alpha=1}^{r} \frac{\partial F_\alpha}{\partial p^k} da^\alpha \quad , \tag{4}$$

is an infinitesimal canonical transformation. The operators which correspond to X_α in Eq. (2) are

$$X_\alpha = \sum_{i=1}^{n} \frac{\partial F_\alpha}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial F_\alpha}{\partial q^i}\frac{\partial}{\partial p^i} = (F_\alpha, \quad) , \tag{5}$$

where $(F_\alpha, \quad)$ are the Poisson bracket operators. Then Eq. (3) becomes

$$(F_\alpha, \quad (F_\beta, \quad)) - (F_\beta, \quad (F_\alpha, \quad)) = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma (F_\gamma, \quad) , \tag{6}$$

from which

$$(F_\alpha, F_\beta) = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma F_\gamma \tag{7}$$

are obtained using the Jacobi identity. Eq. (7) is the necessary and sufficient condition that the infinitesimal transformation (4) generates an r-parameter continuous transformation group on an energy surface.

Now we shall transfer to quantum mechanical case. In quantum mechanical analogy, there exist r linearly independent operators $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots\cdots, \boldsymbol{F}_r$ commuting with the Hamiltonian $\boldsymbol{H}(\boldsymbol{p}^1, \boldsymbol{p}^2, \cdots\cdots, \boldsymbol{p}^n; \boldsymbol{q}^1, \boldsymbol{q}^2, \cdots\cdots, \boldsymbol{q}^n)$, and an infinitesimal unitary transformation :

$$d\boldsymbol{U} = 1 + \sum_{\alpha=1}^{r} \frac{i}{\hbar} F_\alpha(\boldsymbol{p}^1, \boldsymbol{p}^2, \cdots\cdots, \boldsymbol{p}^n; \boldsymbol{q}^1, \boldsymbol{q}^2, \cdots\cdots, \boldsymbol{q}^n) da^\alpha \tag{8}$$

corresponds to the infinitesimal canonical transformation (4). In general, there exist the following commutation relations between the operators $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots\cdots, \boldsymbol{F}_r$

$$[\boldsymbol{F}_\alpha, \boldsymbol{F}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma \boldsymbol{F}_\gamma , \tag{9}$$

where the structure constants $c_{\alpha\beta}{}^\gamma$ are the same as those of Eq. (7). In addition, the generators of infinitesimal transformation $\boldsymbol{F}_\alpha$ tranform an eigenfuction corresponding to an energy eigenvalue into another eigenfunction belonging to the same eigenvalue, as a result of which these eigenfunctions span a representation space of the Lie group generated by $\boldsymbol{F}_\alpha$.

III. STRUCTURE OF INVARIANTS OF n-DIMENSIONAL HARMONIC OSCILLATOR

Now we shall turn our attention to an n-dimensional harmonic oscillator. The Hamiltonian of n-dimensional harmonic oscillator is given by

$$\boldsymbol{H} = \sum_{k=1}^{n} \frac{1}{2}[(\boldsymbol{p}^k)^2+\beta(\boldsymbol{q}^k)^2] \tag{10}$$

Then the following (n^2-1) constants of motion will be found :

$$\boldsymbol{F}^{k'}_{(k)} = \boldsymbol{q}^{k'}\boldsymbol{p}^k - \boldsymbol{q}^k\boldsymbol{p}^{k'} \ , \tag{11a}$$

$$\boldsymbol{G}^{k'}_{(k)} = \frac{1}{\sqrt{\beta}}(\boldsymbol{p}^{k'}\boldsymbol{p}^k+\beta\boldsymbol{q}^{k'}\boldsymbol{q}^k) \ , \tag{11b}$$

$$\boldsymbol{E}_{(k)} = \frac{1}{2\sqrt{\beta}}[(\boldsymbol{p}^{k-1})^2-(\boldsymbol{p}^k)^2+\beta(\boldsymbol{q}^{k-1})^2-\beta(\boldsymbol{q}^k)^2] \ , \tag{11c}$$

where $1 \leqq k \leqq k-1$, $k = 2, 3, \cdots\cdots, n$, and so we have $\frac{(n^2-n)}{2}$, $\frac{(n^2-n)}{2}$, and $(n-1)$ independent operators $\boldsymbol{F}^{k'}_{(k)}$, $\boldsymbol{G}^{k'}_{(k)}$, and $\boldsymbol{E}_{(k)}$ respectively. These operators satisfy the commutations relations given by

$$[\boldsymbol{F}^{k'}_{(k)}, \boldsymbol{F}^{l'}_{(l)}] = -\frac{\hbar}{i}\delta_{kl}\boldsymbol{F}^{k'}_{(l')}-\frac{\hbar}{i}\delta_{k'l'}\boldsymbol{F}^{k}_{(l)}-\frac{\hbar}{i}\delta_{kl'}\boldsymbol{F}^{l}_{(k')}-\frac{\hbar}{i}\delta_{k'l}\boldsymbol{F}^{l'}_{(k)} \ , \tag{12a}$$

$$[\boldsymbol{G}^{k'}_{(k)}, \boldsymbol{G}^{l'}_{(l)}] = -\frac{\hbar}{i}\delta_{kl}\boldsymbol{F}^{k'}_{(l')}-\frac{\hbar}{i}\delta_{k'l'}\boldsymbol{F}^{k}_{(l)}-\frac{\hbar}{i}\delta_{kl'}\boldsymbol{F}^{k'}_{(l)}-\frac{\hbar}{i}\delta_{k'l}\boldsymbol{F}^{k}_{(l')} \ , \tag{12b}$$

$$[\boldsymbol{E}_{(k)}, \boldsymbol{E}_{(l)}] = 0 \ , \tag{12c}$$

$$[\boldsymbol{F}^{k'}_{(k)}, \boldsymbol{G}^{l'}_{(l)}] = -\frac{\hbar}{i}\delta_{k'l'}\boldsymbol{G}^{k}_{(l)}-\frac{\hbar}{i}\delta_{kl}\boldsymbol{G}^{k'}_{(l')}-\frac{\hbar}{i}\delta_{k'l}\boldsymbol{G}^{k}_{(l')}-\frac{\hbar}{i}\delta_{kl'}\boldsymbol{G}^{k'}_{(l)} \ , \tag{12d}$$

$$[\boldsymbol{F}^{k'}_{(k)}, \boldsymbol{E}_{(l)}] = -\frac{\hbar}{i}\delta_{kl}\boldsymbol{G}^{k'}_{(l)}+\frac{\hbar}{i}\delta_{k,l-1}\boldsymbol{G}^{k'}_{(l-1)}+\frac{\hbar}{i}\delta_{k'l}\boldsymbol{G}^{k}_{(l)}-\frac{\hbar}{i}\delta_{k',l-1}\boldsymbol{G}^{k}_{(l-1)} \ , \tag{12e}$$

$$\boldsymbol{G}[^{k'}_{(k)}, \boldsymbol{E}_{(l)}] = +\frac{\hbar}{i}\delta_{kl}\boldsymbol{F}^{k'}_{(l)}-\frac{\hbar}{i}\delta_{k,l-1}\boldsymbol{F}^{k'}_{(l-1)}+\frac{\hbar}{i}\delta_{k'l}\boldsymbol{F}^{k}_{(l)}-\frac{\hbar}{i}\delta_{k',l-1}\boldsymbol{F}^{k}_{(l-1)} \ , \tag{12f}$$

where $\hbar$ is the Plank constant divided by 2π and δ_{kl} are Kronecker's δ. Thus the constants of motion given by Eq. (11) construct a Lie algebra, whose structure constants can be shown to be the same as those of the Lie algebra composed of operators of infinitesimal transformation of n-dimensional unitary unimodular group.

The generators of n-dimensional unitary unimodular group are the following :

$$\boldsymbol{M}^1_{(2)} = \left(\begin{array}{cc:c} 0 & -i & \\ i & 0 & \boldsymbol{0} \\ \hdashline \multicolumn{2}{c:}{\boldsymbol{0}} & \boldsymbol{0} \end{array}\right) ,$$

$$\boldsymbol{M}^1_{(3)} = \left(\begin{array}{ccc:c} 0 & 0 & -i & \\ 0 & 0 & 0 & \boldsymbol{0} \\ i & 0 & 0 & \\ \hdashline \multicolumn{3}{c:}{\boldsymbol{0}} & \boldsymbol{0} \end{array}\right) ,$$

$$
\left.\begin{aligned}
\boldsymbol{M}^{2}_{(3)} &= \left(\begin{array}{ccc:c} 0 & 0 & 0 & \\ 0 & 0 & -i & \mathbf{O} \\ 0 & i & 0 & \\ \hdashline & \mathbf{O} & & \mathbf{O} \end{array}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\boldsymbol{M}^{n-1}_{(n)} &= \left(\begin{array}{c:cc} \mathbf{O} & & \mathbf{O} \\ \hdashline \mathbf{O} & 0 & -i \\ & i & 0 \end{array}\right),
\end{aligned}\right\} \qquad \text{(13a)}
$$

$$
\left.\begin{aligned}
\boldsymbol{N}^{1}_{(2)} &= \left(\begin{array}{cc:c} 0 & 1 & \mathbf{O} \\ 1 & 0 & \\ \hdashline & \mathbf{O} & \mathbf{O} \end{array}\right), \\
\boldsymbol{N}^{1}_{(3)} &= \left(\begin{array}{ccc:c} 0 & 0 & 1 & \\ 0 & 0 & 0 & \mathbf{O} \\ 1 & 0 & 0 & \\ \hdashline & \mathbf{O} & & \mathbf{O} \end{array}\right), \\
\boldsymbol{N}^{2}_{(3)} &= \left(\begin{array}{ccc:c} 0 & 0 & 0 & \\ 0 & 0 & 1 & \mathbf{O} \\ 0 & 1 & 0 & \\ \hdashline & \mathbf{O} & & \mathbf{O} \end{array}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\boldsymbol{N}^{n-1}_{(n)} &= \left(\begin{array}{c:cc} \mathbf{O} & & \mathbf{O} \\ \hdashline \mathbf{O} & 0 & 1 \\ & 1 & 0 \end{array}\right),
\end{aligned}\right\} \qquad \text{(13b)}
$$

$$
\left.\begin{aligned}
\boldsymbol{L}_{(2)} &= \left(\begin{array}{cc:c} 1 & 0 & \mathbf{O} \\ 0 & -1 & \\ \hdashline & \mathbf{O} & \mathbf{O} \end{array}\right), \\
\boldsymbol{L}_{(3)} &= \left(\begin{array}{ccc:c} 0 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{O} \\ 0 & 0 & -1 & \\ \hdashline & \mathbf{O} & & \mathbf{O} \end{array}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\boldsymbol{L}_{(n)} &= \left(\begin{array}{c:cc} \mathbf{O} & & \mathbf{O} \\ \hdashline \mathbf{O} & 1 & 0 \\ & 0 & -1 \end{array}\right),
\end{aligned}\right\} \qquad \text{(13c)}
$$

In fact, the Lie algebra of the preceding characteristic matrices of n-dimensional unitary unimodular group has the same structure constants except $\hbar$ when the following correspondence has been made :

$$\begin{aligned} &\boldsymbol{F}^{k'}_{(k)} \longleftrightarrow \boldsymbol{M}^{k'}_{(k)} \\ &\boldsymbol{G}^{k'}_{(k)} \longleftrightarrow \boldsymbol{N}^{k'}_{(k)} \\ &\boldsymbol{E}_{(k)} \longleftrightarrow \boldsymbol{L}_{(k)} \end{aligned} \tag{14}$$

IV. CONCLUDING REMARKS

The generalization of our consideration in the preceding sections to relativistic case can be easily made. The wave equation of n-dimensional relativistic harmonic oscillater is invariant under an n-dimensional special complex Lorentz group with real invariant quadratic form. In four-dimensional case, we have four-dimensional special complex Lorentz group as the symmetry group, whose infinitesimal operators can be transformed into Cartan's canonical form[1] by taking appropriate linear combinations :

$$\begin{aligned} &[\boldsymbol{H}_i, \boldsymbol{H}_j] = 0 , && i, j = 1, 2, 3, \\ &[\boldsymbol{H}_i, \boldsymbol{E}_\alpha] = r_i(\alpha)\boldsymbol{E}_\alpha , && \alpha = 1, \cdots\cdots, 6, \\ &[\boldsymbol{E}_\alpha, \boldsymbol{E}_{-\alpha}] = r^i(\alpha)\boldsymbol{H}_i , && \\ &[\boldsymbol{E}_\alpha, \boldsymbol{E}_\beta] = N_{\alpha\beta}\boldsymbol{E}_{\alpha+\beta} , && \alpha \neq -\beta. \end{aligned} \tag{15}$$

This group has the Lorentz group and $SU(3)$ as its subgroups, and this property will be used in a treatment of the internal structure of elementary particles from the stand point of the non-local field theory[2),3)]. In other words, the theory of internal structure of elementary particles has been studied by many authors quite independently of the Lorentz group, but we shall extend later the non-local field theory to connect the invariant quantum numbers of internal symmectry group such as isospin and hypercharge, and the invariant quantum numbers of the Lorentz group such as mass, spin, and parity.

REFERENCES

(1) E. Cartan, "Sur la Structure des Groupes de Transformations Finis et Continus", These, Paris, (1894) ; B. L. van der Waerden, *Math. Zeits.*, **37**, 446 (1933) ; L. Pontrjagin, "Topological Groups", Princeton University Press, (1958) ; R. E. Behrends, J. Dreitlein, C. Fronsdal, and W. Lee, *Rev. Mod. Phys.*, **34**, 1 (1962).

(2) H. Yukawa, *Phys. Rev.*, **77**, 219 (1950) ; **80**, 1047 (1950) ; **91**, 415,416 (1953).

(3) H. Yukawa, *Prog. Theor. Phys.*, **16**, 688 (1956) ; *Rev. Mod. Phys.*, **29**, 213 (1957) ; *Prog. Theor. Phys.*, **31**, 1167 (1964) ; Y. Katayama and H. Yukawa, *Prog. Theor. Phys.*, **32**, 366 (1964).