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Three-Particle Relativistic Kinematics

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For reactions leading to three particles in the final state, five independent variables are assigned to the polar and azimuthal angles of two particles and the kinetic energy of one of them. Formulas are given for calculations of other variables from these five. Formulas are also given for transformations of polar and azimuthal angles from the laboratory system to the rest frames of two-particle systems. A FORTRAN programme using these formulas is included in an appendix.

KEY WORDS Three-particle kinematics / Relativistic formulas / FORTRAN programme /

I. INTRODUCTION

For three-particle reactions of the form \( a + b \rightarrow 1 + 2 + 3 \), five independent variables are required to define the final state completely. In experiments of two-particle correlation measurements, two detectors are set at angles \( (\theta_1, \phi_1) \) and \( (\theta_2, \phi_2) \) respectively and kinetic energies \( T_1 \) and \( T_2 \) of particles 1 and 2 are measured in coincidence. Therefore, quantities \( T_1, \theta_1, \phi_1, \theta_2, \) and \( \phi_2 \) are taken as five independent variables and differential cross sections \( d\sigma/dT_1d\Omega_1d\Omega_2 \) are measured as functions of these variables. Kinetic energy \( T_3 \), an extra variable, can be used to identify true events of the reaction.

Coincident energy spectra are characterized by relative kinetic energies between two final-state particles. These energies are calculated from the five independent variables. For the purpose of doing it, nonrelativistic formulas are summarized in Ref. 1. Since two-particle correlation experiments are performed frequently at intermediate incident energies, relativistic calculations are done in this report. Moreover the following matters are examined. First, quasi-free scattering proceeds in two ways. One is that the target \( b \) consists of two particles and one of them collides with the projectile \( a \) with leaving the rest particle as a spectator. The other is that the projectile \( a \) consists of two particles and one of them collides with the target \( b \). In the former, quasi-free scattering occurs near the energy corresponding to the spectator kinetic energy of zero in the target system and in the latter, of zero in the beam system. These two processes of quasi-free scattering are expected to occur especially when particles \( a \) and \( b \) are identical. Then, kinetic energies \( T_1 \) of particles \( i \) are calculated in both the target and

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beam systems. Secondly, in angular-correlation experiments, differential cross sections \( \frac{d^2\sigma}{d\Omega_1 d\Omega_2} \) are measured as functions of angles \( (\theta_2, \phi_2) \) of particle 2 if one assigns particle 1 to the particle emitted from the production process of the \((23)\) system and particle 2 to the succeeding decaying one. Angular correlation functions are obtained through the transformation of differential cross sections to the ones in the rest frame of \((23)\) system. Several times, according to physical or experimental requirements, detector 2 is placed off the reaction plane which is defined by the direction of detector 1 and that of incident beam. Then transformation equations for the azimuthal angle \( \phi_2 \) as well as for the polar angle \( \theta_2 \) are required.

II. CALCULATION FORMULAS

1. Reference Systems

Four types of reference system \(^1\) are used to describe kinetic motions of particles, that is, laboratory system (LS), overall center-of-momentum system (CMS), beam system (BS) and rest frames of two-particle system (R12, R23, and R31). Target b is assumed to rest in the LS and therefore the target system is identical with the LS. The CMS quantities are denoted by an asterisk and the Rij quantities by an index Rij. An index L for the LS quantities is omitted for simplicity.

The z-axis of the LS is defined as the momentum of projectile and the z-axis in the CMS is defined as the direction parallel to the LS one. The zx-plane \( (\phi = 0) \) can be defined arbitrary.

2. Kinetic Energies

Quantities to describe the motion of particle i are as follows: the total energy \( E_i \), the momentum \( P_i \), the rest mass \( m_i \), the kinetic energy \( T_i \), and the velocity \( v_i \). The momentum vector is expressed in terms of its absolute value \( P_i \), the polar angle \( \theta_i \), and the azimuthal angle \( \phi_i \) in the polar coordinate system. Among these quantities exist the following relations:

\[
E_i^2 = P_i^2 + m_i^2, \\
E_i = T_i + m_i, \\
P_i^2 = T_i^2 + 2m_iT_i, \\
v_i = P_i/E_i. 
\]

The similar relations hold for a two-particle system consisting of particles i and j with definition of its energy \( E_{ij} \), momentum \( P_{ij} \) and invariant mass \( M_{ij} \) as

\[
E_{ij} = E_i + E_j, \\
P_{ij} = P_i + P_j, \\
M_{ij} = (E_i + E_j)^2 - (P_i + P_j)^2. 
\]

For the total system its energy \( E_o \), momentum \( P_o \) and invariant mass \( M_o \) are calculated from the quantities of the initial state with assigning the particle a to the projectile and the particle b to the target. They are written as

\[
E_o = E_a + E_b = T_a + m_a + m_b, \\
P_o = P_a + P_b = P_a, 
\]
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\[ P_a = \left( T_a + 2m_a T_a \right)^{1/2}, \quad (10) \]

\[ M_a^2 = T_a^2 - P_a^2 = (m_a + m_b)^2 - 2m_a m_b. \quad (11) \]

If, for the particle 1, the kinetic energy \( T_1 \) and the angles \( (\theta_1, \phi_1) \) are known in the LS, the (23) system consisting of particles 2 and 3 is uniquely determined owing to the energy-momentum conservation equations

\[ E_0 - E_1 = E_2 + E_3, \quad (12) \]

\[ P_0 - P_1 = P_2 + P_3. \quad (13) \]

Then kinematics for two-particle reactions can be applied to solve \( P_2 \) as a function of the angles \( (\theta_2, \phi_2) \). The calculations are performed as follows. First, in the R23, the momentum and the energy of particle 2 are independent of the angles and are expressed in terms of the invariant mass \( M_{23} \):

\[ P_{2R23} = \sqrt{\frac{1}{2} \left( M_{23}^2 - m_2^2 - m_3^2 \right) / (2M_{23})}, \quad (14) \]

\[ E_{2R23} = \frac{M_{23}^2 + m_2^2 - m_3^2}{2M_{23}}, \quad (15) \]

where

\[ M_{23}^2 = \left( E_2 + E_3 \right)^2 \quad (P_2 + P_3)^2 \]

\[ = \left( E_0 - E_1 \right)^2 \quad (P_0 - P_1)^2 \]

\[ = M_0^2 + 2E_0E_1 - 2P_0P_1 \cos \theta_1. \quad (16) \]

and the function \( \lambda(x, y, z) \) is defined as

\[ \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx. \quad (17) \]

Secondly, the momentum of particle 2 in the LS is calculated from the momentum and energy in the R23 through the Lorentz transformation between these systems. The transformation is done along the \( z' \)-axis which is defined as \( P_{23} \) in the LS. The velocity \( v_{23} \) of the R23 in the LS and the associated Lorentz factor \( \gamma_{23} \) are given by

\[ v_{23} = P_{23}/E_{23}, \quad (18) \]

\[ \gamma_{23} = E_{23}/M_{23}, \quad (19) \]

\[ \gamma_{23} v_{23} = P_{23}/M_{23}, \quad (20) \]

where

\[ E_{23} = E_0 - E_1, \quad (21) \]

\[ P_{23} = |P_0 - P_1| = (P_2^2 + P_3^2 - 2P_0 P_1 \cos \theta_1)^{1/2}. \quad (22) \]

The angle \( \theta_{23} \) of the (23) system with respect to the \( z \)-axis is given through the momentum conservation, Eq. (13), as follows:

\[ \sin \theta_{23} = P_3 \sin \theta_1 / P_{23}, \quad (23) \]

\[ \cos \theta_{23} = (P_0 - P_1 \cos \theta_1) / P_{23}. \quad (24) \]

Now, the Lorentz transformation equation solved for \( P_2 \) is written as

\[ E_2^{(23)} = -\gamma_{23} v_{23} P_2 \cos \theta_{23} + \gamma_{23} E_4 \]

\[ = -\gamma_{23} v_{23} P_2 \cos \theta_2 + \gamma_{23} (P_4^2 + m_4^2)^{1/2}. \quad (25) \]

The angle \( \theta_2' \) is measured with respect to the \( z' \)-axis and \( \cos \theta_2' \) is expressed by

\[ \cos \theta_2' = \cos \theta_2 \cos \theta_{23} - \sin \theta_2 \sin \theta_{23} \sin \theta_{23}, \quad (26) \]

where
The \( z'x' \)-plane (\( \phi' = 0 \)) is defined as the plane including the \( z \)-axis (Fig. 1). Finally, the solution \( P_2 \) is given by

\[
P_2^* = P_2^{R23} (B \pm \sqrt{D}) / A,
\]

with

\[
D = \gamma_{23}^2 (1 - g^2) + g^2 (\gamma_{23}/\gamma_{23}^{R23})^2 \cos^2 \theta',
\]

\[
A = \gamma_{23} (1 - v_{23}^2 \cos^2 \theta'),
\]

\[
B = g \cos \theta',
\]

\[
g = v_{23}/v_{23}^{R23},
\]

where \( v_{23}^{R23} \) is the velocity of particle 2 in the \( R23 \) and \( \gamma_{23}^{R23} \) is the associated Lorentz factor and they are given by

\[
v_{23}^{R23} = P_{23}^{R23}/E_{23}^{R23},
\]

\[
\gamma_{23}^{R23} = E_{23}^{R23}/m_2.
\]

Concerning the existence of solutions \( P_2^* \), one has two cases depending on the relative magnitudes of \( v_{23} \) and \( v_{23}^{R23} \). (1) If \( 1 > g \), always \( D > 0 \) as found from Eq. (29) and, however, \( D^{1/2} > |B| \) and consequently \( P_2^- < 0 \). The latter relation is found from the equation

\[
B^2 - D = (g^2 - 1) \gamma_{23}^2 (1 - v_{23}^2 \cos^2 \theta').
\]

Then, one has only the solution \( P_2^+ \). (2) If \( 1 \leq g \), the signs of \( D \) and \( \cos \theta' \) should be examined. (i) If \( D \geq 0 \) and \( \cos \theta' \geq 0 \), one has two solutions \( P_2^+ \). (ii) If \( D \geq 0 \) but \( \cos \theta' < 0 \), one has no physical solution because \( P_2^* \) are always found to be negative as seen from Eqs. (31) and (35). (iii) If \( D < 0 \), regardless of the sign of \( \cos \theta' \), one has no solution. The energy \( E_2 \) is given by

\[
E_2 = (P_2^2 + m_2^2)^{1/2}.
\]

For the particle 3, the energy \( E_3 \) and momentum \( P_3 \) are calculated from the quantities for the particles 1 and 2 through the energy and momentum conservation equations

\[
E_3 = E_4 - E_1 - E_2.
\]
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\[ \mathbf{P}_3 = \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_2. \]  

(38)

From the latter equations, \( P_3 \) is derived as follows.

\[ P_3 = (P_1^2 + P_1^2 + P_2^2 - 2P_1P_2\cos\theta_1 - 2P_1P_2\cos\theta_2 + 2P_1P_2\cos\theta_{1-2})^{1/2}. \]  

(39)

The kinetic energy \( T_i \) of particle \( i \) is given by

\[ T_i = E_i - m_i. \]  

(40)

The kinetic energies of the relative motions between the particles \( j \) and \( k \) and between the particle \( i \) and the \((jk)\) system are given by

\[ T_{j-k} = M_{jk} - m_j - m_k \]  

(41)

and

\[ T_{i-jk} = M_j - m_i - M_{jk} \]  

(42)

respectively. \( M_{jk} \) are obtained by cyclic permutations of indexes in Eq. (16).

Once the energy \( E_i \) is obtained in the LS, the energy \( E_i^B \) in the BS can be calculated through the Lorentz transformation between these systems with \( \gamma_a = E_a / m_a \) and \( \gamma_a v_a = P_a / m_a \) and is given by

\[ E_i^B = (E_a E_i - P_a P_i \cos\theta_i) / m_a. \]  

(43)

3. Polar and Azimuthal Angles

In order to evaluate an azimuthal angle \( \phi \), both values of \(\sin\phi \) and \(\cos\phi \) are needed and on the contrary a polar angle \( \theta \) is uniquely evaluated from a value of \( \cos\theta \). However, in practical calculations with a computer, it is frequently convenient to use both values \(\sin\theta \) and \(\cos\theta \) for evaluating \( \theta \). In the following both formulas for \(\sin\theta \) and \(\cos\theta \) are given.

The LS angles \((\theta_3, \phi_3)\) of particle 3 are calculated from the quantities for the particles 1 and 2 through the momentum conservation, Eq. (38), and are given by

\[ \sin\theta_3 = \left( P_1 \sin\theta_1 + P_2 \sin\theta_2 + 2P_1 P_2 \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1) \right) / P_3, \]  

(44)

\[ \cos\theta_3 = \left( P_1 - P_1 \cos\theta_1 - P_2 \cos\theta_2 \right) / P_3, \]  

(45)

\[ \sin\phi_3 = \left( - P_1 \sin\theta_1 \sin\phi_1 - P_2 \sin\theta_2 \sin\phi_2 \right) / (P_3 \sin\theta_3), \]  

(46)

\[ \cos\phi_3 = \left( - P_1 \sin\theta_1 \cos\phi_1 - P_2 \sin\theta_2 \cos\phi_2 \right) / (P_3 \sin\theta_3). \]  

(47)

Practically the factor \( P_3 \sin\theta_3 \) in Eqs. (46) and (47) is not needed because \( \phi_3 \) is evaluated through the ratio \(\tan\phi_3 = \sin\phi_3 / \cos\phi_3 \) associated with an examination of the sign of \(\sin\phi_3 \) and \(\cos\phi_3 \) \((\sin\phi_3 \geq 0)\). For calculation of \( \theta_3 \) through Eq. (45) only, the value of \( P_3 \) is given by Eq. (39).

The CMS angles \((\theta_1^*, \phi_1^*)\) are calculated from the LS ones through the Lorentz transformation equations with \( \gamma_0 = E_0 / M_0 \) and \( \gamma_0 v_0 = P_0 / M_0 \) which are evaluated with Eqs. (8), (10) and (11), and are given by

\[ \sin\theta_1^* = P_3 \sin\theta_1 / P_1^*, \]  

(48)

\[ \cos\theta_1^* = (E_0 P_1 \cos\theta_1 - P_3 E_1) / (M_0 P_1^*), \]  

(49)

\[ \phi_1^* = \phi_1. \]  

(50)

If one calculates \( \theta_1^* \) through Eq. (49) only, the value of \( P_1^* \) is needed and is given by

\[ P_1^* = \frac{M_1^2}{2M_0^2} \left( M_{jk}^2, M_4^2, m_i^2 \right) / (2M_0). \]  

(51)
In the Rjk, the z-axis is defined as the direction opposite to the momentum of particle i, that is, \(-\mathbf{p}_i^{\text{Rjk}}\). Then the polar angle \(\theta_i^{Rjk}\) equals \(\pi - \theta_{ij}^{Rjk}\) and is expressed in terms of the invariant mass \(M_{ij}\) and the energies and momenta of particles i and j in the Rjk as follows:

\[
\cos \theta_j^{Rjk} = \frac{(m_2^2 - m_1^2)}{2E_i^RikE_j^{Rjk}} \left( 2P_i^{Rjk}P_j^{Rjk} \right),
\]

where

\[
E_i^{Rjk} = \frac{(M^2 - M_{jk}^2 - m_i^2)}{2M_{jk}},
\]

\[
P_{1Rik} = \frac{a_{1/2}(M_{12}^2, M_{jk}^2, m_i^2)}{2M_{jk}}.
\]

and \(E_i^{Rjk}\) and \(P_j^{Rjk}\) are obtained by cyclic permutations of indexes in Eqs. (14) and (15). The Rjk azimuthal angle \(\phi_j^{Rjk}\) of particle j is calculated from the LS angles through three steps of transformations. The first is the Lorentz transformation between the LS and the CMS along the z-axis. The azimuthal angle is invariant, that is, \(\phi_1^* = \phi_1^*\). The second is a rotation of coordinate system in the CMS which transform the z-axis to the \(z''\)-axis. The latter axis is defined as \(\mathbf{P}_{jk}^* = -\mathbf{P}_j^*\) and the rotation angle equals to the angle \(\theta_{jk}^* = \pi - \theta_{ij}^*\). By using formulas of spherical trigonometry, \(\phi_1^*\) is transformed to \(\phi_j^{R''}\) through the following equations:

\[
\sin \theta_j^{R''} \sin \phi_j^{R''} = \sin \theta_j^* \sin (\phi_j^* - \phi_1^*),
\]

\[
\sin \theta_j^{R''} \cos \phi_j^{R''} = \cos \theta_j^* \sin \phi_j^* - \sin \theta_j^* \cos \phi_j^* \cos (\phi_1^* - \phi_1^*).
\]

The \(z''\)-plane (\(\phi'' = 0\)) is defined as the plane including the z-axis (Fig. 2). The last is a Lorentz transformation between the CMS and the Rjk along the \(z''\)-axis and the equations are as follows:

\[
P_j^{Rjk} \sin \theta_j^{Rjk} = P_j^* \sin \theta_j^{R''},
\]

\[
\phi_j^{Rjk} = \phi_j^{R''},
\]

By substitution of Eqs. (48), (49), (50), (57) and (58) into Eqs. (55) and (56), the following equations are obtained:

\[
\sin \phi_j^{Rjk} = P_j^* \sin \theta_j^* \sin (\phi_1^* - \phi_1^*) / (P_j^{Rjk} \sin \theta_j^{Rjk}),
\]

\[
\cos \phi_j^{Rjk} = [(E_2^jP_j^* \cos \theta_j^* - P_2^jE_1^j) P_j^* \sin \theta_j^* - (E_2^jP_j^* \cos \theta_j^* - P_2^jE_1^j) P_j^* \sin \theta_j^* \cos (\phi_1^* - \phi_1^*)] \\
\times (M_{12}^2)^{-1}(P_j^{Rjk} \sin \theta_j^{Rjk})^{-1}.
\]

Concerning to the factor \((P_j^{Rjk} \sin \theta_j^{Rjk})\), is noted the same fact as mentioned about Eqs. (46) and (47).

Fig. 2. Angle coordinate in the center-of-momentum system.
4. Phase Space Factor

If the differential cross section is written in a form as follows:

\[ \frac{d^5\sigma}{dT_1d\Omega_1d\Omega_2} = \frac{(2\pi/\hbar_o)^4}{M^2} \rho_1(E_i) |M|^2, \]  

(61)

then the phase space factor \( \rho_1(E_i) \) is given by

\[ \rho_1(E_i) = \frac{(2\pi\hbar)^{-4}}{P_1P_2E_1E_2E_3(E_2+E_3-E_2-P_1-P_2)^2} \]  

(62)

This equation corresponds to the following definition:

\[ \rho_1(E_i) dT_2d\Omega_1d\Omega_5 = \frac{(2\pi\hbar)^{-6}}{Sd^3P_1d^3P_2d^3P_3} \]  

(63)

with the integrations over the variables \( P_3, \theta_3, \phi_3, \) and \( P_2 \). This expression is a non-invariant form and differs from a Lorentz-invariant one by inclusion of a factor of \( 8E_1E_2E_3 \). In the R23, the phase space factor is calculated in the form:

\[ \rho_1(E_{R23}) = \frac{(2\pi\hbar)^{-4}}{P_1^{R23}P_2^{R23}E_1^{R23}E_2^{R23}E_3^{R23}/M_{23}}. \]  

(64)

Then the Jacobian \( \partial(T_1, \Omega_1, \Omega_2)/\partial(T_1^{R23}, \Omega_1^{R23}, \Omega_2^{R23}) \) for the transformation of the differential cross section \( d\sigma/dT_1d\Omega_1d\Omega_2 \) from the LS to the R23 is given by

\[ \partial(T_1, \Omega_1, \Omega_2)/\partial(T_1^{R23}, \Omega_1^{R23}, \Omega_2^{R23}) = \rho_1(E_{R23})/\rho_1(E_i). \]  

(65)

III. PROGRAMME

A FORTRAN programme is given in an appendix. The input quantities are the masses of five particles participating in the reaction: \( m_a, m_b, m_1, m_2, \) and \( m_3 \) (in AMU), the kinetic energy of the projectile in the laboratory system: \( T \) (in MeV) and the polar and azimuthal angles of the particles 1 and 2 in the laboratory system: \( \theta_1, \phi_1, \) and \( \phi_2 \) (in degrees). In calculations the kinetic energies of particle 1: \( T_1 \) (in MeV) are given successively by a starting value plus an increment value multiplied by integers. These values are also given in the input data. In the programme is put a restriction that a given value of \( T_1 \) should satisfy a limit \( M_{23} - m_2 - m_3 > 0 \), before the examinations mentioned in the preceding section. The output quantities are represented in a matrix form as used in Ref. 1.

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The calculations were made with FACOM 230–48 computer at the Institute for Chemical Research of Kyoto University.

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