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Valuing Corporate Growth Using Real Options

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The corporate DCF model is often criticized for its limitations in valuing investments with significant multi-period growth opportunities. The current paper offers a practical methodology that employs recombining binomial trees and real option techniques to value companies pursuing organic growth strategies in case of demand uncertainties. A specially constructed recombining tree is adapted to valuations of multiple American-type compound growth options. Computational complexity is considerably reduced; this makes these solutions feasible for implementation in business practice and empirical testing.

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1. Introduction

Traditional Discounted Cash Flow (DCF) analysis based on shareholder value maximization is widely used in corporate budgeting practice as a criterion for evaluating projects and companies. Despite its popularity, the corporate DCF model has inherent limitations in valuing investments with significant flexibility options (Mun [2004]) and is often criticized for undervaluation of investment opportunities that may lead to wrong decisions, underinvestment problems, and potential loss of competitive position. Instead, alternative methods have been developed to effectively capture management flexibility to respond to future uncertain developments, namely: (1) the revised (dynamic) DCF model, (2) decision analysis, and (3) contingent claim analysis (real options). All methods value risky alternatives using backward recursion; however, the probabilities and discount rates used in each type of analysis differ, and hence principally the results may substantially differ depending on the employed approach. Option theory offers a significant improvement from other pricing methods because it adopts perspectives based on market equilibrium and determines the value of investment opportunity using the values of other traded assets. The famous classic option model is that of Black and Scholes [1973]. Cox, Ross, and Rubinstein [1979] later developed a binomial option pricing method that converges in the limit to Black and Scholes' [1973] prominent model. Subsequently, based on Myers' [1977] seminal idea that one can view a firm's discretionary investments as a call option on real assets, the real option approach (McDonald and Siegel [1986], Dixit and Pindyck [1994], Trigeorgis [1996]) extended the classic option theory to apply to partially or completely irreversible investments with uncertain future payoffs and flexibility around investment timing. Recent developments have extended the real option analysis in a broader context of corporate strategy with a focus on portfolio of options, and studied irreversible investments in building strategic capabilities such as R&D, mergers and acquisitions, and development of new technologies. (Kogut [1991], Trigeorgis [1993], Campa [1994], Copeland and Antikarov [2001], Mun [2002], Folta and O'Brien [2004], Smit and Trigeorgis [2004], Schwartz and Trigeorgis [2004]). A number of recent papers in the strategic management literature have investigated the influence of uncertainty on the value of specific growth options (Kogut [1991], Berger et al. [1996], Hall [2000], Kogut and Kulatilaka [2001], Levitas and Chi [2001], McGrath and Nerkar [2004]). For instance, Kogut [1991] and Kogut and Kulatilaka [2001] explored the option value associated with investing in platform capabilities based on which a firm may better respond to uncertain changes in external environments. Several empirical studies have also examined how financial markets value real growth options possessed by a firm. Kester [1984] measured the value of growth options as the difference between a firm's current market value and the value of its assets-in-place and estimated that the value of growth options is more than half of the market value of equity for many firms and as much as 70 to 80% for volatile industries. Following the work of Kester, other authors (Danbolt, Hall and Jones [2002], Tong and Reuer [2006]) also suggested that the value of growth options can represent a sizable portion of a firm's market value; however, even after adopting the real option framework none of these studies tested an analytic model of multiple growth options. Pindyck [1988] employed numerical simulation and suggested that if demand volatility exceeds 0.2 the value of growth options occupies more than half of a firm's total value.

Most of the above mentioned real option literature provides a compelling framework for capturing management flexibility under uncertainty and suggests implications for its quantitative valuation in case of individual investment projects; however, it is important to recognize several pitfalls in practical applications of the theory when valuing corporations (Lander and Pinches [1998], Kellogg and Charnes [2000], Philippe [2005], Triantis [2005]). Several studies have shown that the use of real options remains limited in business practice. Graham and Harvey [2001] and Triantis [2001] found that real option techniques are often being used only as a strategic qualitative tool rather than an instrument for exact quantitative estimation. According to another investigation conducted by the Japanese Ministry of Economy Trade and Industry (METI [2003–04]), no corporate finance officers were actually using real option techniques for investment appraisal in

Japan in 2003/04. Further information about real options was published in *Economist* in April, 2000: 46% of American companies that tried real option techniques gave up because those methods were too complicated to be used in practical evaluation of real investments.

Regarding the application of quantitative models to growth reinvestments and corporate valuations, the complexities of real-world investments create difficulties in specifying a proper mathematical model for capturing the sequential interdependence among multiple growth reinvestments over time. Traditionally, real option analysis of multiple growth options focuses on multi-stage investment opportunities. In this framework, first-stage investment is undertaken to create growth options, whereas second-stage investment may occur in order to exercise this growth option (Copeland and Tufano [2004], Panayi and Trigeorgis [1998], Luehrman [1998]). In case of multiple growth opportunities, this model setting is generally limited to European-type options. In contrast, practical real option problems usually involve American-type options with several state variables and path-dependent pricing dynamics which are extremely difficult to solve analytically. With multiple American-type growth options that give owners the right to exchange one asset for another at any time prior to expiration, complexity accompanying real investments can make it difficult to solve the problem computationally. The few attempts (Smit and Trigeorgis [2004], Copeland and Antikarov [2005], Brandao et al. [2005]) to model a portfolio containing a large number of growth options have utilized non-recombining binomial trees and face the problem of exponential complexity, which makes these solutions difficult to implement in practice. Hence, testing propositions derived from applications of real option theory to research in strategic management presents a challenge since the value of growth option portfolios embedded in a company's value cannot be directly measured.

The challenges related to real option applications motivated the author to develop a computationally efficient valuation technique capable of directly measuring the value of a firm with regard to the value of its portfolio of multiple growth options. Specifically, to overcome the gap between real option theory and corporate finance practice, the current paper utilizes a specially constructed lattice with recombining properties and applies a Marketed Asset Disclaimer concept as in Copeland and Antikarov [2001] to properly value interacting multiple options embedded in corporate growth. The results show that for high/moderate growth firms operating in a highly dynamic business environment, the value of a growth option portfolio is significantly undervalued by the traditional corporate Discounted Cash Flow (DCF) model. This paper offers some contributions to existing corporate finance literature and practitioners. First, it builds a computationally effective model of a firm's flexibility that explicitly includes a potentially very large number of compound American-type growth options. Second, it validates an arbitrage-free valuation tool as an alternative to the traditional corporate DCF model. The author believes this tool has practical applications.

The following section provides a detailed description of the model for arbitrage free valuation of corporations with careful attention paid to type of exogenous uncertainty and treatment of risk. The analysis will display the problem of choosing the optimal reinvestment strategy in a discrete-time compound option framework in order to obtain numerical arbitrage-free solutions of a corporate value with strategic flexibility. Simple numerical examples, discussed in Section II, illustrate the results. Section III concludes the discussion.

2. An Outline of the Model

The strategic interest in any real option lies primarily in its ability to truncate the distribution of unfavorable moves of the uncertain environment in the value of investment. This section investigates the effects of introducing multiple growth options to the value of corporations, and assesses the optimal operating rules for companies that possess many American-type growth options.

Building on Myers' [1977] original concept of treating investment opportunities as "growth options," traditional corporate finance theory considers a firm's stand-alone value to comprise two components: the value of its assets in place and the value of growth opportunities (Damodaran [2002]; Copeland, Koller, Murrin [2000]). Corporate DCF values both components by dividing the infinite life of the firm into two periods: (1) a period of abnormal growth during which the company enjoys extra rents due to competitive advantage (e.g., economies of scale/scope, absolute cost advantage, or product differentiation), and/or high entry/exit barriers in an industry (e.g., capital intensity, minimum efficient scale, or advertising intensity); and (2) a period of stable growth during which the state of competitive equilibrium has been achieved and the firm cannot earn excess return given current conditions such as technology or customer tastes, thus earning only market average return on its capital.

The DCF model further assumes that a firm's management executes growth reinvestments in each period (Copeland, Weston and Shastri [2005], Damodaran [2001]). In contrast, real option theory treats the reinvestment pattern in a different manner since each reinvestment can be considered as exercising a "growth option" similar to a financial option with the underlying asset being "real" cash flows. Real option theory maintains that the expansion investments made by corporations are often sunk in the form of firm- or market-specific production/distribution infrastructure, brand enhancement, or knowledge base, and executing such (partly) irreversible reinvestment is the right and not the obligation of management (Dixit and Pindyck [1994]); therefore, these rights should be exercised only if the market turns out to be favorable. Hence, before making a final investment decision a firm faces a critical trade-off between flexibility and commitment. If a firm makes investments to develop a market it will actually install additional expansion options that may set the path for future corporate development. Conversely, despite the fact that earlier investments may induce immediate cash flow and provide a platform for future follow-on investments, they may "kill" the value of the option to "wait and see." As a result, a traditional cut-off rule like a positive net present value (NPV) is in itself an insufficient criterion for the viability of individual investment; instead of accepting all projects with positive NPVs, it is necessary to choose the NPV-maximizing "wait-reinvest" strategy, recognizing that future decisions and appropriate discount rates will be conditioned on the realizations of stochastic variables and previously executed decisions. Thus, the apparatus of real option theory enables a formal analysis of these sequential discretionary reinvestment decisions.

Several issues must be considered in the direct practical implementation of real option theory to corporate growth valuation. Traditionally, real option models treat the value of a project (or a company) itself as the exogenous stochastic variable. The underlying assumption of modeling the stochastic process of the value of the whole project lies in Samuelson's [1965] proof that multiple sources of uncertainty driving the project's value can be eventually combined into a single random walk. Although this is theoretically appealing because it greatly reduces the dimensional complexity of the valuation model to a single uncertainty, this approach also aggregates information about individual risk factors into a single number of dispersion of a company's value; hence, based on the current models it is very difficult to provide clear threshold values of individual uncertainty drivers that trigger investments. Moreover, Smith [2005] showed that procedures for calculating integrated volatility may overstate the actual uncertainty in the cash flows and thus overstate the value of many options. The presence of multiple real options embedded in corporate growth also influences the company's value in a nonlinear way. If a company has only one growth option, it is possible to characterize the company's value with a single growth option as following, suppose, a lognormal process. However, if the company adds two or more subsequent growth options, the company's value process results in being nonlinearly transformed from the original and no longer follows the same process. Hence, due to truncation of the distribution of option pay-off, it is very difficult to consistently estimate the volatility of underlying cash flows of individual growth options. Moreover, it is not possible to separately identify the effects of other company-specific factors (e.g., tax shields) on the value of compound growth options. Based on these considerations, in the current model it is assumed that an individual output factor, rather then a company's value, is an exogenous stochastic variable. This uncertainty is then further incorporated into the fundamental variable of a company's value: the company's cash flows. These cash flows will therefore be the function of the driving exogenous uncertainty and represent the underlying stochastic process that drives the valuation model. In addition, the use of company cash flows provides a greater level of detail when modeling a firm's business processes and allows for accounting of non-linear effects of marginal income taxes, depreciation tax shields or capital structure.

Consider a corporation operating under endogenous uncertainty is faced with a problem of choosing optimal capital expansion policy, such as investments in new production lines, more efficient equipment, plant expansions or reengineering, or new production sites. Realistically, the firm cannot increase its capital capacity instantaneously. This means that some favorable market opportunities cannot be pursued due to the impossibility of expanding the production beyond a given threshold. Hence, instalment of additional capacity can be viewed as exercising a growth option with the exercise price equal to the investment costs and option value being represented by expected future revenues (cash flows) from operations, including possible follow-up expansions. The effect of increasing the capital capacity may also be interpreted as increasing not just future cash flow stream but also the availability of future options to further expand production in order to take advantage of potentially favorable demand changes.

Now consider a no-growth case in which the firm constantly produces and sells the same output S_0. Due to exogenous uncertainty, the future value of this output is also uncertain. A common assumption regarding the exogenous risk variable is that its current level incorporates all public information available at this time point, and that all future fluctuations will be purely random and, thus, modeled as a random walk. Specifically, the exogenous risk faced by a firm with a constant output is characterized as demand uncertainty, ${S^{nom}}_t$, which follows a continuous-time diffusion process of the form:

$$\frac{dS^{nom}}{S_t^{nom}} = \mu \cdot dt + \sigma \cdot dW \, (S_0^{nom} = S_0), \tag{1}$$

where dW is a standard Wiener process, $dW \in N(0, 1)$, μ is inflation rate in the firm's product market, and σ is volatility of demand changes. Next, for the general case in which the firm has the rights to expand capacity, assume that the life of distinctive competitive advantage during which the firm earns extra economic rents is T periods, after which shareholders obtain continuing value depending on a reinvestment policy during high growth periods[1]. To simplify the model, the share of variable cost in sales is assumed to be state-independent (although this assumption can be relaxed to incorporate cost uncertainty). Demand uncertainty is resolved and market opens at times $t=1, \ldots, T$. One traditional restriction is placed on the form of demand volatility σ. At any future time t, σ must be modeled as non-stochastic and known with certainty at the time of analysis. Let $S^{nom} = \{{S^{nom}}_1, {S^{nom}}_2, \ldots, {S^{nom}}_T\}$ denote the (finite) set of possible demand realizations. At each time t, a firm possesses all the information ω_t about past states of demand prior to this time. The firm's beliefs about future states based on information ω_t available at the current time t are captured by objective probabilities p. Given the state of information ω_t the expected value of firm's cash flows is written as $E(\ldots|\omega_t)$ or simply $E_t(\ldots)$.

Following Tong and Reuer [2006], who found that firm-specific effects and heterogeneity in the firm's proprietary options always dominate over industry

[1] See Copeland, Weston, Shastri [2005] for deviation of the formula for continuing value.

effects in the value of a firm's growth options, it is assumed that when considering a reinvestment strategy a firm is engaged only in a game with nature (i.e., exogenous environmental uncertainty) and agents formulate investment exercise strategies in isolation as a response to changes in uncertain market demand. Reinvestment decisions are made at the beginning of the period and uncertainty surrounding demand is fully resolved at the end of the period after the investment installation.

Let t be the valuation date and T the growth option expiration date which is bound by the length of abnormal growth stage for any growth option. To fit the general model of corporate flexibility, I characterize the firm that undertakes only replacement investment to maintain sales and postpones net expansion reinvestment as being in mode $m_t=1$ ("wait to reinvest"). Once a firm makes both a replacement investment and an additional net reinvestment to increase sales, the firm will be operating in mode $m_t=2$ ("reinvest"). By making a net reinvestment of $I^{new}(\xi)$, a firm will expand the scale of its operations by factor ξ. Here, the exercise price of growth option which is represented by net reinvestment costs of capital installation plus necessary increase in working capital $d\overline{WC}(\xi)$ is assumed to be non-stochastic and invariant to changes in the exogenous demand. At each state, a firm's management may execute either strategic decision (wait or reinvest), with each mode having its own free cash flow function $FCF^{m}{}_{t}$ for $m_t \in (1, 2)$. Operating in mode 1 entails no switching costs, hence $c_1=0$; switching from mode "wait" to mode "reinvest" has a cost in the form of net reinvestment and increase of working capital caused by sales growth, hence, $c_2=I^{new}(\xi)+d\overline{WC}(\xi)$. The cash flow in the interval $(t, t+\Delta t)$ depends on the value of stochastic variable $S^{nom}{}_{t}$ and on past and current decision modes m_t, hence $FCF^{m}{}_{t}=FCF(S^{nom}{}_{t}, m_t, m_{t-1}, \ldots, m_0, t)$ or simply $FCF(S^{nom}{}_{t}, m_t, t)$. After adjusting for inflation effect, the value of free cash flows from operations (excluding switching costs between modes) that are dedicated to mode m_t can be expressed as[2]:

$$FCF_t(S_t, m_t, t)=EBIT_t(S_t, m_t, t)\cdot(1-tax)+A'_t(m_t)-I_t^{replace}(m_t)-dWC'_t(m_t) \quad (2)$$

where:

$EBIT_t(S_t, m_t, t)-$ real earnings before interest and taxes which are calculated as:

$EBIT_t(S_t, m_t, t)=S_t(m_t, t)\cdot(1-VC)-A'_t(m_t)$

VC is share of variable costs in real sales S_t,

tax is marginal corporate tax rate applied to EBIT,

A'_t is adjusted real depreciation[3],

$I_t^{replace}$ is real replacement investment,

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[2] Here I maintain the capital budgeting representation of cash flows belonging to a firm as commonly accepted in corporate finance practice.

dWC'_t is adjusted change in working capital due to inflation[4].

At the end of a high-growth stage, future cash flows are represented by continuing value of the firm CV_t that can be calculated using a simplified formula approach (see the Gordon growth model in Copeland, Weston, and Shastri [2005]).

The valuation algorithm is fulfilled working backwards in a stochastic dynamic-programming fashion (Dreyfus [1965]) and is similar to numerical methods for valuing American call option on a stock. Generally, a closed form solution for a value of a company with multiple American growth options is not available; therefore, I solve the outlined optimization problem numerically using a recombining binomial-decision tree that specifies cash flows associated with management decisions at each uncertain state. The binomial model of Cox, Ross, and Rubinstein [1979] gives an accurate approximation of continuous Geometric Brownian motion of demand uncertainty with the added advantage of providing necessary flexibility to adapt a model to real world situations and allowing a solution for the value of embedded American options. However, working through a traditional recombining lattice becomes computationally cumbersome and non-intuitive if a project involves several compound options and is not suitable for dealing with multiple American growth options. Hence, the tree was modified to accommodate the dynamic of uncertain demand and then cash flows both with and without associated American-type growth options. In order for the modified tree to recombine, I imposed the following model restrictions:

1. Companies use the straight-line amortization method both for financial reporting and tax purposes.
2. No losses are carried forward.
3. Firms do not have financial constraints and can raise additional debt or equity capital to finance profitable investments.

One additional condition is fulfilling all calculations in real monetary terms. Strictly speaking, this is not a restriction; if done properly, calculations in real

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[3] In order to match financial statements in nominal and real terms for the firms operating in high-inflation environments, it is critical to adjust the levels of depreciation, interest expense, and changes in working capital (see Copeland, Koller, and Murrin [2000] for a detailed discussion of adjustment techniques).

[4] If changes in working capital are estimated from real cash flows in high-inflation markets, the actual cash inflow will be overstated. For example, in the case of a no-growth in real sales, real changes in working capital would equal zero, whereas nominal changes in working capital adjusted by the inflation index will be always positive:

$dWC_{nom} = \Delta_{WC} \cdot i \cdot S$, where

dWC_{nom} are changes in nominal working capital,

Δ_{WC} is ratio of net working capital to sales,

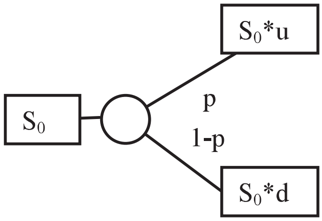
i is consumer inflation rate, % per period,

S is level of sales in previous period.

Respectively, inflation adjusted changes in working capital dWC' are:

$dWC' = \frac{dWC_{nom}}{1+i}$, which does not equal zero if $i > 0$.

Figure 1 Demand uncertainty experienced by a firm with constant output S_0.



terms match estimations in nominal terms. This is a technical condition that is necessary in order for the tree to recombine and, thus reduce the model's dimension. For example, if the same real volume of invested funds is calculated in nominal monetary terms, the amount of nominal reinvestment will depend on the time of investing due to inflation effect. Hence, level of capital calculated in nominal units at each node will be path dependent and the tree will not recombine. However, if all calculations are done in real monetary terms, path-dependence is eliminated. Applied to all other cash inflows and outflows, this logic serves as the basis for the whole methodology and secures the practical applicability of the Bellman algorithm.

First, by applying a conventional binomial model in the case of a firm producing constant output under demand uncertainty, in the next period nominal sales can increase by factor u with objective probability p or decrease by factor d with probability $1-p$ (Figure 1):

Factors u and d are the functions of dispersion in demand changes σ[5]:

$$u=\exp(\sigma\cdot\sqrt{k^{-1}}),\ d=1/u, \tag{3}$$

where k is number of time steps in one period (used to increase a sample of the binomial distribution in order to improve the approximation to geometric Brownian motion (GBM)).

Objective transition probability among states p is the function of consumer inflation rate μ and up/down factors:

$$p=\frac{1+\mu/k-d}{u-d} \tag{4}$$

Next, let $S_t(j, q)$ indicate the volume of real sales at period t when demand increased j times and decreased $t-j$ times, and management executed growth

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5) Although it is recognized that the parameterization of u and d is not unique in the Cox et al. model, it gives an accurate approximation of the continuous geometric Brownian motion of demand uncertainty in the limit.

reinvestment q times before the period t. Assume that management has the rights (growth options) to execute net reinvestments in any period during the abnormal growth phase of length T; therefore, q may range from zero (no growth case) to $t\,(t \leq T)$. Mathematically, sales at each state (j, q) are equal the initial level of sales S_0 multiplied by factor u^j and d^{t-j} due to market uncertainty and by expansion factor $\xi(q)$ due to reinvestments executed by the firm's management (and adjusted finally by inflation index $1+\mu$):

$$S_t(j, q) = \frac{S_o \cdot u^j \cdot d^{t-j} \cdot \prod_{i=0}^{q} \xi_i}{(1+\mu)^t}, \tag{5}$$

where $0 \leq j \leq t, 0 \leq q \leq t$, $\xi_i -$ expansion factor of i-th growth option (with $\xi_0 = 1$ for a "no-growth" case).

Figure 2 shows the fragment of the multinomial recombining tree containing information about currently possible decision modes at each state of uncertain demand conditioned upon the past states and decision policies. Due to the fact that all cash flows in the model are estimated in real terms, the final tree of underlying cash flows from operations possesses the recombining property similar to the tree depicted in Figure 2; that in turn dramatically reduces the computational complexity of the model from order $2^t \cdot (k+1)^t$ of the non-recombining tree to order $(t+1) \cdot (t \cdot k+1)$ (where t is the number of periods and k is the number of time-steps in one period).

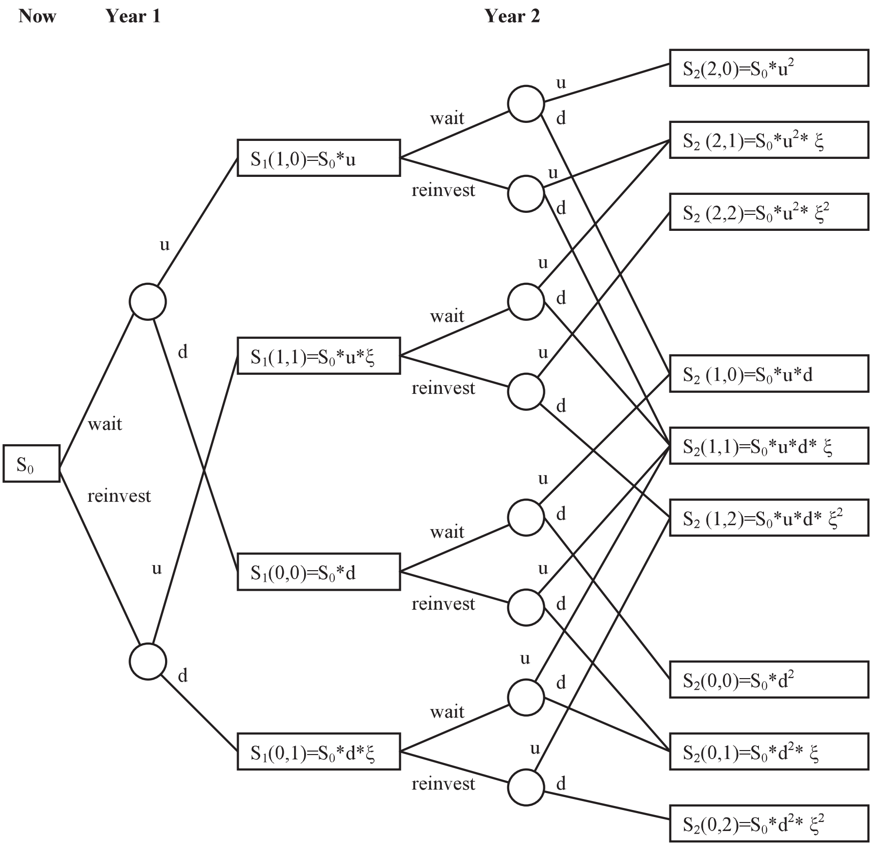
Given that a firm at each state is operating in a particular mode, the value of future cash flows given optimal behavior is henceforth denoted as $\Lambda_t(S_t, m_t, t)$. Optimal behavior assumes that a firm will maximize the value of the current payoff plus the present value of expected future payoffs' net of switching costs. At each uncertain state the optimization algorithm compares the conditional expected value from waiting with the conditional expected value of immediate reinvesting. In my notation, the dynamic Bellman equation can be written as (where it is assumed that the firm arrives at time t operating at mode m_t with possible switching to mode $l_{t+\Delta t} = 1, 2$):

$$\Lambda_t(S_t, m_t, \mathrm{t}) = \max_l \{FCF(S_t(j, q), m_t, t) - c_l + \rho \cdot E_t(\Lambda(S_{t+\Delta t}, l, t+\Delta t))\} \tag{6}$$

where ρ denotes the appropriate discount rate.

At this step, in order to value future cash flows that depend on the uncertain state variable S and can be modified by reinvestment decisions q, it is necessary to derive the appropriate discount rates in order to properly reflect the changing risk levels of these cash flows. The traditional corporate DCF model discounts expected future cash flows by a risk-adjusted rate which is usually represented by the weighted average cost of capital (WACC); however, this approach does not provide a correct valuation for growth options. The reason is that if management

Figure 2 Fragment of recombining decision/ binomial tree for reinvestment policy in sales expansion*.



*Inflation adjustment parameter is hidden to preserve space.
"Wait" means execution of replacement investment to maintain sales and postponement of net reinvestment.
"Reinvest" means execution of both replacement investment and net reinvestment to increase sales.
For expositional simplicity and without loss of the generality, sales expansion factor ξ is represented by a constant.

decides to switch to the "reinvest" mode, this decision will alter the expected future cash flows and the original risk profile of the company. Thus, the original WACC can no longer be considered a suitable measure of future corporate risks and it is necessary to apply a different discount rate to adjust for the changed risk of cash flows. The best method is to "make" these cash flows riskless and then discount them by a risk free rate. The traditional procedure in the option theory is to construct a replicating portfolio with B units of risk-free asset and Y units of a traded underlying asset that generates cash flows which exactly match an option's future uncertain payoffs at all times and in all states; hence, in order to avoid arbitrage possibilities, the present value of the option must equal the present value of the replicating portfolio[6]. To justify general methodology for arbitrage-free valuation of a company with many growth options, the author used the Marketed Asset Disclaimer (MAD) approach as discussed in Copeland and Antikarov [2001]. The fundamental assumption underlying the MAD approach is that the fair value of a non-traded underlying asset without flexibility options is "the price the asset would have if it were traded" (Mason and Merton [1985]) and various embedded options are treated as if they were assets whose cash flows are derived from the value of that underlying asset without options. Hence, the present value of a company's cash flows without growth options is characterized as a traded security in order to construct a portfolio that replicates the payoffs of cash flows with growth options.

Using the developed framework of a recombining tree, it is easy and intuitive to construct replicating portfolios for estimating the value of different managerial alternatives. I provide an example of computational procedure with two growth options for a two-period tree; this offers enough information to understand the algorithm, which can be extended to any number of periods and options. Let me again present a recombining tree similar to the one described in Figure 2, but now each node will include the Free Cash Flow (FCF) estimated in Formula (2) (Figure 3). At node (j, q), the slightly changed notation $FCF_t\,(u^j,\, d^{t-j},\, \xi^q)$ characterizes cash flow in case market demand increased j times and decreased $t-j$ times, and new reinvestments to bring this cash flow were executed q times (t is period of estimation, $j=0$ to t). First, consider a terminal node belonging to a "no-growth" path where management was choosing the "wait" mode in every state of nature (e.g., node $FCF_1(u)$). This is exactly the path where associated expected cash flows are generated by the company's initial assets-in-place. There are basically two methods by which to estimate fair value of the firm's assets-in-place. The first is to employ a fundamental DCF analysis relying on the company's WACC as a risk adjusted discount rate and actual probabilities of cash flow realizations. The second is to use a traded asset correlated with the company's assets-in-place. In this case, dynamic spanning holds and it is possible to use the risk neutral probability measure and risk free discount rate to obtain the value of the firm's assets-in-place. It should be noted that in the case of pricing the firm's current assets-in-place, the DCF valuation yields the same results as the risk-neutral estimation since the risk of the assets-in-place does not change during the time (giving traditional assumption of constant volatility) and the application of constant risk-adjusted discount rate to value the firm's assets-in-place is perfectly correct. Respectively, to generalize the model both for public and private firms, the present value of a firm's assets-in-place is determined using the traditional corporate DCF model by taking the real expectations E_t over possible realizations

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[6] See Copeland and Antikarov [2001], which provides a detailed introductory discussion of real option replication.

of FCF and discounting them by a company's WACC at initial time $t=0$:

$$PV^{AIP}(FCF_1(u)) = \frac{FCF_2(u^2)\cdot p + FCF_2(u, d)\cdot(1-p)}{1+WACC} \tag{7}$$

Now consider the "reinvest" mode for the current node ($FCF_1(u)$). Under the MAD assumption, the expected present value of FCF without a growth option calculated in Formula (7) is considered as if it were a traded asset; hence, it can be used to replicate the payoffs (cash flows) of a "reinvest" decision to obtain an arbitrage free estimation of their present value. The algorithm then moves on the lower branch, where management has already executed one growth option (a "reinvest" decision) at a previous time point ($FCF_1(u, \xi)$). Similarly, in the current node it is necessary to estimate the expected present value of FCF in case of a "wait" mode (value of the company *without* the second growth option) and to apply the MAD concept to replicate the cash flows of the "reinvest" mode (value of the company *with* the second growth option). However, due to the tree's recombining property, the value of cash flows expected from a "wait" decision at the current node $FCF_1(u, \xi)$ exactly equals the value of cash flows corresponding to the "reinvest" mode of the node at the upper branch, $FCF_1(u)$. The latter value has already been calculated using the arbitrage free principle at the previous step and is thus used to effectively replicate the payoffs of the "reinvest" mode for the current node. The similar procedures presented in Figure 3 are then applied sequentially to other nodes and the fair present values of all other cash flows can be estimated.

Finally, moving backwards and repeating all the steps, one can solve for the arbitrage free value of the company at the initial node and the optimal strategic mode at each uncertain state. The calculated amount at the initial node will represent the value of the company whose management has the flexibility to choose an optimal reinvestment strategy in case of uncertainty.

3. Numerical Example

I illustrate the valuation of a company with many growth options by applying it to a hypothetical Corporation XYZ. The abnormal growth phase is assumed to be seven periods during which the company's management has the right to execute seven growth options. To achieve the desired probability distribution, the number of time-steps is set to 150 per period. Numerical assumptions for three growth scenarios are summarized in Appendix A. The corporate DCF model's estimation of a moderate growth scenario with 20% volatility of demand changes produces the following valuations[7]:

Assets in place (no growth case): 41,160 million

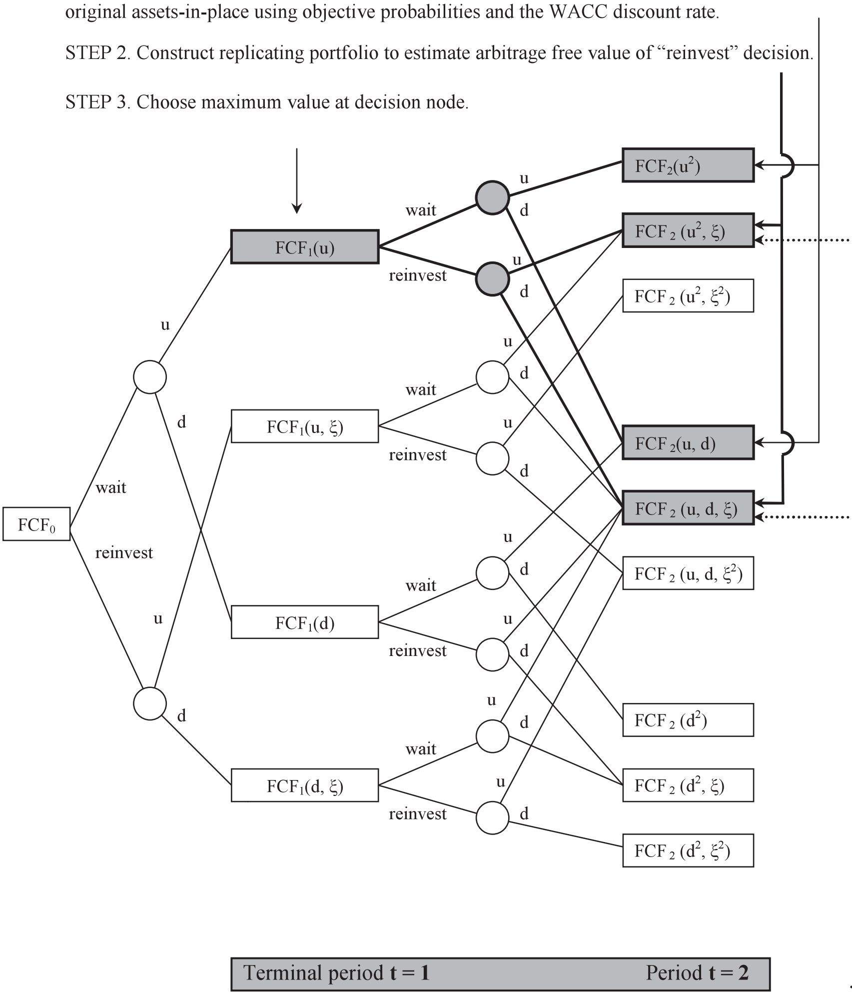
[7] Continuing value was estimated simplistically by assuming no growth in cash flows in stable period.

Figure 3 Applying MAD to properly value forecasted cash flows for two-period model with two American-type growth options.

STEP 1. Calculate the value of the "wait" decision for a path with cash flows generated by the firm's original assets-in-place using objective probabilities and the WACC discount rate.

STEP 2. Construct replicating portfolio to estimate arbitrage free value of "reinvest" decision.

STEP 3. Choose maximum value at decision node.



STEP 4. Go to a lower branch decision node with already installed first growth option: here the value of the "wait" decision has been calculated at step 2.

STEP 5. Repeat steps 2 and 3.

STEP 6. Repeat steps 1 to 5 for decision nodes in all other states and periods.

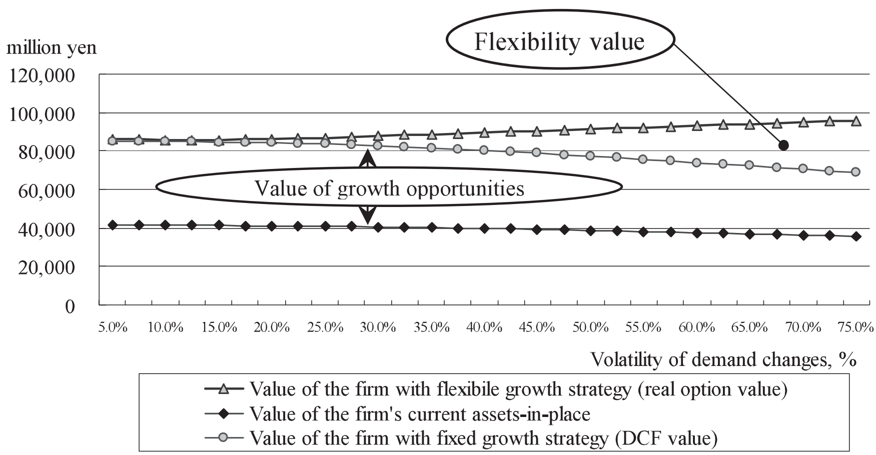
yen
Assets in place plus net reinvestments (growth case): 84,191 million yen

The calculations based on the described model give the company a value of 86,297 million yen, which is 2.5% larger than the DCF "growth" case. If only the value of growth opportunities is accounted for, which in the case of the DCF model equals 43,031 million yen and in the case of the proposed real option model equals 45,137 million yen, the DCF results in a 4.9% undervaluation of the company's growth options because of the inability to account for management flexibility.

I now examine the impact of greater uncertainty on the value of a corporation. Following the concept of mean-preserving spread as discussed in Rothschild and Stiglitz [1970], the increase of uncertainty over the demand parameter S is defined in such a way that it does not affect its mean. In both the fields of real option theory and strategic management it has been noted that the value of real options is positively related to the underlying uncertainty. In the case of increasing market uncertainty, the value of installed growth options should also increase since the possible loss is limited to the initial investment and the potential gain from future growth opportunities has no upper limit (Kulatilaka and Perotti [1998]). Hence, the firm's overall value should be positively linked to the value and uncertainty of the underlying cash flows that can be acquired through potential exercise of the growth options included in them.

Figure 4 summarizes the results of simulations of the mean-preserving increases in volatility of demand changes and their effects on the values of the firm and its growth options. The results suggest that the value of the portfolio of compound growth options strictly increases with the uncertainty over demand S. Figure 5 further clarifies that the impact of volatility also depends on the degree of future potential growth gained by exercising strategic growth options. It shows

Figure 4 Value of the firm as a function of volatility of demand changes.



Inputs: Moderate sales growth scenario.

that companies which expect high sales growth with high volatility have a much higher flexibility premium defined as:

$$Flexibility\ premium = \frac{Value\ of\ the\ firm\ with\ flexibility}{DCF\ Value\ of\ the\ firm} - 1 \quad (8)$$

Next, I investigate critical values of demand S that trigger option execution.

Figure 5 Flexibility premium of different growth scenarios as a function of volatility of demand changes.

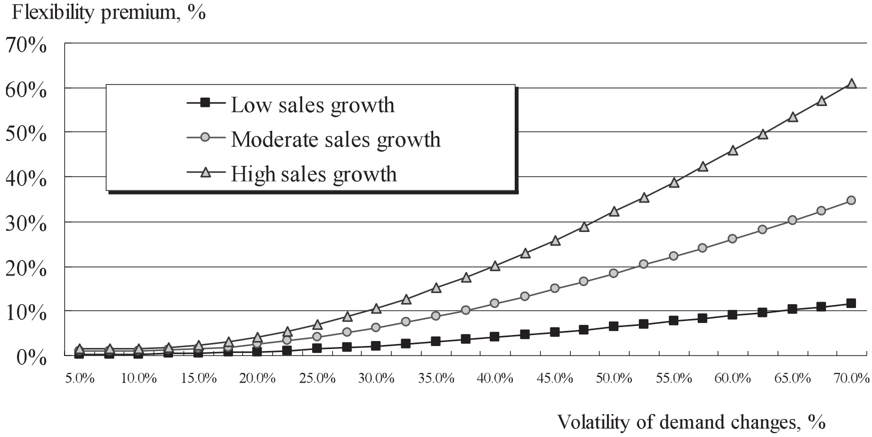
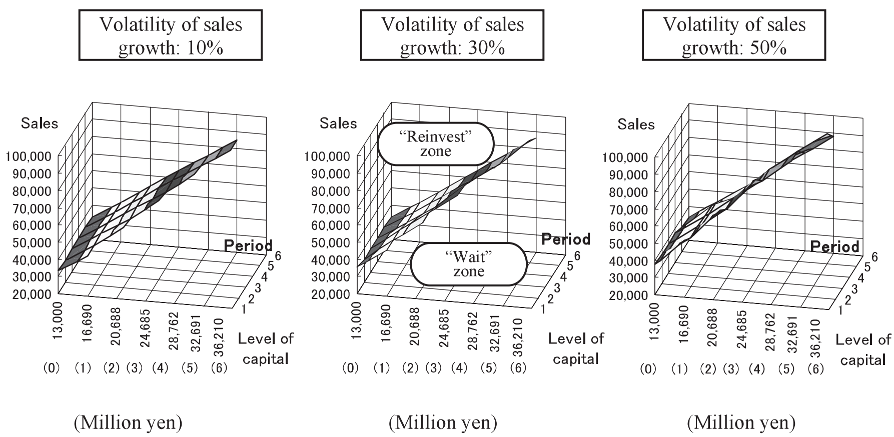


Figure 6 Boundary space of critical values of sales that trigger reinvestments as a function of capital level (previously installed options) and the time of option execution (number of growth options executed in previous periods shown in parentheses).



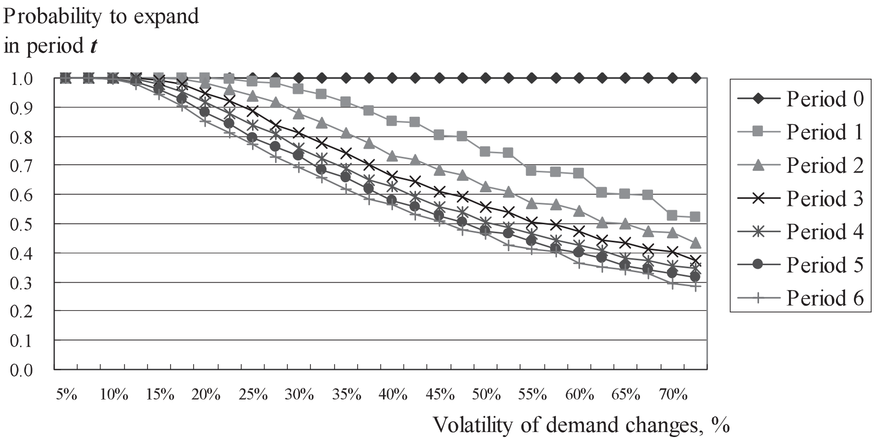
Inputs: Moderate sales growth scenario.

Figure 6 shows a moderate sales growth scenario for different values of demand volatility with an appropriate critical boundary at which it is optimal to execute growth reinvestment conditioned upon the previous waiting/reinvesting decisions (i.e., on the level of capital stock at time t).

The optimal investment policy given the capital level at each period t is to "wait" when actual level of sales lies below critical boundary space and to "reinvest" if actual level of demand exceeds the boundary. Clearly, higher demand levels favor reinvestments and reduce the value of waiting, whereas higher volatility increases the critical threshold and induces postponement of reinvestments. These results extend the findings of numerous previous studies that investigated the negative influence of uncertainty on individual real options for firms that possess many growth options in their business portfolios. Figure 7 further highlights the impact of uncertainty on firms' investment behavior and shows that the probability of investing in any period decreases with underlying uncertainty. Note also that in the framework of the analyzed numerical example, it is always optimal to exercise first an American growth option at initial period $t=0$ since the high level of current sales justifies the economic viability of the first reinvestment, both in terms of generating operating cash flows and building up platforms for future follow-on expansion investments.

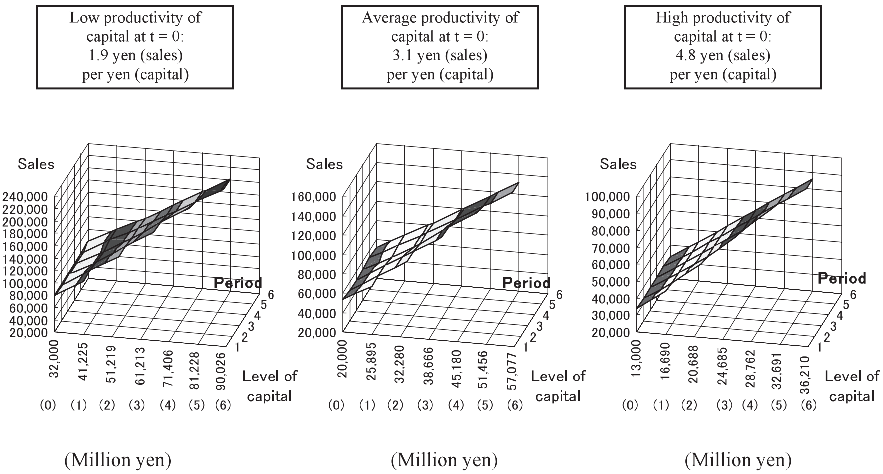
I now examine the effect of heterogeneity in resources and capabilities on investment patterns in relation to option creation and exercise. From a resource-based view, one may argue that the optimal management of real options requires managerial efforts that are enabled and constrained by firm-specific resources and capabilities. Using a numerical example, I show that different firms facing the same opportunity may display different investment behavior. Assume that three

Figure 7 Probability of reinvesting in each period as a function of volatility of demand changes.



Inputs: Moderate sales growth scenario.

Figure 8 Effect of heterogeneity of firm's capabilities on investment execution and demand threshold values.



Inputs: Moderate sales growth scenario, Volatility of demand changes-20%, Number of available options in the whole high growth period-7, Length of high growth phase-7 periods, Number of simulated time steps in one period-150.

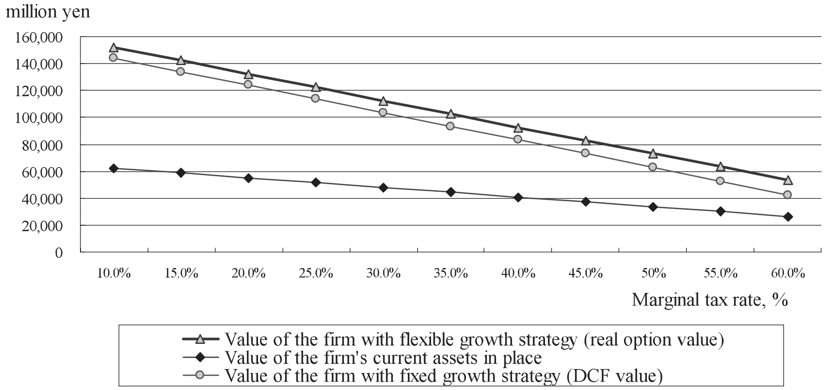
firms face the same demand uncertainty and the same growth prospects, but have differences in capital productivity, e.g., high, average, and low capital productivity. The results of the simulations are presented in Figure 8.

Figure 8 shows that a firm with high capital productivity invests at much lower levels of demand compared to its less productive competitors; hence, a capital-effective firm is much more alert in a dynamic environment and reacts quicker to changes in uncertain market demand. Respectively, one of the conclusions of the current study is the proposition that when managers of highly productive companies have a bundle of successive growth options, they may be willing to accept lower levels of current performance thresholds in order to gain access to future valuable growth options.

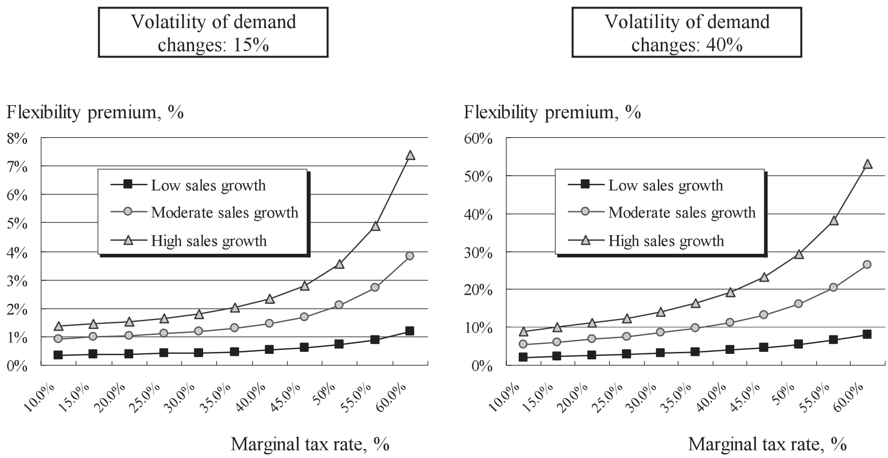
Finally, I show the effect of corporate tax rates on the value of a firm's growth options. Taxes impact cash flows in a non-linear manner since they are levied in case there is taxable income and not if the firm incurs current loss. Figure 9 shows that although the absolute value of a portfolio of growth options linearly decreases with a marginal tax rate, the value of management flexibility is much more profound at higher levels of uncertainty and in high tax regime environments.

It can be seen from the analysis that the presence of firm-specific effects expands the critical parameters in the real option analysis of determinants of a firm's investment behavior. The critical parameters are expanded from a focus on demand volatility to include: taxes; market potential which can be measured by attainable sales expansion ratios; and marginal efficiency of capital and new

Figure 9 Value of firm and flexibility premium as a function of corporate tax rate and expansion rates.



Inputs: Moderate sales growth scenario, Volatility of demand changes-40%.



investment. The current framework integrates all these firm-specific factors and effectively captures the portfolio effects of a firm's growth options.

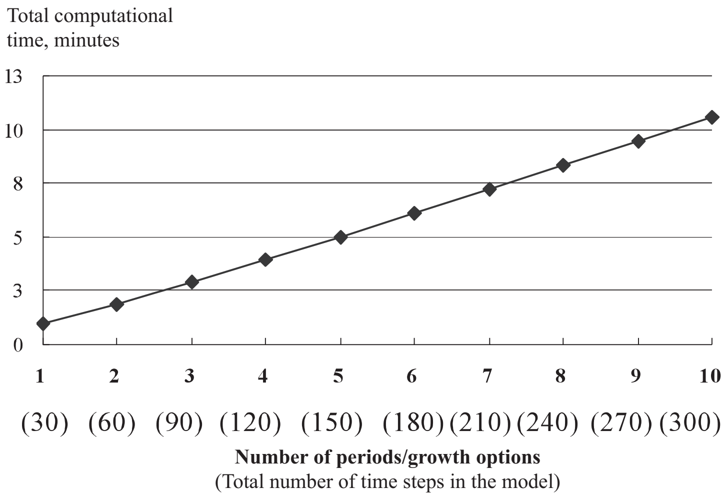
4. Discussion

The following discussion focuses on comparison between the proposed algorithm and alternative numerical methods of valuing corporations and multiple American options embedded in firm's growth: namely, corporate DCF, decision binomial trees, and binomial lattices.

The key advantage of the current paper is that it integrates corporate valuation with the real option framework and allows quantitative estimation of how capital allocation made in any year can enhance the company's value under uncertainty. Such "option-based" thinking provides a new focus for corporate CFOs by shifting the emphasis from choosing projects with maximum static NPV to paying attention to NPV-maximizing flexible reinvesting policy. It was shown that the values and insights from the proposed analysis may be significantly different from the DCF model, which assumes a predetermined reinvestment pattern and uses a subjectively defined risk-adjusted discount rate to calculate the present value of corporate cash flows. Concerning real option applications, use of the MAD assumption (Copeland and Antikarov [2001]) has been criticized because the value of a non-traded asset without flexibility, which is used to create a complete market, cannot be tested empirically and thus may lead to significant errors in estimation of the value of option on that asset. However, the described model recognizes the value of the company's current assets-in-place as the basis for arbitrage-free replication of the company's growth options. This value is usually observable on the market or can be estimated by direct comparisons with peer companies; hence, the model's results can be empirically tested.

The main drawback of traditional binomial lattices is that they do not deal with multiple American growth options that are compound and path-dependent. As an alternative, non-recombining binomial decision trees that can potentially handle any number of American-type growth options were proposed. The current model is able to capture the same number of American-type growth options; however, its computational procedure is more efficient. Generally, when estimating the value of a firm, the length of the abnormal growth period is assumed to be 10 to 20 years. Hence, if the standard one-period tree is used to model uncertainty, the next period's risky value can take only two possible values (whether going up by factor U or down by factor D). This is very restrictive because in reality the level of demand or any other stochastic variable may take several (many) values. Therefore, it is necessary to model the probability distribution of possible demand realization by dividing the period into smaller time steps (bearing in mind an investment decision will be executed at the beginning of the period). In this case the number of terminal nodes of the non-recombining tree will be $2^t \cdot (k+1)^t$, where t is the number of periods and k is the number of time steps in each period. At the same time, the number of end nodes in the provided model is $(t+1) \cdot (t \cdot k+1)$. For example, a simplistic 10-period model without division into subperiods will have the following: non-recombining case — 1,048,576 nodes, and recombining case — 121 nodes. A 10-period model with 100 time steps in each period to obtain fine distribution of stochastic variable will have the following: non-recombining case — approximately 10^{23} nodes, and recombining case — 11,011 nodes. Even the simplistic case will take hours to compute using the non-recombining algorithm, not to mention the more realistic case of a finer probability space which is unattainable for even the most powerful personal computer. In contrast, the proposed recombining model can be implemented in Excel by writing a VBA code or by computing directly in a spreadsheet. I further

Figure 10 Computational efficiency of the proposed model*.



**Number of simulated time steps in one period-30; Total number of time steps is shown in parenthesis.*

conducted a computational performance test on the real option evaluation using the proposed recombining binomial tree. According to Hull [2003], solving a lattice with 30 or more periods gives reasonable results; therefore, I used it as an upper limit for the testing range. The results were obtained using a Notebook with an Intel(R) Core(TM)2 Duo 1.20 GHz processor and 0.99 Gb RAM. As indicated in Figure 10, a recombining binomial tree is computationally very efficient, especially in cases with a large number of options or when a high degree of accuracy is required. In contrast, the binomial tree used by Brandao, Dyer, and Hahn [2005] requires several hours to evaluate using the most advanced professional decision tree software.

Finally, it is necessary to mention that the real option theory, with its origin in stochastic calculus, does not focus on a firm's organizational and behavioral issues. Hence, no quantitative decision-making technique can guarantee the optimal policy and substitute for proper managerial efforts and discretion. In future research it will be crucial to consider various organizational and managerial issues that can stimulate or, on the contrary, limit the optimal execution of real growth options in business practice.

5. Conclusion

A major source of value from growth reinvestments arises from their ability to enhance the upside cash generating capacity of a corporation during favorable market conditions by making follow-on expansion reinvestments. Myers [1987] observed that “the standard discounted cash flow techniques will tend to understate the option value attached to growing, profitable lines of business” and that “the most promising line of research is to try to use option pricing theory to model the time-series interactions between investments.” In an attempt to address this issue, the current article contributes to existing corporate finance literature in several manners:

1. It develops a general model of corporate flexibility in which a company pursuing organic growth uses capacity as a strategic variable to adjust discretely to a stochastic demand. During each period, a company may operate in two modes (wait or reinvest) with costly switching that provides a set of multiple American-type compound options.
2. It provides a modeling framework for valuing multiple growth options embedded in corporate growth, which are essentially compound American options on American options.
3. It validates a computationally effective tool for valuing corporations, providing a real option alternative to the traditional corporate DCF model.

The described methodology shows that the combined value of multiple growth options can be quite large and that corporate DCF criteria, which do not account for these options, can be misleading. In general, generation of growth options should drive corporate value. However, as growth reinvestments are costly and may be (partially) irreversible, it is not certain that this generating process is always conducted efficiently and creates value for shareholders. The applied dynamic-programming framework enables a computationally feasible evaluation of the effects of many such growth options. The proposed model explicitly accounts both for exogenous (market demand) and endogenous (net reinvestments) determinants of the firm’s growth and shows that the binomial model combined with a decision tree analysis allows for quantitative estimation of the optimal reaction of a firm to uncertain nature moves. The primary advantage of this approach is that it provides greater transparency in the modeling of a company’s real growth options by focusing on the individual uncertainty factor that drives a company’s future cash flows. The methodology also provides clear threshold information for management (e.g., based on currently employed capital, invest if and only if sales reach the level of X yen), because the model explicitly accounts for the values of each individual cash flow in every period. Hence, applying a Marketed Asset Disclaimer using a firm’s current assets-in-place as a base for replication, the model offers an arbitrage free algorithm for corporate valuation that properly accounts for changing risk profiles in each period depending on reinvestment decisions.

As noted in the introductory sections, a notable challenge to empirical research in strategic management under the real option framework is difficulty measuring the value of a firm’s growth opportunities. The current study addresses this issue and provides a theoretically consistent and computationally effective algorithm for future studies to improve the precision in measuring corporate growth options using public accounting data. The recombining property of the final Free Cash

Flow tree allows for improvement in accuracy of the model output by increasing the number of subperiods without placing excessive burden on computer memory. Moreover, the algorithm, although complicated, does not use higher mathematics; therefore, it can be effectively presented to decision-makers. Besides including other options (for example, the option to abandon a firm), the presented model may also be used to incorporate other risk factors such as cost uncertainty for companies with volatile variable costs or foreign exchange rate uncertainty for firms with high export/import exposure[8)]. The methodology has also a parallel application; when slightly modified it can be used to value switching options when the number of alternative assets is (infinitely) large.

This paper is not without limitations. It is based on several restrictive assumptions concerning real cash flows and does not account for the effects of competitive interactions in the value of portfolio of growth real options and optimal investment decisions. Future research should address the mentioned limitations, as well as empirically test the arguments presented.

[8)] Generally, a recombining multinomial lattice can potentially handle up to three uncertainty factors simultaneously.

Appendix A

List of assumptions for numerical examples in the text:

	Sales expansion factor of i-th growth option, i:						
Scenario of demand conditions	**1**	**2**	**3**	**4**	**5**	**6**	**7**
High sales growth	162. 0%	150. 0%	145. 0%	142. 0%	129. 0%	114. 0%	109. 0%
Moderate sales growth	130. 0%	125. 0%	120. 0%	117. 0%	114. 0%	111. 0%	108. 0%
Low sales growth	107. 0%	106. 5%	106. 0%	105. 5%	105. 0%	104. 5%	104. 0%

Capital efficiency	**Assumption**
Average productivity of initial assets-in-place	4. 73 yen of sales per 1 yen of capital
Marginal productivity of new investment	5 yen of sales per 1 yen of capital
Income Statement	
Initial level of sales	61, 500 million yen
Length of high growth phase	7 periods
Variable cost of sales	90. 0%
Marginal income tax	40. 7%
Balance Sheet	
Working capital, % of sales	4. 9%
Initial fixed capital	13, 000 million yen
Remaining usage life	7 periods
Capital structure	
Net debt ratio	19. 8%
After-tax nominal cost of debt	0. 6%
Nominal cost of equity	9. 0%
Real WACC	6. 9%
Market	
Consumer inflation rate	0. 5%
Inflation of capital assets	1. 0%
Real risk free rate	1. 0%

Appendix B

For readers unfamiliar with real option techniques, this Appendix outlines a simple example to demonstrate the applications of the methodology. Suppose a firm that has three years of abnormal growth is facing demand uncertainty with 20% volatility. The real risk free rate r is 1% and the consumer inflation rate μ is 0.5%. During the following three years the firm has three options to expand its sales by 1.3, 1.25, and 1.2 times by investing 5,426.5 yen, 5,878.7 yen, and 5,878.7 yen of additional net capital, respectively. To finance sales growth, the firm would also need to inject 895.5 yen, 970.1 yen, and 970.1 yen, respectively, of additional funds in working capital whenever it executes its growth options. The company's real after-tax $WACC$ is estimated to be 7.288%. After three years of abnormal growth, the firm will be receiving a constant stream of cash flows depending on the previous installed options and state of demand. To calculate the value of the firm, I show the steps necessary to fulfill the analysis using the proposed recombining model.

Table 1 shows the dynamics of real sales $S_t(j, q)$ conditioned on the management decisions and stochastic realizations. The top part shows the model assumptions estimated based on formulas (3) and (4). The current level of sales $S_0(0, 0)$ is assumed to be 61,500 yen, and in order to maintain expositional simplicity the time step k is set to one year. The up-factor u is given by $u = exp(0.2) = 1.2214$, and down factor by $d = 1/1.2214 = 0.8187$. Objective up-move probability among states p is the function of consumer inflation rate μ and up/down factors: $p = (1 + \mu/k - d)/(u - d) = 0.4626$. Management decisions are incorporated in symbol q, so a "reinvest" decision at the current node corresponds to the next period q increasing by 1, and "wait" corresponds to the q being unchanged. In turn, demand uncertainty is incorporated in symbol j, sales increasing by factor u correspond to the next period value of j increasing by 1, and sales decreasing by factor d correspond to the next period j is unchanged. The values $S_t(j, q)$ with $q = 0$ show the possible demand realizations for the company without any installed options (i.e., the value of sales generated by the firm's initial assets-in-place). Respectively, the company that replaces its capital without making any expansion reinvestments expects a constant level of real sales in any period (e.g., expected real sales in period 1 are $74{,}742.6 \cdot 0.4626 + 50{,}101.4 \cdot (1 - 0.4626) = 61{,}500$ yen).

In each node I further evaluate real free cash flows defined in Equation (2) as after-tax operating cash flows net of replacement investment and effect of working capital changes due to inflation[9]. Taking node $S_3(2, 2)$ with sales of 120,251.1 yen as an example, Table 2 provides the necessary estimations of the company's income statement, balance sheet, and cash flows calculated as 73,069.5 yen. The

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[9] See footnotes to Equation (2) for an explanation.

Table 1 Modeling demand uncertainty. Dynamics of real sales $S_t(j, q)^*$.

Initial level of realized demand	¥61,500	Length of abnormal growth period	3
Volatility of demand changes	20.0%	k	1
		Number of available growth options	3
Inflation rate	0.5%	p	0.4626
Real risk free rate	1.0%	u	1.2214
Real after tax WACC	7.6%	d	0.8187

Real yen

Period

0	**1**	**2**	**3**
$S_0(0, 0)=61,500.0$	$S_1(1, 0)=74,742.6$	$S_2(2, 0)=90,836.6$	$S_3(3, 0)=110,396.1$
	$S_1(1, 1)=97,165.3$	$S_2(2, 1)=118,087.6$	$S_3(3, 1)=143,514.9$
	$S_1(0, 0)=50,101.4$	$S_2(2, 2)=147,609.4$	$S_3(3, 2)=179,393.6$
	$S_1(0, 1)=65,131.9$	$S_2(1, 0)=60,889.6$	$S_3(3, 3)=215,272.3$
		$S_2(1, 1)=79,156.5$	$S_3(2, 0)=74,000.7$
		$S_2(1, 2)=98,945.6$	$S_3(2, 1)=96,200.9$
		$S_2(0, 0)=40,815.5$	$\mathbf{S_3(2, 2)=120,251.1}$
		$S_2(0, 1)=53,060.2$	$S_3(2, 3)=144,301.4$
		$S_2(0, 2)=66,325.2$	$S_3(1, 0)=49,604.2$
			$S_3(1, 1)=64,485.4$
			$S_3(1, 2)=80,606.7$
			$S_3(1, 3)=96,728.1$
			$S_3(0, 0)=33,250.7$
			$S_3(0, 1)=43,225.9$
			$S_3(0, 2)=54,032.3$
			$S_3(0, 3)=64,838.8$

* *See Figure 2 for expanded presentation of the sales dynamics.*

lattice reported in Table 3 shows the possible values of free cash flows $FCF_t(j, q)$ calculated similarly for all other state nodes (with the same notation as for the sales dynamics).

At the next step I utilize dynamic programming to determine optimal strategies and values of appropriate cash flows. I start from the terminal decision node $FCF_2(2, 0)$ whose cash flows correspond to the firm's assets-in-place. The present value of the expected cash flows generated by the assets-in-place in the next period is computed via a traditional DCF algorithm as:

$$\begin{aligned} PV_{AIP} &= (FCF_3(3, 0)\cdot p + FCF_3(2, 0)\cdot(1-p))/(1+WACC) \\ &= (82{,}685.9\cdot 0.4626 + 43{,}656.7\cdot(1-0.4626))/(1+0.07288) \\ &= \mathbf{57{,}519.6\ yen} \end{aligned} \tag{9}$$

Table 3 Dynamics of real cash flows $FCF_t(j, q)$.

Period

0	**1**	**2**	**3**
$FCF_0(0, 0) = 1,701.7$	$FCF_1(1, 0) = 2,465.7$	$\mathbf{FCF_2(2, 0) = 3,415.7}$	$FCF_3(3, 0) = 82,685.9$
	$FCF_1(1, 1) = 3,261.6$	$FCF_2(2, 1) = 4,496.0$	$FCF_3(3, 1) = 108,478.1$
	$FCF_1(0, 0) = 1,004.3$	$FCF_2(2, 2) = 5,670.4$	$FCF_3(3, 2) = 136,492.1$
	$FCF_1(0, 1) = 1,361.7$	$FCF_2(1, 0) = 1,639.6$	$FCF_3(3, 3) = 164,579.3$
		$FCF_2(1, 1) = 2,187.0$	$FCF_3(2, 0) = 43,656.7$
		$FCF_2(1, 2) = 2,784.2$	$FCF_3(2, 1) = 57,740.1$
		$FCF_2(0, 0) = 449.0$	$\mathbf{FCF_3(2, 2) = 73,069.5}$
		$FCF_2(0, 1) = 639.3$	$FCF_3(2, 3) = 88,472.3$
		$FCF_2(0, 2) = 849.5$	$FCF_3(1, 0) = 17,494.6$
			$FCF_3(1, 1) = 23,729.4$
			$FCF_3(1, 2) = 30,556.2$
			$FCF_3(1, 3) = 37,456.2$
			$FCF_3(0, 0) = -403.4$
			$FCF_3(0, 1) = 931.3$
			$FCF_3(0, 2) = 2,058.6$
			$FCF_3(0, 3) = 3,259.0$

Hence, if in the current decision node management executes its first growth option, the firm will receive $FCF_3(3, 1)$ in case demand goes up and $FCF_3(2, 1)$ in case demand goes down. Applying the MAD principle and considering the value of assets-in-place as an underlying asset, the present value of the "expanded" cash flows can be estimated by replication via following familiar equations[10]:

$$\begin{aligned} Y &= (FCF_3(3, 1) - FCF_3(2, 1))/(FCF_3(3, 0) - FCF_3(2, 0)) \\ &= (108{,}478.1 - 57{,}740.1)/(82{,}685.9 - 43{,}656.7) = 1.3 \end{aligned}$$

$$\begin{aligned} B &= (FCF_3(3, 1) - Y \cdot FCF_3(3, 0))/(1+r) = (108{,}478.1 - 1.3 \cdot 82{,}685.9)/(1+0.01) \\ &= 976.7 \end{aligned}$$

$$PV_{reinvest} = Y \cdot PV_{AIP} + B = 1.3 \cdot 57{,}519.6 + 976.7 = \mathbf{75{,}752.2\ yen} \tag{10}$$

Finally, I add the current node's cash flow of 3,415.7 yen to obtain the total value of cash inflows expected in the current state. Hence, the optimal value of cash flows and optimal strategy at the current node can be determined as:

[10] I refer interested readers to Copeland and Antikarov [2001], wherein a detailed introductory discussion of real option replication is provided.

Corresponds to node $S_2(2, 0)$ with 90,836.6 yen in sales without any installed growth options

$$\Lambda_2(2, 0)=\max\{\underbrace{3{,}415.7+57{,}519.6}_{\text{Continuation Value (waiting)}}\ ;\ \underbrace{\mathbf{3{,}415.7+75{,}752.2-5{,}426.4-895.5}}_{\text{Termination Value (reinvesting)}}\}$$
$$=\mathbf{72{,}846.0\ yen}\ (\text{“reinvest”}), \tag{11}$$

where the continuation value in the maximum corresponds to the present expected value of cash flows generated by the firm's current assets-in-place, and termination value equals the present expected value of cash flows in case the company executes its first growth option less the "exercise price" in form of 5,426.5 yen of net reinvestment in fixed assets and 895.5 yen of investment in working capital necessary to expand sales. In this case the continuation value (60,935.3 yen) is less than the termination value (72,846.0 yen) and the optimal policy is to reinvest.

Next, I evaluate the present value of cash flows for a decision node at a lower branch, $FCF_2(2, 1)$, with the already-installed first growth option. Future cash flows that correspond to a "wait" decision in the current node are $FCF_3(3, 1)$ and $FCF_3(2, 1)$. The "arbitrage-free" present value PV_{wait} of these cash flows has already been estimated in Equation (10) as 75,752.2 yen. Thus, one can apply the MAD concept and use the value of the firm with the first installed option PV_{wait} as an underlying asset to replicate future cash flows of the firm's second growth option. Hence, their present value is given as:

$$Y=(FCF_3(3, 2)-FCF_3(2, 2))/(FCF_3(3, 1)-FCF_3(2, 1))$$
$$=(136{,}492.1-73{,}069.5)/(108{,}478.1-57{,}740.1)=1.25$$

$$B=(FCF_3(3, 2)-Y\cdot FCF_3(3, 1))/(1+r)=(136{,}492.1-1.25\cdot 108{,}478.1)/(1+0.01)$$
$$=885.6$$

$$PV_{reinvest}=Y\cdot PV_{wait}+B=1.25\cdot 75{,}752.2+885.6=\mathbf{95{,}575.9\ yen} \tag{12}$$

And the optimal state value is given by:

Corresponds to node $S_2(2, 1)$ with 118,087.6 yen in sales with the first growth option installed

$$\Lambda_2(2, 1)=\max\{\underbrace{4{,}496.0+75{,}752.2}_{\text{Continuation Value (waiting)}}\ ;\ \underbrace{\mathbf{4{,}496.0+95575.9-5{,}878.7-970.1}}_{\text{Termination Value (reinvesting)}}\}$$
$$=\mathbf{93{,}223.1\ yen}\ (\text{“reinvest”}) \tag{13}$$

Similar calculations are applied in the node $FCF_2(2, 2)$ to estimate the present value of expected cash flows in case the firm executes its third growth option. Respectively, using the values of the firm's assets-in-place cash flows estimated using the DCF model as a basis for replicating the option payoffs, the described algorithm can be further incorporated for all other nodes and periods. The final lattice presented in Table 4 shows the dynamic programming rollback values $\Lambda_t(j, q)$ and optimal policies in each node. The optimal strategies are indicated in the lattice: bold font in gray cells indicates that reinvesting is optimal and normal font in white cells indicates that waiting is an optimal decision policy.

Table 4 Optimal values of real cash flows $\Lambda_t(j, q)$ and optimal investment strategies of the firm*.

Period

0	**1**	**2**	**3**
$\Lambda_0(0, 0) = 38,767.7$	**$\Lambda_1(1, 0) = 57,241.4$**	**$\Lambda_2(2, 0) = 72,846.0$**	$\Lambda_3(3, 0) = 82,685.9$
	$\Lambda_1(1, 1) = 72,061.1$	**$\Lambda_2(2, 1) = 93,223.1$**	$\Lambda_3(3, 1) = 108,478.1$
	$\Lambda_1(0, 0) = 18,709.4$	**$\Lambda_2(2, 2) = 114,293.5$**	$\Lambda_3(3, 2) = 136,492.1$
	$\Lambda_1(0, 1) = 25,321.2$	**$\Lambda_2(1, 0) = 32,156.7$**	$\Lambda_3(3, 3) = 164,579.3$
		$\Lambda_2(1, 1) = 42,272.8$	$\Lambda_3(2, 0) = 43,656.7$
		$\Lambda_2(1, 2) = 53,037.8$	$\Lambda_3(2, 1) = 57,740.1$
		$\Lambda_2(0, 0) = 7,789.9$	$\Lambda_3(2, 2) = 73,069.6$
		$\Lambda_2(0, 1) = 11,420.9$	$\Lambda_3(2, 3) = 88,472.3$
		$\Lambda_2(0, 2) = 15,212.1$	$\Lambda_3(1, 0) = 17,494.6$
			$\Lambda_3(1, 1) = 23,729.4$
			$\Lambda_3(1, 2) = 30,556.2$
			$\Lambda_3(1, 3) = 37,456.2$
			$\Lambda_3(0, 0) = -403.4$
			$\Lambda_3(0, 1) = 931.3$
			$\Lambda_3(0, 2) = 2,058.6$
			$\Lambda_3(0, 3) = 3,259.0$

* *Grey cells with bold font correspond to the states of demand in which it is optimal to exercise growth options.*

Finally, the value of the company with flexibility is estimated to be 38,767.7 yen, which is 5% larger then the value calculated via the traditional corporate DCF model which simply assumes reinvestment in each state of nature and then computes the present value of expected future cash flows by discounting them using the weighted average cost of the firm's capital.

In Appendix I illustrated the practical implementation of the developed methodology for solving a real option problem of valuing multiple growth opportunities of a firm. This simple valuation example demonstrates several

Table 2 Projections of financial statements in state node of recombining tree.

State of demand: $S_3(2, 2) = 120,251.1$ yen			
Period	3	Number of Up jumps	2
Number of previous expansions	2	Number of Down jumps	1

Management forecast of Sales and Investments expected in period 3

Cumulative expansion factor (2 expansions)	162. 5%
Expected real sales after 2 expansions	99, 937. 5
Expected change in real sales from t=0	38, 437. 5
Necessary net real reinvestment	11, 305. 1
Expected change in Working Capital (inflationary effect)	24. 3

Management forecast of Revenues and Cash Flows

Income Statement	Actual (time t=0)	Forecasted (current state)
Real Sales	61, 500. 0	120, 251. 1
Variable Cost of Sales (excluding amortization)	55, 350. 0	108, 226. 0
Earnings before interest, taxes, and amortization (EBITA)	6, 150. 0	12, 025. 1
Nominal depreciation adjusted for inflation	3, 333. 3	5, 165. 3
EBIT	2, 816. 7	6, 859. 9
Interest expense	76. 5	121. 0
EBT	2, 740. 2	6, 738. 9
Income tax	1, 115. 0	2, 742. 0
Net Income	1, 625. 2	3, 996. 8

Cash Flows	Actual (time t=0)	Forecasted (current state)
EBIT less taxes	1, 701. 7	4, 117. 8
+Depreciation	3, 333. 3	5, 165. 3
- Capital expenditures (Replacement of depreciated Capital)	(3, 333. 3)	(5, 217. 5)
- / +(Increase)/ Decrease in Working Capital (inflationary effect)		(24. 3)
Free Cash flow	1, 701. 7	4, 041. 3
Continuing value		73, 069. 5

At the end of the abnormal growth period cash flows are represented by continuing value given as: $CV_t(j, q) = FCF_t(j, q)/(1-1/(1+WACC_{stable}))$, where $WACC_{stable}$ is the weighted average cost of capital expected by the firm's management during the stable period.

Management forecast of Capital requirements

Capital (at the beginning of the period)	Actual (time t=0)	Forecasted (current state)
Expected Working Capital	3, 000. 0	4, 923. 9
Fixed assets	20, 000. 0	31, 458. 7
Total invested capital	23, 000. 0	36, 382. 6
Reconciliation:		
Net debt	4, 554. 0	7, 203. 8
Equity	18, 446. 0	29, 178. 8
Total invested capital	23, 000. 0	36, 382. 6

Management forecast of Property, Plant, and Equipment

	Forecast Period			
	0	1	2	3
PPE, Gross at the beginning	20, 000. 0	25, 426. 5	31, 397. 3	31, 458. 7
Nominal depreciation	3, 333. 3	4, 237. 7	5, 232. 9	5, 243. 1
Real Depreciation	3, 333. 3	4, 237. 7	5, 217. 5	5, 217. 5
CAPEX:				
Replacement investment	3, 333. 3	4, 237. 7	5, 217. 5	5, 217. 5
Net reinvestment	5, 426. 5	5, 878. 7	0. 0	0. 0
PPE, at the end	25, 426. 5	31, 397. 3	31, 458. 7	31, 543. 9

Other forecast assumptions

Consumer inflation rate, % per period	0. 5%
Inflation rate of capital goods, % per period	1. 0%
Variable Cost of Sales, % of sales	90. 0%
Marginal income tax, % of pre- tax income	40. 7%
Net Debt ratio	19. 8%
Interest rate on debt	1. 7%
Real after tax WACC during high-growth-phase	7. 3%
Real after tax WACC during stable period	5. 9%
Working Capital, % of sales	4. 9%
Remaining usage life of fixed capital, periods	6
Capital turnover, yen (sales) per yen (capital)	3. 1
Net reinvested capital turnover, yen (sales) per yen (capital)	3. 4

important features. First, if no traded asset perfectly correlates with a firm's current assets-in-place, traditional DCF valuation is a necessary step to obtain the fair value of the firm's current assets. However, although DCF valuation of the firm's assets-in-place is essential at the first step, the successive replication of real option payoffs using a dynamic portfolio composed of a risk-free asset and the firm without option (and considering the firm's assets-in-place as a base underlying asset) provides economically correct arbitrage-free valuation for the firm's multiple growth opportunities. Further, estimations of the firm's growth opportunities using recombining multinomial trees yield precise values for the company's flexibility options, whereas the traditional DCF model fails to properly account for managerial discretion in optimal capital allocation. Finally, the provided algorithm is fairly easy to understand and convey to practical decision-makers who may not possess technical background in real options.

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