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Noncooperative Game in Cooperation: Reformulation of Correlated Equilibria[1)]

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In the usual Nash equilibrium of strategic noncooperative games, the mixed strategies of all the players are assumed to be stochastically independent. In order to relax this stochastic independence, two essentially different formulations are proposed. In this paper, by using random variables defined on an abstract and universal probability space, I introduce two equilibrium concepts, one is for the framework of a noncooperative game in which a third person, a mediator, influences each player's strategy independently or plays the role of a publicly observable random device. The other concept is for the framework of a noncooperative game in which the restriction of stochastic independence is removed, which could imply possible communications between players. Consequently, we have two different theorems that characterize all the strategies in the correlated equilibrium for each framework. These two theorems show that the two equilibrium concepts, which we shall call "exogenous correlated equilibrium" and "endogenous correlated equilibrium" are entirely different concepts. Finally, we provide some comments on previous related studies[2)].

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1. Introduction

Luce and Raiffa (1957, p.91 and p.94) were the first to discuss preplay communication or binding agreements between players of non-zero sum games,

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and they also gave an example (also see the comment in Section 7.1 of the present paper).

In practical situations, noncooperative games may be influenced by a person other than the game players themselves, for example, a nurse can influence the patient's decision when a doctor seeks an informed consent from a patient, and the U. S. Government influencing the relation between Japan and North Korea. In some cases, the players may be able to communicate with each other without any mediator.

In this paper, by using random variables, we will formulate two types of "correlated equilibria" for noncooperative games.

Many examples for correlated equilibria have been given by Aumann (1974, 1987) and Fudenberg and Tirole (1991). It is possible to analyze some of those examples using our formulations. In the coming paper, we will generalize our formulation so that all other examples can be investigated.

2. Reformulation of Noncooperative Games Defined by Random Variables

Aumann (1974, p. 68) argues that "it is best to view a randomized strategy as a random variable with values in the pure strategy space, rather than as a distribution over pure strategies."

Throughout this paper, we will completely follow his argument. However, unlike Aumann our "random variables" are rigorously defined on an abstract and universal probability space $(\Omega, \mathcal{F}, P)$[3], where Ω is an abstract set that is sometimes called the sample space, $\mathcal{F}$ is a σ-field whose element is called an event, and P is a probability measure on $\mathcal{F}$. The sample space Ω can be assumed to be rich enough, if necessary. Fortunately, in this paper all our calculations on probability are so elementary that any advanced probability theorems are not required; as Aumann (1987, p. 2) says, "Once one has the formulation, it is indeed embarassingly easy to prove, as we shall see." The essential point is that one should recognize that all random variables under consideration are defined on this abstract and universal probability space $(\Omega, \mathcal{F}, P)$[4]. Irrespective of whether or not the random variables are defined on this space, our final concern is always the distributions of these random variables[5].

First, we will reformulate a noncooperative strategic game Γ by using random variables and give the general definition of Nash equilibrium. In this paper, we

[3] Most textbooks on probability theory include the axioms of probability, and therefore, we omit the details.

[4] A function X from Ω to a finite set M is called a random variable if X satisfies $\{\omega \in \Omega ; X(\omega) = m\} \in \mathcal{F}$ for all $m \in M$. According to the conventions of probability theory, we will omit Ω and ω hereafter.

[5] Two random variables defined on two different probability spaces with the same distributions should not be distinguished; however, their joint distribution can not be discussed.

will focus on a game involving only two players say Player 1 and Player 2; with this assumption, a formal extension to an n-person game becomes very easy.

Let $S_i (i=1, 2)$ be a finite set of Player i's pure strategies and let set $S := S_1 \times S_2$. Let $u_i(s_1, s_2) (i=1, 2)$ and $(s_1, s_2) \in S$, denote a real-valued function defined on S, and let us call it Player i's pay-off function. Let $\mathcal{R}(S_i)$ be the set of all S_i-valued random variables and $\mathcal{R}(S)$ be the set of all S-valued random variables, that is, $\mathcal{R}(S) = \mathcal{R}(S_1) \times \mathcal{R}(S_2)$. A strategy of Player i is an element of $\mathcal{R}(S_i)$. An element $(X, Y) \in \mathcal{R}(S)$ is called a strategy profile of the game, that is, a strategy for Player i is the selection of a random variable from $\mathcal{R}(S_i)$. If $P(X=s_1, Y=s_2)$[6] $=1$ for some $s_1 \in S_1$ and $s_2 \in S_2$, then (X, Y) is called a pure strategy profile, otherwise it is called a mixed strategy profile. Player i's utility is represented by $E[u_i(X, Y)]$, which is the mathematical expectation of a real-valued random variable $u_i(X, Y)$[7].

Set $\mathcal{R}_{ind}(S) := \{(X, Y) \in \mathcal{R}(S) ; X \in \mathcal{R}(S_1)$ and $Y \in \mathcal{R}(S_2)$ are independent$\}$, that is, $(X, Y) \in \mathcal{R}_{ind}(S)$ if and only if

$$P(X=s_1, Y=s_2) = P(X=s_1) P(Y=s_2)$$

holds for all $s_1 \in S_1$ and $s_2 \in S_2$.

Now we can redefine a general noncooperative game and its Nash equilibrium.

Definition 1. A game Γ is called a noncooperative game when any strategy profile $(X, Y) \in \mathcal{R}(S)$ belongs to the set $\mathcal{R}_{ind}(S)$.

Definition 2. A strategy profile $(X^*, Y^*) \in \mathcal{R}_{ind}(S)$ is a Nash equilibrium if and only if

$$\text{(i)} \quad E[u_1(X^*, Y^*)] \geq E[u_1(X, Y^*)]$$

holds for all $X \in \mathcal{R}(S_1)$ such that $(X, Y^*) \in \mathcal{R}_{ind}(S)$, and

$$\text{(ii)} \quad E[u_2(X^*, Y^*)] \geq E[u_2(X^*, Y)]$$

holds for all $Y \in \mathcal{R}(S_2)$ such that $(X^*, Y) \in \mathcal{R}_{ind}(S)$.

The distribution of a strategy profile $(X, Y) \in \mathcal{R}_{ind}(S)$ is determined by the marginal distributions of the random variables, that is, the distribution of X on S_1 and that of Y on S_2 because X and Y are independent. Therefore, at present, there

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[6] $(X=s_1, Y=s_2) := \{\omega \in \Omega ; X(\omega) = s_1\} \cap \{\omega \in \Omega ; Y(\omega) = s_2\}$ is an event, that is, an element of σ-field $\mathcal{F}$ by the definiton of random variables and the axiom of $\mathcal{F}$.

[7] In this paper, we always focus on discrete-valued random variables, so that there is no difficulty concerning measurability of a real-valued function.

is no advantage in redefining the general noncooperative game by using random variables, but if one wants to relax the independence of the two players' strategies X and Y, some difficulties, e.g., how can a strategy profile (X, Y) whose distribution on $S = S_1 \times S_2$ is not determined by the marginal distributions of the strategies be characterized, arise.

Further, in order to describe the relationship between the random variables and their distributions, we need to define some notations for representing probability measures. $\mathcal{P}(M)$ denotes the set of all probability measures on a measurable space M. In this study, the set M is always a finite set, so $\mathcal{P}(M)$ is equal to the set of all vectors $\mathrm{p} = \{p_i\}_{i \in M}$ such that $p_i \geq 0$ for all $i \in M$ and $\sum_{i \in M} p_i = 1$. As is well known, a random variable X with values in M induces a probability measure on M ; the probability measure is called the distribution of X. We recall that for two elements σ_1, σ_2 in $\mathcal{P}(M)$ and for a real number t such that $0 \leq t \leq 1$, $t\sigma_1 + (1-t)\sigma_2$ is well defined[8] and is a probability measure on M, that is, an element of $\mathcal{P}(M)$. In general, for a finite subset $\mathcal{P}_0 = \{\sigma_1, ..., \sigma_n\}$ of $\mathcal{P}(M)$, the subset $\mathcal{P}_0^{ch} := \{\sum_{k=1}^{n} t_k \sigma_k \text{ ; } t_k \geq 0 \text{ and } \sum_{k=1}^{n} t_k = 1\}$ of $\mathcal{P}(M)$ is called *the convex hull* of $\mathcal{P}_0$, that is, although the set $\mathcal{P}(M)$ is not a linear space, we can define the convex hull of its subset.

Now, we can state the equivalent definition of a Nash equilibrium by using a probability measure as usual.

Definition 2′. A pair of probability measures $\{x_i^*\}_{i \in S_1} \in \mathcal{P}(S_1)$ and $\{y_j^*\}_{j \in S_2} \in \mathcal{P}(S_2)$ is a Nash equilibrium if and only if

$$\text{(i)} \quad \sum_{i \in S_1} \sum_{j \in S_2} u_1(i, j) x_i^* y_j^* \geq \sum_{i \in S_1} \sum_{j \in S_2} u_1(i, j) x_i y_j^*$$

holds for all $\{x_i\}_{i \in S_1} \in \mathcal{P}(S_1)$, and

$$\text{(ii)} \quad \sum_{i \in S_1} \sum_{j \in S_2} u_2(i, j) x_i^* y_j^* \geq \sum_{i \in S_1} \sum_{j \in S_2} u_2(i, j) x_i^* y_j$$

holds for all $\{y_i\}_{i \in S_2} \in \mathcal{P}(S_2)$.

Remark 1. Recall that the probability measure $\{x_i^* y_j^*\}_{i \in S_1, j \in S_2}$ on S is the distribution of the strategy profile (X^*, Y^*) in Definition 2.

We use $\mathcal{D}_{Nash}(S)$ to denote the set of distributions on S induced by all Nash equilibria of the noncooperative game Γ defined in Definition 2. It is well known that $\mathcal{D}_{Nash}(S)$ is a nonempty subset of $\mathcal{P}(S)$. We sometimes identify a strategy profile $(X, Y) \in \mathcal{R}(S)$ and its distribution on S. The representation of a Nash equilibrium involving random variables is helpful for a formal description of the

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[8] $\sigma_t(A) := t\sigma_1(A) + (1-t)\sigma_2(A)$ for all measurable subsets A of M defines a new probability measure on M.

equilibrium concept, whereas the representation involving distributions induced by the random variables is useful for obtaining the actual equilibrium points.

3. Formulation of a Noncooperative Game with a Mediator

In this section we will introduce a noncooperative game with a mediator; the equilibrium concept of the noncooperative game with a mediator seems to be closely related to Aumann's (1974) "correlated equilibrium" or Vanderschraaf's (1995a) "exogenous correlated equilibrium".

Let T be a finite set of choices for the third person, a mediator[9], and let $\mathcal{R}(T)$ be a set of all T-valued random variables. A T-valued random variable could be interpreted as the mediator's suggestion.

Let us first formulate a framework for a noncooperative game with a mediator. First, let us fix $Z \in \mathcal{R}(T)$, and let $\mathcal{R}_Z(S)$ be the subset of $\mathcal{R}(S)$ that satisfies Condition A[10].

Condition A: $(X, Y) \in \mathcal{R}_Z(S)$ is conditionally independent relative to Z[11], that is,

$$P(X=i,\ Y=j/Z=k) = P(X=i/Z=k)P(Y=j/Z=k)$$

holds for all $i \in S_1$, $j \in S_2$ and $k \in T$. Here, $P(A/B)$ refers to the conditional probability of an event $A \in \mathcal{F}$ relative to an event $B \in \mathcal{F}$ defined by $P(A/B) = \dfrac{P(A, B)}{P(B)}$. When $P(B)=0$, then $P(A/B)$ is not defined or is supposed to be zero for convenience.

Now, we can formulate the equilibrium concept of a noncooperative game with a mediator.

Definition 3. A game Γ is called a noncooperative game with a mediator who suggests Z if and only if any strategy profile $(X, Y) \in \mathcal{R}(S)$ belongs to the set

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9) In a mathematical model, it is not essential to know whether a mediator is a decision maker (such as an arbiter of Luce and Raiffa (1957, p.121), or just an objective chance mechanism of Aumann (1974, p.72). In this sense, it might be better to call our model a "noncooperative game with exogenous correlation", adopting Vanderschraaf's (1995a) terminology.

10) Fudenberg and Tirole's (1991, p.53) following statement may describe our situation. "Now consider players who may engage in preplay discussion, but then go off to isolated rooms to choose their strategies."

11) cf. Loève (1955, p.351) and Chung (1968, p.284), where the conditioning σ-field is a σ-field generated by the partition $\{\omega\ ;\ Z(\omega)=k\}$, $k \in T$. As for a more elementary textbook, see, for example, Stirzaker (2003, p.57). In our case, independence or conditional independence is a rather elementary notion.

$\mathcal{R}_Z(S)$.

Definition 4. A strategy profile $(X^*, Y^*)\in\mathcal{R}_Z(S)$ is an equilibrium of the noncooperative game with a mediator who suggests Z if and only if

$$\text{(i)} \quad E[u_1(X^*, Y^*)]\geq E[u_1(X, Y^*)]$$

holds for all $X\in\mathcal{R}(S_1)$ such that $(X, Y^*)\in\mathcal{R}_Z(S)$, and

$$\text{(ii)} \quad E[u_2(X^*, Y^*)]\geq E[u_2(X^*, Y)]$$

holds for all $Y\in\mathcal{R}(S_2)$ such that $(X^*, Y)\in\mathcal{R}_Z(S)$.

We shall call this equilibrium an *exogenous correlated equilibrium* relative to Z, adopting the terminology used by Vanderschraaf (1995a).

Now, we propose our theorem.

Theorem 1. The set $\mathcal{D}_{Z,exo}(S)$ of the distributions induced by all the exogenous correlated equilibria relative to Z is

$$\mathcal{D}_{Z,exo}(S)=\{\sum_{k\in T} z_k\sigma_k\ ;\ \sigma_k\in\mathcal{D}_{Nash}(S),\ k\in T\},$$

where $z_k:=P(Z=k)$, $k\in T$.

As for the proof of Theorem 1, see Section 6.

Remark 2. By taking account of all the same $\sigma=\sigma_k$, $k\in T$ we have the inclusion $\mathcal{D}_{Nash}(S)\subset\mathcal{D}_{Z,exo}(S)$.

Remark 3. Theorem 1 suggests that any point of the convex hull of $\mathcal{D}_{Nash}(S)$ can be achieved by using an exogenous correlated equilibrium relative to some mediator who gives an appropriate suggestion. Aumann's following statements seem to be very similar to Theorem 1. However, we define the equilibrium for each Z. On the other hand, it is not clear how Aumann's definition of an equilibrium concept depends on the "randomizing structure."

> (Aumann 1974, p.78) In the case of two-person games, when the randomizing structure is standard, then it is easily verified that the set of equilibrium payoffs is precisely the convex hull of the Nash equilibrium payoffs.
> (Aumann 1987, p.4) By similar methods, it may be seen that any convex combination of Nash equilibria is a correlated equilibrium.

4. Formulation of Endogenous Correlated Equilibrium

In Section 3, we introduced the equilibrium concept of a noncooperative game with a mediator. In this section, we will characterize another equilibrium concept of a noncoopearative game in which the players receive "endogenous" correlated signals (Fudenberg and Tirole, 1991, pp.58-59).

Definition 5. A game Γ is called a noncooperative game with an endogenous correlation if and only if any strategy profile $(X, Y) \in \mathcal{R}(S)$ has no restrictions.

Definition 6. A strategy profile $(X^*, Y^*) \in \mathcal{R}(S)$ is an equilibrium of the noncooperative game with an endogenous correlation if and only if

$$\text{(i)} \quad E[u_1(X^*, Y^*)] \geq E[u_1(X, Y^*)]$$

holds for all $X \in \mathcal{R}(S_1)$, and

$$\text{(ii)} \quad E[u_2(X^*, Y^*)] \geq E[u_2(X^*, Y)]$$

holds for all $Y \in \mathcal{R}(S_2)$.

We shall call this equilibrium an *endogenous correlated equilibrium*, adopting the terminology used by Vanderschraaf (1995a).

Remark 4. Although any distribution on S can be obtained as a distribution of a certain strategy profile $(X, Y) \in \mathcal{R}_Z(S)$ by a properly chosen mediator suggestion Z, the exogenous correlated equilibrium and the endogenous correlated equilibrium are entirely different concepts, as shown by Theorems 1 and 2.

Set

$$U_{1j} := \{i \in S_1 \,;\, \max_{l \in S_1} u_1(l, j) = u_1(i, j)\},$$
$$U_{2i} := \{j \in S_2 \,;\, \max_{l \in S_2} u_2(i, l) = u_2(i, j)\}, \text{ and}$$
$$S_0 := \{(i, j) \,;\, U_{1j} \ni i \text{ holds if and only if } j \in U_{2i}\}.$$

We note that S_0 coincides with the set of all pure Nash equilibria of the noncooperative game Γ.

Theorem 2. The set of distributions induced by all the endogenous correlated equilibria coincides with $\mathcal{P}(S_0)$, that is, it coincides with the convex hull $\mathcal{D}^{ch}_{pure\ Nash}(S)$ of the set of distributions $\mathcal{D}_{pure\ Nash}(S)$ of all the pure Nash equilibria.

Corollary. When a noncooperative game Γ has no pure Nash equilibrium, then there does not exist any endogenous correlated equilibrium.

As for the proofs of Theorem 2 and the corollary, see Section 6.

As shown in Theorem 2, the set of endogenous correlated equilibria has three types of sets: (1) an infinite number of equilibria, in the case where more than one pure Nash equilibrium exists, (2) just one pure Nash equilibrium, or (3) an empty set, in the case where no pure Nash equlibrium exists. On the other hand, for each Z, the set of all the exogenous correlated equilibria relative to Z is always a nonempty set, and also a finite set if the noncooperative game has a finite number of Nash equilibria.

5. An Example

We illustrate our Theorems with an example given in Fudenberg and Tirole's textbook (p.54) which can be referred to as battle of sexes whose pay-off matrix is given below. Here Player 1 picks the row, while Player 2 selects the column. The numeral on the left side in the parenthesis represents Player 1's pay-off and that on the right side denotes Player 2's pay-off.

There are three Nash equilibria whose distributions on $S=S_1\times S_2$ can be represented as follows; where a 2×2 matrix (p_{ij}) means $p_{ij}:=P(X=i,\ Y=j)$.

$$\sigma_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\ \sigma_3=\begin{pmatrix}1/4 & 1/4\\ 1/4 & 1/4\end{pmatrix}.$$

When a mediator joins the game with his or her choice set $T=\{head,\ tail\}$, then he/ she can get the following three essentially different distributions as the exogenous equilibria relative to Z:

$$\sigma_4=\begin{pmatrix}1-3z_i/4 & z_i/4\\ z_i/4 & z_i/4\end{pmatrix}=z_i\sigma_3+(1-z_i)\sigma_1,$$

$$\sigma_5=\begin{pmatrix}z_i/4 & z_i/4\\ z_i/4 & 1-3z_i/4\end{pmatrix}=z_i\sigma_3+(1-z_i)\sigma_2,$$

$$\sigma_6=\begin{pmatrix}z_i & 0\\ 0 & 1-z_i\end{pmatrix}=z_i\sigma_1+(1-z_i)\sigma_2.$$

Table 1 Pay-off matrix of the game

	1	2
1	(5, 1)	(0, 0)
2	(4, 4)	(1, 5)

Here, $i=1$ or 2 and $z_1 := P(Z=head)$, *and* $z_2 := P(Z=tail)$. (Z denotes a trial of a coin flip, and is not necessarily fair.)

On the other hand, the set of distributions of all the endogenous correlated equilibria is represented as follows:

$$\left\{\begin{pmatrix} z & 0 \\ 0 & 1-z \end{pmatrix} = z\sigma_1 + (1-z)\sigma_2 \ ;\ 0 \leq z \leq 1\right\}.$$

Remark 5. By using Aumann's (1974) suggestion, Fudenberg and Tirole (1991, p.54) show that the players can perform better if they use the following distribution for a strategy profile:

$$\sigma = \begin{pmatrix} 1/3 & 0 \\ 1/3 & 1/3 \end{pmatrix}.$$

Fudenberg and Tirole do not give the precise definition of the quilibrium concept from which this distribution follows. To define the equilibrium concept, the framework of a strategy space available for the given game must be defined clearly; however, they do not define it. In the coming paper, we will generalize our framework and obtain one that makes this distribution an equilibrium.

The above distribution can be represented by a strategy profile (X, Y) belonging to $\mathcal{R}_Z(S)$ or, of course, belonging to $\mathcal{R}(S)$; however, Theorems 1 and 2 show that it does not represent either an exogenous correlated equilibrium relative to any Z or an endogenous correlated equilibrium.

Let us first show how σ is not an exogenous correlated equilibrium distribution relative to Z by performing a fair coin flip, that is, $P(Z=1)=1/2$, $P(Z=2)=1/2$. (X^*, Y^*) denotes a random variable whose distribution is σ; further, set

$$x^*_{1/k} := P(X^*=1/Z=k),\ y^*_{1/k} := P(Y^*=1/Z=k),\ k=1, 2.$$

Since (X^*, Y^*) must belong to $\mathcal{R}_Z(S)$ and

$$P(X^*=2/Z=k) = 1-x^*_{1/k},\ P(Y^*=2/Z=k) = 1-y^*_{1/k},\ k=1, 2,$$

applying Condition A, we have

$$\sigma_{11} = P(X^*=1,\ Y^*=1) = \frac{1}{2}x^*_{1/1}y^*_{1/1} + \frac{1}{2}x^*_{1/2}y^*_{1/2} = \frac{1}{3} \tag{1}$$

$$\sigma_{12} = P(X^*=1,\ Y^*=2) = \frac{1}{2}x^*_{1/1}(1-y^*_{1/1}) + \frac{1}{2}x^*_{1/2}(1-y^*_{1/2}) = 0 \tag{2}$$

$$\sigma_{21} = P(X^*=2,\ Y^*=1) = \frac{1}{2}(1-x^*_{1/1})y^*_{1/1} + \frac{1}{2}(1-x^*_{1/2})y^*_{1/2} = \frac{1}{3} \tag{3}$$

$$\sigma_{22}=P(X^*=2,\ Y^*=2)=\frac{1}{2}(1-x^*_{1/1})(1-y^*_{1/1})+\frac{1}{2}(1-x^*_{1/2})(1-y^*_{1/2})$$
$$=\frac{1}{3}. \tag{4}$$

It follows from the equation (2) that

$$x^*_{1/1}(1-y^*_{1/1})=0 \text{ and } x^*_{1/2}(1-y^*_{1/2})=0 \tag{5}$$

Four possible cases can be considered for the solution of equation (5): (1) $x^*_{1/1}=0$ and $y^*_{1/2}=1$, (2) $x^*_{1/2}=0$ and $y^*_{1/1}=1$, (3) $x^*_{1/1}=0$ and $x^*_{1/2}=0$, and (4) $y^*_{1/1}=1$ and $y^*_{1/2}=1$. Upon inspection, it is clear that the last two cases are incompatible with the other equations, while the following two cases are compatible with equations (1) to (4).

Case 1. $x^*_{1/1}=0$, $x^*_{1/2}=2/3$, $y^*_{1/1}=1/3$, $y^*_{1/2}=1$, or

Case 2. $x^*_{1/1}=2/3$, $x^*_{1/2}=0$, $y^*_{1/1}=1$, $y^*_{1/2}=1/3$.

First let us check case 1. In this case, set $y_{1/1}:=0$ and $y_{1/2}:=1$ and $x^*_{1/1}$ and $x^*_{1/2}$ are unchanged. These conditional distributions are the conditional distributions of $(X^*, Y)\in\mathcal{R}_Z(S)$ relative to Z for some $Y\in\mathcal{R}(S_2)$. Then, by calculation, the distribution on S induced by (X^*, Y) is

$$\begin{pmatrix} 1/3 & 0 \\ 1/6 & 1/2 \end{pmatrix},$$

and we have

$$\frac{10}{3}=E[u_2(X^*, Y^*)]<\frac{7}{2}=E[u_2(X^*, Y)],$$

which implies that (X^*, Y^*) is not an exogenous correlated equilibrium relative to Z.

As for the second case, set $x_{1/1}:=1$ and $x_{1/2}:=0$ and $y^*_{1/1}$ and $y^*_{1/2}$ are unchanged. These conditional distributions are the conditional distributions of $(X, Y^*)\in\mathcal{R}_Z(S)$ relative to Z for some $X\in\mathcal{R}(S_1)$. Then, by calculation, the distribution on S induced by (X, Y^*) is

$$\begin{pmatrix} 1/2 & 0 \\ 1/6 & 1/3 \end{pmatrix},$$

and we have

$$\frac{10}{3}=E[u_1(X^*, Y^*)]<\frac{7}{2}=E[u_1(X, Y^*)],$$

which implies that (X^*, Y^*) is not an exogenous correlated equilibrium relative to Z.

Finally, we will show how the distribution σ is not an endogenous correlated equilibrium. Again, let (X^*, Y^*) denote a random variable whose distribution is σ and set $x_1^* := P(X^*=1)=1/3$, $x_2^* := P(X^*=2)=2/3$, $y_{1/1}^* := P(Y^*=1/X^*=1)=1$, and $y_{1/2}^* := P(Y^*=1/X^*=2)=1/2$. Recall that the distribution

$$\sigma_0 := \begin{pmatrix} 1/3 & 0 \\ 1/6 & 1/2 \end{pmatrix}.$$

Since the marginal distribution of σ_0 on S_1 is just the same as X^*, we can choose a random variable $(X^*, Y) \in \mathcal{R}(S)$ whose conditional probabilities relative to X^* are $y_{1/1} := P(Y=1/X^*=1)=1$ and $y_{1/2} := P(Y=1/X^*=2)=1/4$, that is, the distribution of (X^*, Y) is equal to σ_0.

Then, obviously, we have

$$\frac{10}{3} = E[u_2(X^*, Y^*)] < \frac{7}{2} = E[u_2(X^*, Y)],$$

which implies that (X^*, Y^*) is not an endogenous correlated equilibrium.

6. Mathematical Proofs of Theorems 1 and 2

First, we will start with the proof of Theorem 1. Without loss of generality, we assume that $P(Z=k)=z_k>0$ for all $k \in T$.

Though the definition of the exogenous equilibrium in terms of random variables is simple, in order to obtain the strategy profiles in equilibrium, we have to represent the conditions (i) and (ii) of Definition 4 by using the distributions of (X^*, Y^*), (X, Y^*), and (X^*, Y) on S, which are all elements of $\mathcal{R}_Z(S)$.

Now, the joint probability $P(X=i, Y=j) =: p_{ij}$ for $(X, Y) \in \mathcal{R}_Z(S)$ is calculated by virtue of Condition A as follows:

$$\begin{aligned} p_{ij} &= \sum_{k \in T} P(X=i, Y=j, Z=k) \\ &= \sum_{k \in T} P(X=i, Y=j/Z=k) z_k \\ &= \sum_{k \in T} P(X=i/Z=k) P(Y=j/Z=k) z_k \quad \text{(by applying Condition A).} \end{aligned}$$

For (X^*, Y^*), (X, Y^*), and (X^*, Y) in Definition 4, set

$$x_{i/k}^* := P(X^*=i/Z=k),\ x_{i/k} := P(X=i/Z=k),\ i \in S_1,\ k \in T,$$

and

$$y^*_{j/k}:=P(Y^*=j/Z=k),\ y_{j/k}:=P(Y=j/Z=k),\ j\in S_2,\ k\in T.$$

Since a simple calculation yields

$$\begin{aligned}E[u_1(X^*,\ Y^*)]&=\sum_{i\in S_1}\sum_{j\in S_2}u_1(i,\ j)P(X^*=i,\ Y^*=j)\\&=\sum_{k\in T}\sum_{i\in S_1}\sum_{j\in S_2}u_1(i,\ j)x^*_{i/k}y^*_{j/k}z_k\end{aligned}$$

and

$$\begin{aligned}E[u_2(X^*,\ Y^*)]&=\sum_{i\in S_1}\sum_{j\in S_2}u_2(i,\ j)P(X^*=i,\ Y^*=j)\\&=\sum_{k\in T}\sum_{i\in S_1}\sum_{j\in S_2}u_2(i,\ j)x^*_{i/k}y^*_{j/k}z_k,\end{aligned}$$

conditions (i) and (ii) of Definition 4 turn out to be the following:

$$\text{(i)}\quad \sum_{k\in T}\sum_{i\in S_1}\sum_{j\in S_2}u_1(i,\ j)x^*_{i/k}y^*_{j/k}z_k\geq\sum_{k\in T}\sum_{i\in S_1}\sum_{j\in S_2}u_1(i,\ j)x_{i/k}y^*_{j/k}z_k \tag{6}$$

holds for all $\{x_{i/k}\}_{i\in S_1}\in\mathcal{P}(S_1)$ and for all $k\in T$, and

$$\text{(ii)}\quad \sum_{k\in T}\sum_{i\in S_1}\sum_{j\in S_2}u_2(i,\ j)x^*_{i/k}y^*_{j/k}z_k\geq\sum_{k\in T}\sum_{i\in S_1}\sum_{j\in S_2}u_2(i,\ j)x^*_{i/k}y_{j/k}z_k \tag{7}$$

holds for all $\{y_{j/k}\}_{j\in S_2}\in\mathcal{P}(S_2)$ and for all $k\in T$.

Since each player can arbitrarily determine his or her distribution $\{x_{i/k}\}_{i\in S_1}\in\mathcal{P}(S_1)$ in (6) and $\{y_{j/k}\}_{j\in S_2}\in\mathcal{P}(S_2)$ in (7) for each $k\in T$ and $z_k>0$, we have the following lemma.

Lemma 1.

Condition (i) of Definition 4 is equivalent to the following condition.

(i) For each $k\in T$,

$$\sum_{i\in S_1}\sum_{j\in S_2}u_1(i,\ j)x^*_{i/k}y^*_{j/k}\geq\sum_{i\in S_1}\sum_{j\in S_2}u_1(i,\ j)x_iy^*_{j/k} \tag{8}$$

holds for all $\{x_i\}_{i\in S_1}\in\mathcal{P}(S_1)$. Condition (ii) of Definition 4 is equivalent to the following condition.

(ii) For each $k\in T$,

$$\sum_{i\in S_1}\sum_{j\in S_2}u_2(i,\ j)x^*_{i/k}y^*_{j/k}\geq\sum_{i\in S_1}\sum_{j\in S_2}u_2(i,\ j)x^*_{i/k}y_j \tag{9}$$

holds for all $\{y_j\}_{j\in S_2}\in\mathcal{P}(S_2)$.

The final step of the proof of Theorem 1: Looking at Definition **2′** and Remark 1, inequalities (8) and (9) are the necessary and sufficient conditions for the distribution $\sigma_k := \{x^*_{i/k} y^*_{j/k}\}_{i\in S_1,\, j\in S_2} \in \mathcal{P}(S)$ to be a Nash equilibrium of the noncooperative game Γ for each $k \in T$. Hence, taking into account the inequalities (6) and (7), we conclude

$$\sigma = \sum_{k\in T} z_k \sigma_k$$

to be the distribution of (X^*, Y^*) in equilibrium according to Definition 4. This completes the proof of Theorem 1.

In order to prove Theorem 2, we also have to restate Definition 6 by using the distributions of (X^*, Y^*), (X^*, Y), and (X, Y^*).

Set

$$\begin{aligned}
&x^*_i := P(X^* = i),\ y^*_j := P(Y^* = j),\\
&x^*_{i/j} := P(X^* = i/Y^* = j) \text{ if } y^*_j > 0\ (:= 0 \text{ if } y^*_j = 0),\\
&y^*_{j/i} := P(Y^* = j/X^* = i) \text{ if } x^*_i > 0\ (:= 0 \text{ if } x^*_i = 0),\\
&x_{i/j} := P(X = i/Y^* = j) \text{ if } y^*_j > 0\ (:= 0 \text{ if } y^*_j = 0) \text{ and}\\
&y_{j/i} := P(Y = j/X^* = i) \text{ if } x^*_i > 0\ (:= 0 \text{ if } x^*_i = 0).
\end{aligned}$$

We note that the distribution of (X^*, Y^*) is uniquely determined by $\{x^*_{i/j} y^*_j\}_{i\in S_1,\, j\in S_2}$ or $\{x^*_i y^*_{j/i}\}_{i\in S_1,\, j\in S_2}$. Therefore, we can write

$$P(X^* = i,\ Y^* = j) = x^*_{i/j} y^*_j = x^*_i y^*_{j/i}.$$

Then, conditions (i) and (ii) of Definition 6 turn out to be the following.

Condition (i) of Definition 6 is equivalent to stating that

$$\text{(i)} \quad \sum_{i\in S_1}\sum_{j\in S_2} u_1(i, j) x^*_{i/j} y^*_j \geq \sum_{i\in S_1}\sum_{j\in S_2} u_1(i, j) x_{i/j} y^*_j \tag{10}$$

holds for all $\{x_{i/j}\}_{i\in S_1} \in \mathcal{P}(S_1)$ and $j \in S_2$, where $\{x_{i/j} y^*_j\}_{i\in S_1,\, j\in S_2}$ is the distribution of (X, Y^*) on S.

Condition (ii) of Definition 6 is equivalent to stating that

$$\text{(ii)} \quad \sum_{i\in S_1}\sum_{j\in S_2} u_2(i, j) x^*_i y^*_{j/i} \geq \sum_{i\in S_1}\sum_{j\in S_2} u_2(i, j) x^*_i y_{j/i} \tag{11}$$

holds for all $\{y_{j/i}\}_{j\in S_2} \in \mathcal{P}(S_2)$ and $i \in S_1$, where $\{x^*_i y_{j/i}\}_{i\in S_1,\, j\in S_2}$ is the distribution of (X^*, Y) on S.

Since each player can arbitrarily determine his or her distribution $\{x_{i/j}\}_{i \in S_1} \in \mathcal{P}(S_1)$ for each j such that $y_j^* > 0$ in (10) and $\{y_{j/i}\}_{j \in S_2} \in \mathcal{P}(S_2)$ for each i such that $x_i^* > 0$ in (11), we have the following lemma.

Lemma 2.

Condition (i) of Definition 6 is equivalent to stating that

$$\text{(i)} \quad \text{for each } j \text{ such that } y_j^* > 0, \quad \sum_{i \in S_1} u_1(i, j) x_{i/j}^* \geq \sum_{i \in S_1} u_1(i, j) x_i \tag{12}$$

holds for all $\{x_i\}_{i \in S_1} \in \mathcal{P}(S_1)$.

Condition (ii) of Definition 6 is equivalent to stating that

$$\text{(ii)} \quad \text{for each } i \text{ such that } x_i^* > 0, \quad \sum_{j \in S_2} u_2(i, j) y_{j/i}^* \geq \sum_{j \in S_2} u_2(i, j) y_j \tag{13}$$

holds for all $\{y_j\}_{j \in S_2} \in \mathcal{P}(S_2)$.

Proof of Theorem 2. First, take any distribution $\{p_{ij}\}_{i \in S_1, j \in S_j} \in \mathcal{P}(S)$ such that $\{(i, j)\ ;\ p_{ij} > 0\} \subset S_0$, that is, $\{p_{ij}\}_{i \in S_1, j \in S_j}$ belongs to $\mathcal{P}(S_0)$.

Set $y_j^* := \sum_{i \in S_1} p_{ij}$ and $x_{i/j}^* := p_{ij}/y_j^*$ if $y_j^* > 0$, otherwise $x_{i/j}^* := 0$. Obviously, it follows that

$$\max_{l \in S_1} u_1(l, j) \geq u_1(i, j)$$

holds for all $i \in S_1$. Since $\sum_{k \in U_{1j}} x_{k/j}^* = 1$ for $y_j^* > 0$, from the definition of U_{1j} and S_0, for each j such that $y_j^* > 0$, we have

$$\begin{aligned} \max_{l \in S_1} u_1(l, j) &= \sum_{k \in U_{1j}} (\max_{l \in U_{1j}} u_1(l, j)) x_{k/j}^* \\ &= \sum_{k \in U_{1j}} u_1(k, j) x_{k/j}^* \\ &= \sum_{k \in S_1} u_1(k, j) x_{k/j}^*. \end{aligned}$$

Combining the previous estimation, it follows that

$$\sum_{k \in S_1} u_1(k, j) x_{k/j}^* \geq u_1(i, j)$$

holds for all $i \in S_1$. Multiply the both sides of the above inequality by x_i of any distribution $\{x_i\}_{i \in S_1} \in \mathcal{P}(S_1)$ and sum with respect to i. Then, it follows from $\sum_{i \in S_1} x_i = 1$ that

$$\sum_{i \in S_1} u_1(i, j) x^*_{i/j} \geq \sum_{i \in S_1} u_1(i, j) x_i.$$

This inequality is same as the inequality (12) of Lemma 2. We can check inequality (13) of Lemma 2 in a similar manner. This means that any distribution on S_0 is an equilibrium according to Definition 6.

Conversely, if a distribution $\{p_{ij}\}_{i \in S_1, j \in S_2}$ of (X^*, Y^*) does not belong to $\mathcal{P}(S_0)$, then there exists at least one pair $(i_0, j_0) \notin S_0$ such that $p_{i_0 j_0} > 0$, which implies $i_0 \notin U_{1j_0}$ or $j_0 \notin U_{2i_0}$, $x^*_{i_0} > 0$, and $y^*_{j_0} > 0$.

First, we assume $i_0 \notin U_{1j_0}$. Since, from the definition of U_{1j_0},

$$\max_{l \in S_1} u_1(l, j_0) > u_1(i_0, j_0)$$

holds, it follows that

$$\sum_{i \in S_1} u_1(i, j_0) x^*_{i/j_0} < \sum_{i \in S_1,\, i \neq i_0} u_1(i, j_0) x^*_{i/j_0} + \max_{l \in S_1} u_1(l, j_0) x^*_{i_0/j_0}.$$

Now, let $\max_{l \in S_1} u_1(l, j_0) = u_1(i_1, j_0)$, and define a new probability measure $\{x_i\}_{i \in S_1}$ on S_1 by $x_{i_1} := x^*_{i_1/j_0} + x^*_{i_0/j_0}$, $x_{i_0} := 0$. All other components are unchanged with respect to x^*_{i/j_0}. Then, obviously, the above inequality yields

$$\sum_{i \in S_1} u_1(i, j_0) x^*_{i/j_0} < \sum_{i \in S_1} u_1(i, j_0) x_i.$$

From condition (i) of Lemma 2, this inequality implies that (X^*, Y^*) is not an equilibrium in the sense of Definition 6.

Second, we assume $j_0 \notin U_{2i_0}$. Since, from the definition of U_{2i_0},

$$\max_{l \in S_2} u_2(i_0, l) > u_2(i_0, j_0)$$

holds, it follows that

$$\sum_{j \in S_2} u_2(i_0, j) y^*_{j/i_0} < \sum_{j \in S_2,\, j \neq j_0} u_2(i_0, j) y^*_{j/i_0} + \max_{l \in S_2} u_2(i_0, l) y^*_{j_0/i_0}.$$

Now, let $\max_{l \in S_2} u_2(i_0, l) = u_2(i_0, j_1)$, and define a new probability measure $\{y_j\}_{j \in S_2}$ on S_2 by $y_{j_1} := y^*_{j_1/i_0} + y^*_{j_0/i_0}$, $y_{j_0} := 0$. All other components are unchanged with respect to y^*_{j/i_0}. Then, obviously, the above inequality yields

$$\sum_{j \in S_2} u_2(i_0, j) y^*_{j/i_0} < \sum_{j \in S_2} u_2(i_0, j) y_j.$$

From condition (ii) of Lemma 2, this inequality implies that (X^*, Y^*) is not an equilibrium in the sense of Definition 6.

We note that the above proof is valid even if set S_0 is empty; this completes the proof of Theorem 2 and the corollary.

7. Some Remarks or Comments

1. Vanderschraaf (1995a, p.82) says "This (Luce and Raiffa, 1957) is the earliest discussion of a correlated equilibrium that I am aware of Stergious Sapardas first alerted me to this section of Game and Decisions."

Although Luce and Raiffa (1957, p.89) classify a strategic game providing complete freedom of preplay communication to have joint binding agreements as a cooperative game, in our view, it would be better to call such a game a noncooperative game under specific circumstances so that we can formulate such a game by relaxing the conditions regarding the players' choices to be independent.

2. Our exogenous correlated equilbrium concept (Definition 4) seems to correspond to the notion of Aumann's 1974 paper, and our endogeneous correlated equilibrium concept (Definition 6) obviously corresponds to that of Aumann's 1987 paper. However, page 6 of Aumann's 1987 paper says, "Definition 2.1 (in his paper) appears a little different from our 1974 definition of correlated equilibrium... it turns out, though, that practically speaking, the two definitions are equivalent."

Fudenberg and Tirole (1991) have given two different definitions of the correlated equilibrium concept. One is Definition 2.4A (in their book), which corresponded to that in Aumann's 1974 paper, and the other one is Definition 2.4B (in their book), which corresponded to that in Aumann's 1987 paper. They write on page 57 that "Let us explain why the two definitions of correlated equilibrium are equivalent. Clearly an equilibrium in the sense of definition 2.4B is an equilibrium according to definition 2.4A — just take $\Omega = S$, and $h_i(s) = \{s' \mid s'_i = s_i\}$. Conversely,..." Inspection of this proof seems to indicate that they are confused between the information structure $(\Omega, \{H_i\}, p)$ and the probability distribution over the pure strategy space $S = S_1 \times S_2$ in Definition 2.4B. The main problem in their formulations seems to be the inadequacy of the definition of mixed strategies, though they argue on page 56 that "Mixed strategies can be defined in the obvious way..."

In Definition 4, if we take Z as a trivial random variable, that is, $P(Z=k)=1$ for some $k \in T$, then our exogenous correlated equilibrium turns out to be the usual Nash equilibrium, for Condition A in this case is the usual independence of the strategies of the two players.

Fudenberg and Tirole's (1991) following ambiguous statement indicates that their "public correlating devices" could be equivalent to our "mediator."

> (Fudenberg and Tirole, 1991 p.58) Inspection of the definition shows that the set of correlated equilibria is convex, so the set of correlated equilibria is

at least as large as the convex hull of the Nash equilibria. This convexification could be attained by using only public correlating devices.

3. Rosenthal (1974, p.119) gave a definition of correlated equilibrium. Lemma 1 in the present paper is equivalent to his definition when it is assumed that $C_k = D_k = \{Z = k\}$ for all k in his paper, though he does not assume Condition A. He also argues that "through such correlated equilibria, any payoff in the convex hull of the Nash equilibrium payoffs can be achieved (p.120)." This statement is essentially equivalent to Theorem 1 in the present paper (see also Remark 3). In the next study, we shall generalize our formulation to include Rosenthal's model by using random variables in a similar manner to that in Definitions 3 and 4.

4. Moulin and Vial (1978) introduced a "correlation scheme" that seems to correspond to our endogenous correlation concept because of their argument (p.203) that "this correlation scheme does not require any commitment of the players. Equilibria can be viewed as self-enforcing agreements." They also give the necessary and sufficient conditions for a pair of passive strategies to be an equilibrium (p.203). Their condition also exactly reflects Aumann's Proposition 2.3 (1987, p.6). However, this proposition does not seem to be equivalent to Definition 2.1 (in his paper) of "correlated equilibrium."

5. Hart and Schmeidler (1989) proved the existence of correlated equilibria but used Aumann' (1987) criteria stated in Proposition 2.3 for proving the existence of a solution. Owen's textbook (1995, VIII. 4.1 Definition, p.184) and Calvó-Armengol (2006) also adopt this proposition as the definition of correlated equilibrium.

As is pointed out in 4 above, Proposition 2.3 is not an appropriate criterion for our endogenous correlated equilibria, and that is why the corollary of Theorem 2 and Hart and Schmeidler's (1989) argument are incompatible. Myerson (1986), Evangelista and Raghavan (1996), Mailath et al. (1997), Hart and MasColell (2000) and Nau et al. (2003) use Aumann's Proposition 2.3 or its n-player version as the definition of correlated equilibrium.

6. There are many other studies concerning "correlated equilibria," such as those of Myerson (1985), Moulin (1986, chapter 8), Brandenburger and Dekel (1987), Aumann and Brandenburger (1995), Vanderschraaf (1995a, b), Skyrms (1996, chapter 4), Neyman (1997) and Mandler (2007).

In these studies, investigations of extended Nash-type equilibria are not sufficiently clear. In order to define Nash-type equilibria, it is necessary to set up a framework that considers all strategy profiles of the players; it is also necessary to define how to control the situation in which each player is able to choose his or her strategy by deciding the strategies of other players without using the concept where the strategies of the different players are independent.

One reason why their definitions are unclear and inappropriate seems to be that their fundamental notion of randomness is not necessarily rigorous from a mathematical viewpoint. That is, their definitions of the concepts of random variables, marginal distributions, conditional probabilities, and conditional independence in multidimensional probability distributions are not well defined, and they use many intuitive explanations instead. For example, Vanderschraaf (1995a, p.68) says, "The endogenous correlated equilibrium concept generalizes the Nash equilibrium concept simply by dropping the assumptions that the actions of each player's opponents are probabilistically independent and...", but his formulation is nevertheless considerably different from ours.

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