Fundamental Studies on the Synthesis of Sampled-Data Control Systems

Hiroshi FUKAWA

March 1960

SUMMARY

Synthetic studies of sampled-data control system are dealt in this paper. The first five chapters are concerned with linear systems and the remainder with non-linear systems.

It is the main purpose of Chapter 1 to show an evaluating method of the quadratic control area of sampled-data control systems using the advanced Z transform method. As one of direct methods of evaluating the quadratic control area, the integrated-square error is also calculated in this chapter. The influence of the sampling period upon the control performance (stability, quadratic control area and integrated-square error) is discussed in Chapter 2 using results of Chapter 1. As the results, it can be shown that there exists the optimum value of the sampling period. This investigation is performed as the first step for the development of many point control systems.

The following three chapters deal with finite-settling-time systems. In Chapter 3 the synthesis of the systems by the method of minimizing the integrated-square error is discussed. The case of finite-settling-time system with a stationay random noise treated statistically by the R.S.M.-Criterion in Chapter 4. In addition, when the system designed statistically is exited by step, ramp and acceleration inputs, the integrated-square error of the system is also considered in this chapter. Chapter 5 is devoted to the fundamantal study of the finite-settling-time response. In order to make the mechanism of such response clear, the indicial response of the controlled element is investigated in the time domain.

In Chapter 6, the evaluating method of the response of non-linear sampled-data control systems is described and the method can be also applied to the continuous control system with non-linear elements.

The experimental results for response of the control system with the optimum non-linear controller using sampled-data is described in Chapter 7 and the method proposed in Chapter 6 is used for determining the optimum switching line of the system.

CONTENTS

Chapter 1

QUADRATIC CONTROL AREA OF SAMPLED DATA CONTROL SYSTEMS

1.1 Introduction

As a useful criterion for sampled data control systems, we often employ the method of the quadratic control area. In sampled data control systems, this quantity can be obtained by using the advanced Z transform method.

Let $e(t)$ be the control error, then the quadratic control area is of the form

$$\int_0^{\infty}\{e(t)\}^2\,dt \tag{1.1}$$

This integral can be expressed as follows:

$$\sum_{n=0}^{\infty}\int_{nt_0}^{(n+1)t_0}\{e(t)\}^2\,dt \tag{1.2}$$

where t_0 is the sampling period. Transforming the integral variable t to Δ by $t = nt_0 + \Delta t_0$ ($1 \geq \Delta \geq 0$) and exchanging the order of the integration and summation, the expression (1.2) becomes

$$\int_0^1 t_0 \sum_{n=0}^{\infty}\{e(nt_0+\Delta t_0)\}^2 d\Delta \tag{1.3}$$

Let $E(z, \Delta)$ be the advanced Z transform of the control error, then the integrand of (1.3) can be expressed in the complex integral form as follows:(1)

$$Q(\Delta) \equiv t_0 \sum_{n=0}^{\infty}\{e(nt_0+\Delta t_0)\}^2 = \frac{t_0}{2\pi j}\int_C \frac{E(Z,\Delta)\,E(Z^{-1},\Delta)}{Z}\,dZ \tag{1.4}$$

where C is the unit circle in the Z plane. Therefore, the quadratic control area can be calculated by the following integral

$$\int_0^{\infty}\{e(t)\}^2\,dt = \int_0^1 Q(\Delta)\,d\Delta \tag{1.5}$$

1.2 Evaluation of the complex integral

The complex integral (1.4) can be evaluated only when the system is stable. Generally $E(z, \triangle)$ is a ratio of polynomials in Z, and the order of the numerator is lower than or equal to that of the denominator.

Therefore, the integrand of (1.4) is of the form

$$\frac{E(Z, \triangle) \cdot E(Z^{-1}, \triangle)}{Z} = \frac{g_n(Z)}{f_n(Z) \cdot f_n^*(Z)} \tag{1.6}$$

where $f_n(Z)$ is a polynomial of order n, $f_n^*(Z) = Z^n f_n\left(\frac{1}{Z}\right)$ and $g_n(Z)$ is a polynomial of order 2n-2. In the stable system, all zeroes of $f_n(Z)$ lies inside of the unit circle in the Z plane, therefore all those of $f_n^*(Z)$ lies outside of the unit circle.

The right-hand side of (1.6) can be expressed by the summation of two rational functions as follows:

$$\frac{g_n(Z)}{f_n(Z) \cdot f_n^*(Z)} = \frac{h_n(Z)}{f_n(Z)} + \frac{h'_n(Z)}{f_n^*(Z)} \tag{1.7}$$

where $h_n(Z)$ and $h'_n(Z)$ are both polynomials of order n-1 or lower. As $h'_n(Z)/f_n^*(Z)$ is regular in $|Z| \leq 1$, the integration of $h'_n(Z)/f_n^*(Z)$ along the unit circle vanishes. Therefore,

$$I_n = \frac{1}{2\pi j} \int_c \frac{g_n(Z)}{f_n(Z) f_n^*(Z)} \, dZ = \frac{1}{2\pi j} \int_c \frac{h_n(Z)}{f_n(Z)} \, dZ \tag{1.8}$$

As the singularities of $h_n(z)/f_n(z)$ all lie inside the unit circle in the Z plane, the integration (1.8) becomes

$$I_n = \sum_{\nu=1}^{n} R(\alpha_\nu) \tag{1.9}$$

where $R(\alpha_\nu)$ is the residue at α_ν which is a zero of polynomial $f_n(Z)$.

On the other hand, the total sum of residues of a rational function vanishes. Let $R(\infty)$ be the residue of $h_n(Z)/f_n(Z)$ at $Z = \infty$, then

$$I_n = \sum_{\nu=1}^{n} R(\alpha_\nu) = -R(\infty) = \lim_{Z\to\infty} Z \frac{h_n(Z)}{f_n(Z)} \tag{1.10}$$
$$= \frac{\text{coefficient of } Z^{n-1} \text{ of } h_n(Z)}{\text{coefficient of } Z^n \text{ of } f_n(Z)}$$

Therefore, we have

$$I_n = \frac{1}{2\pi j} \int_c \frac{g_n(Z)}{f_n(Z)\, f_n^*(Z)}\, dZ = \frac{\text{coef. of } Z^{n-1} \text{ of } h_n(Z)}{\text{coef. of } Z^n \text{ of } f_n(Z)} \tag{1.11}$$

Now, let us evaluate the integral I_n when

$$\left.\begin{aligned} f_n(Z) &= a_0 Z^n + a_1 Z^{n-1} + \cdots\cdots + a_{n-1} Z + a_n \\ f_n^*(Z) &= a_n Z^n + a_{n-1} Z^{n-1} + \cdots\cdots + a_1 Z + a_0 \\ g_n(Z) &= b_0 Z^{2n-2} + b_1 Z^{2n-3} + \cdots\cdots b_{2n-3} Z + b_{2n-2} \end{aligned}\right\} \tag{1.12}$$

If

$$\left.\begin{aligned} h_n(Z) &= p_0 Z^{n-1} + p_1 Z^{n-2} + \cdots\cdots + p_{n-1} \\ h_n'(Z) &= q_0 Z^{n-1} + q_1 Z^{n-2} + \cdots\cdots + q_{n-1} \end{aligned}\right\} \tag{1.13}$$

then, according to (1.10),

$$I_n = \frac{1}{2\pi j} \int_c \frac{g_n(Z)}{f_n(Z) \cdot f_n^*(Z)}\, dZ = \frac{p_0}{a_0} \tag{1.14}$$

Then we must calculate p_0. Multiplying both sides of (1.7) by $f_n(Z) \cdot f_n^*(Z)$ and substituting (1.12) and (1.13) in it, there results

$$\sum_{k=0}^{2n-2} b_k Z^{2n-2-k} = \sum_{k=0}^{n-1} p_k Z^{n-1-k} f_n^*(Z) + \sum_{k=0}^{n-1} q_k Z^{n-1-k} f_n(Z) \tag{1.15}$$

namely,

$$\left.\begin{array}{ll} p_0 Z^{n-1} \cdot f_n^*(Z) = & p_0 (a_n Z^{2n-1} + a_{n-1} Z^{2n-2} + \cdots\cdots + a_0 Z^{n-1}) \\ p_1 Z^{n-2} \cdot f_n^*(Z) = & \qquad p_1 (a_n Z^{2n-2} + \cdots\cdots + a_0 Z^{n-2}) \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_{n-1} \cdot f_n^*(Z) = & \qquad\qquad p_n (a_n Z^n + \cdots\cdots + a_0) \\ q_0 Z^{n-1} \cdot f_n(Z) = & q_0 (a_0 Z^{2n-1} + a_1 Z^{2n-2} + \cdots\cdots + a_n Z^{n-1}) \\ q_1 Z^{n-2} \cdot f_n(Z) = & \qquad q_1 (a_0 Z^{2n-2} + \cdots\cdots + a_n Z^{n-2}) \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ q_{n-1} \cdot f_n(Z) = & \qquad\qquad q_n (a_0 Z^n + \cdots\cdots + a_n) \\ \hline g_n(Z) = & \qquad b_0 Z^{2n-2} + \cdots\cdots\cdots\cdots\cdots + b_{2n-2} \end{array}\right\} \tag{1.16}$$

Setting the terms of the same power in Z on both sides of (1.15) equal to each other, we obtain the following simultaneous linear equations in the matrix form:

$$
\begin{pmatrix}
a_n & & & & a_0 & & & \\
a_{n-1} & a_n & & & a_1 & a_0 & & \\
\cdot & \cdot & \vdots & & \cdot & \cdot & \cdot & \\
a_1 & \cdot & \cdot & a_n & a_{n-1} & \cdot & \cdot & a_0 \\
a_0 & a_1 & \vdots & a_{n-1} & a_n & a_{n-1} & \cdot & a_1 \\
 & a_0 & \vdots & \cdot & & a_n & \cdot & \cdot \\
 & & \cdot & \cdot & & & \cdot & \cdot \\
 & & & a_0 & & & & a_n
\end{pmatrix}
\begin{pmatrix} p_0 \\ p_1 \\ \cdot \\ p_{n-1} \\ q_0 \\ q_1 \\ \cdot \\ q_{n-1} \end{pmatrix}
=
\begin{pmatrix} 0 \\ b_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_{2n-2} \end{pmatrix}
\tag{1.17}
$$

where elements in blank spaces are all equal to zero.

By the Cramer's rule, we can easily obtain p_0,

$$
I_n = \frac{p_0}{a_0} = \frac{R_b(f_n, f_n^*)}{a_0 R(f_n, f_n^*)} \tag{1.18}
$$

where $R(f_n, f_n^*)$ is the determinant formed from coefficients of (1.17) and $R_b(f_n, f_n^*)$ is the determinant formed by replacing the first row of $R(f_n, f_n^*)$ by the right hand side of (1.17)

$$
R(f_n, f_n^*) = \begin{vmatrix}
a_n & & & & a_0 & & & \\
a_{n-1} & a_n & & & a_1 & a_0 & & \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \\
a_1 & \cdot & \cdot & a_n & a_{n-1} & \cdot & \cdot & a_0 \\
a_0 & a_1 & \cdot & a_{n-1} & a_n & a_{n-1} & \cdot & a_1 \\
 & a_0 & \cdot & \cdot & & a_n & \cdot & \\
 & & \cdot & \cdot & & & \cdot & \\
 & & & a_0 & & & & a_n
\end{vmatrix} \tag{1.19}
$$

$$
R_b(f_n, f_n^*) = \begin{vmatrix}
0 & & & & a_0 & & & \\
b_0 & a_n & & & a_1 & a_0 & & \\
b_1 & \cdot & \cdot & & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & a_n & a_{n-1} & \cdot & \cdot & a_0 \\
\cdot & a_1 & \cdot & a_{n-1} & a_n & a_{n-1} & \cdot & a_1 \\
\cdot & a_0 & \cdot & \cdot & & a_n & \cdot & \cdot \\
\cdot & & \cdot & \cdot & & & \cdot & \cdot \\
b_{2n-2} & & & a_0 & & & & a_n
\end{vmatrix} \tag{1.20}
$$

A few values of I_n are as in the following:

$$I_1 = \frac{b_0}{a_0^2 - a_1^2} \tag{1.21}$$

$$I_2 = \frac{-a_1 b_0 + (a_0 + a_2) b_1 - a_1 b_2}{(a_0 - a_2)(a_0 + a_1 + a_2)(a_0 - a_1 + a_2)} \tag{1.22}$$

$$\left.\begin{aligned} I_3 &= \frac{m_0 b_0 + m_1 b_1 + m_2 b_2 + m_3 b_3 + m_4 b_4}{(a_0 + a_1 + a_2 + a_3)(a_0 - a_1 + a_2 - a_3)(a_0^2 - a_0 a_2 + a_1 a_3 - a_3^2)} \\ \text{where} \quad m_0 &= a_1^2 - a_0 a_2 + a_1 a_3 - a_2^2 \\ m_1 &= a_2 a_3 - a_0 a_1 \\ m_2 &= a_0^2 + a_0 a_2 - a_1 a_3 - a_3^2 \\ m_3 &= m_1 \\ m_4 &= m_0 \end{aligned}\right\} \tag{1.23}$$

1.3 Evaluation of quadratic control area

Integrating (1.18) in Δ from zero to unity, the quadratic control area can be obtained. The function of Δ is the coefficient of $g_n(Z)$ only i.e. b_0, b_1, ..., and b_{2n-2}, and these coefficients are only in the first row of $R_b(f_n \cdot f_n^*)$. Therefore, the integration in Δ can be easily evaluated and there results

$$\int_0^\infty \{e(t)\}^2 dt = \frac{B(f_n \cdot f_n^*)}{a_0 R(f_n \cdot f_n^*)} \tag{1.24}$$

where $B(f_n \cdot f_n^*)$ is the determinant as follows:

$$B(f_n \cdot f_n^*) = \begin{vmatrix} 0 & & & & a_0 & & & \\ \int_0^1 b_0 d\Delta & a_n & & & a_1 & a_0 & & \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & a_n & a_{n-1} & \cdot & \cdot & a_0 \\ \cdot & a_1 & \cdot & a_{n-1} & a_n & a_{n-1} & \cdot & a_1 \\ \cdot & a_0 & \cdot & \cdot & & a_n & \cdot & \cdot \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \int_0^1 b_{2n-2} d\Delta & & & a_0 & & & & a_n \end{vmatrix} \tag{1.25}$$

1.4 Approximation of quadratic control area

As mentioned above, the quadratic control area of the sampled data control is given by the following expression:

$$\int_0^\infty \{e(t)\}^2\, dt = \int_0^1 Q(\Delta)\, d\Delta \tag{1.26}$$

However, the higher the order of the control system, the more difficult it becomes to evaluate the advanced Z transform and the integration in Δ .

We may then adopt the following approximations:

$$\int_0^\infty \{e(t)\}^2 dt \fallingdotseq t_0 \sum_{n=0}^{\infty} e_n^2 = Q(0) \tag{1.27}$$

or

$$\int_0^\infty \{e(t)\}^2 dt \fallingdotseq \frac{1}{2} e_0^2 t_0 + t_0 \sum_{n=1}^{\infty} e_n^2 = Q(0) - \frac{1}{2} e_0^2 t_0 \tag{1.28}$$

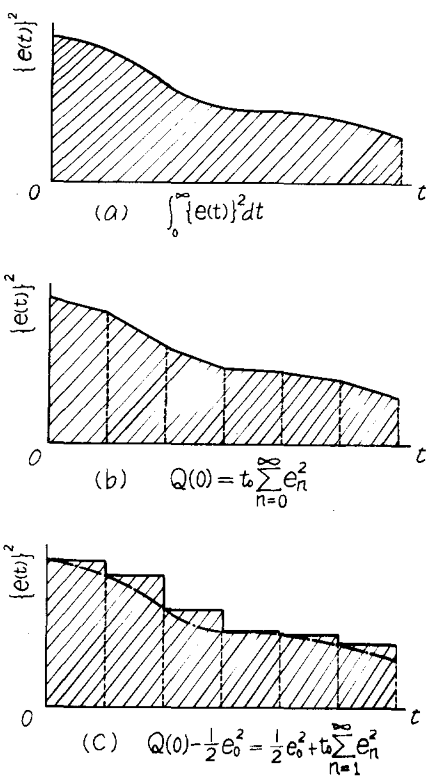


Fig.1.1 Quadratic control area and its approximations.

These expressions give a good approximation when the sampling period is small. Their geometrical meaning is shown in Fig. 1.1.

1.5 Supplement

(1) The evaluating method of the complex integral mentioned in the section 1.2 can be also applied to the continuous system, if we set $f^*_n(Z) = f(-Z)$.

(2) The determinant $R(f_n, f^*_n)$ is identical to the resultant formed from $f_n(Z)$ and $f^*_n(Z)$. On the other hand, $f_n(Z) = 0$ and $f^*_n(Z) = 0$ have a common root at the boundary of the stability, so $R(f_n, f^*_n) = 0$ (Appendix I).

Therefore, the series (1.4) diverges at the boundary of the stability.

Chapter 2

ON THE INFLUENCE OF SAMPLING PERIOD UPON CONTROL PERFORMANCE

2.1 Introduction

Introducing the digital controller into processes, we will come to adopt a many point control system in order to make up for the cost of the digital controller. Observing one controlled element only, we can regard the many point control system as a single sampled data control system. From the economical point of view, we would like to enlarge a number of controlled elements which are controlled by one digital controller concurrently. However, as a sampling period becomes large, the control performance may become inferior.

Therefore, in this chapter we investigate the influence the sampling period has upon the control performance (stability, and quadratic control area).

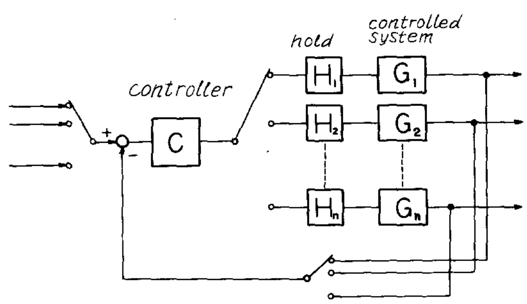


Fig. 2.1 Block diagram of many point control system.

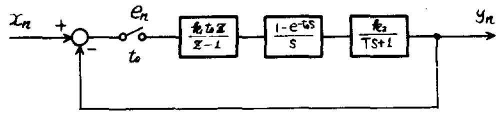
2.2 Digital integration + 0th order hold + 1st order element

Let us consider the control system with the controlled element whose transfer function is

$$\frac{k_2}{TS + 1} \tag{2.1}$$

as shown in Fig. 2.2. The advanced Z transform of the system error to a unit step input is

Fig. 2.2 Block diagram of system with controlled element $k_2/(Ts+1)$.



$$E(Z,\triangle) = X(Z,\triangle) - \frac{C(Z)\,HG(Z,\triangle)}{1+C(Z)\,HG(Z)}\,X(Z) \tag{2.2}$$

where X(z), C(z) and HG(z) are Z transforms of an input, the controller and the controlled element with a data hold element. $X(z,\triangle)$ and $HG(z,\triangle)$ are the advanced Z transforms of X(z) and HG(z). These values of the system shown in Fig. 2.2 are as follows:

$$\left.\begin{aligned}
X(Z) &= X(Z,\triangle) = \frac{Z}{Z-1} \\
C(Z) &= \frac{k_1 t_0 Z}{Z-1} \\
HG(Z) &= \frac{k_2(1-d)}{Z-d} \\
HG(Z,\triangle) &= \frac{k_2\{(1-d^{\triangle})Z + d^{\triangle} - d\}}{Z-d} \\
d &= \exp\left(-\frac{t_0}{T}\right)
\end{aligned}\right\} \tag{2.3}$$

Substituting (2.3) in (2.2) and arranging it, there results

$$E(Z,\Delta) = \frac{\{1-k_1k_2t_0(1-d^{\Delta})\}Z^2-dZ}{Z^2+\{k_1k_2t_0(1-d)-(1+d)\}Z+d} \tag{2.4}$$

Integrated-square error $Q(\Delta)$ can be obtained by (1.22); namely,

$$Q(\Delta) = \frac{(1+d)\{[1-k_1k_2t_0(1-d^{\Delta})]^2+d^2\}+2[1-k_1k_2t_0(1-d^{\Delta})]d\{k_1k_2t_0(1-d)-(1+d)\}}{k_1k_2t(1-d)[2(1+d)-k_1k_2t_0(1-d)]} \tag{2.5}$$

and especially

$$Q(0) = \frac{(1-d^2)+2dk_1k_2t_0}{k_1k_2(1-d)[2(1+d)-k_1k_2t_0(1-d)]} \tag{2.6}$$

By integrating $Q(\Delta)$ from zero to unity, the quadratic control area becomes as follows:

$$\frac{1}{T}\int_0^{\infty}\{e(t)\}^2dt = \frac{\kappa^2\lambda\{\lambda(1-d+2d^2)-\frac{1}{2}(1-d)(3-2d+3d^2)\}+2\kappa(1-d)(1-d^2-\lambda)+(1+d)(1-d)^2}{(1-d)^2\kappa[2(1+d)-\kappa\lambda(1-d)]} \tag{2.7}$$

where κ is the loop gain

$$\kappa = k_1k_2T, \quad \text{and} \quad \lambda = \frac{t_0}{T} \tag{2.8}$$

By setting the denominator of $Q(\Delta)$ or $Q(0)$ to zero, the boundary of stability is given by

$$\kappa\{2(1+d)-\kappa\lambda(1-d)\}=0 \tag{2.9}$$

therefore, we have the region of stability as

$$\frac{\coth\frac{\lambda}{2}}{\frac{\lambda}{2}} > \kappa > 0 \tag{2.10}$$

The region of stability is shown in Fig. 2.3. This figure shows that the region of stability decreases monotonously, as the sampling period becomes large. The quadratic control area (2.7) is shown in Fig. 2.4, where the parameter is the sampling period : $\lambda = \frac{t_0}{T}$.

The integrated-square error (approximate quadratic control area) Q(0) in (2.6) is shown in Fig. 2.5. Digital integration + 0th

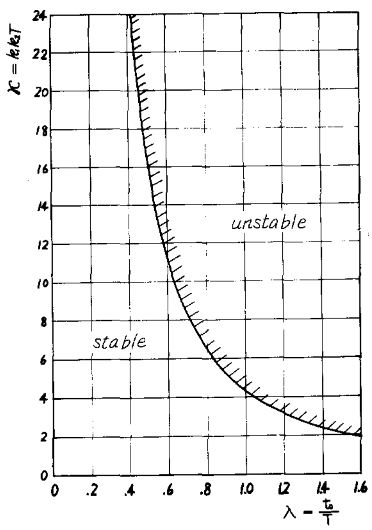


Fig. 2.3 Region of stability of system of Fig. 2.2.

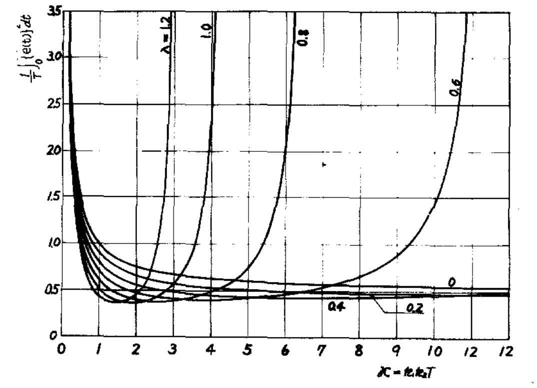


Fig. 2.4 Quadratic control area of system of Fig. 2.2.

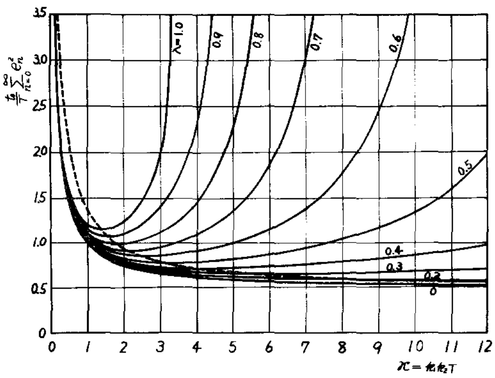


Fig. 2.5 Integrated-square error of system of Fig. 2.2.

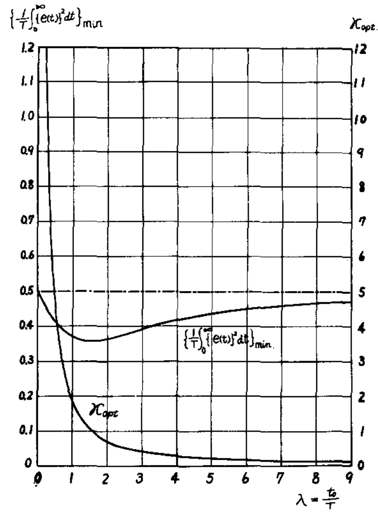


Fig. 2.6 Minimized quadratic control area and optimum loop gain system of Fig. 2.2.

order hold element tends to k_1/s as $t_0 \to 0$(i.e. continuous). From Fig. 2.4 and Fig. 2.5, we can recognize that the sampling system has almost equal control performance to that of the continuous control system when $t_0/T \leq 0.1$. The minimum value of the quadratic control area and κ at the minimum point (i.e. κ_{opt}) are shown in Fig. 2.6. The curve of the former takes a minimum value at $t_0 \doteqdot 1.5T$. Unexpectedly, the continuous case (i.e. $t_0 = 0$) and the open case (i.e. $t_0 \to \infty$) are both inferior. However, the gain margin is infinity at t = 0 and zero at $t_0 = \infty$. Fig. 2.7 shows the transient responses for $t_0 = 0$ and $t_0 = 1$ (where T = 1). From this figure, we can recognize that the case of $t_0 = 1$ is better than the case of $t_0 = 0$.

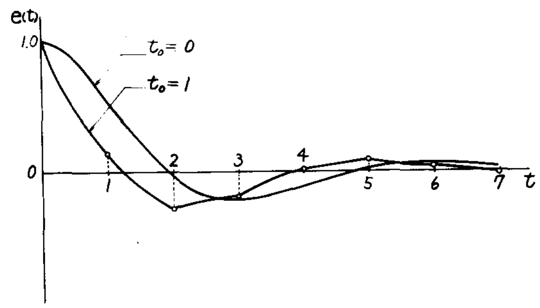


Fig. 2.7 Indicial responses of system of Fig. 2.2 for $t_0 = 0$ and $t_0 = 1$ (T = 1, $k_1 k_2 = 1.37$)

2.3 Digital integration + 0th order hold + 2nd order element with non-oscillatory indicial response.

Next, let us consider the control system with the controlled element whose transfer function is

$$\frac{k_2}{(T_1 S+1)\ (T_2 S+1)} \tag{2.11}$$

as shown in Fig. 2.8. The boundary of stability can be obtained by setting the denominator of I_3 in (1.23) to zero, and the integrated square error Q(0) of this system can be evaluated by $t_0 I_3$.

But the following values must be substituted in a_i and b_i : namely,

$$
\left.
\begin{aligned}
a_0 &= \mu - 1 \\
a_1 &= k_1 k_2 T_1 \lambda_1 (\mu - 1 + d_1 - \mu d_2) - (\mu - 1)\ (1 + d_1 + d_2) \\
a_2 &= k_1 k_2 T_1 \lambda_1 (d_2 - \mu d_1 + (\mu - 1)\, d_1 d_2) + (\mu - 1)\ (d_1 + d_2 + d_1 d_2) \\
a_3 &= -(\mu - 1)\, d_1 d_2 \\
b_0 &= b_4 = (\mu - 1)^2 d_1 d_2 \\
b_1 &= b_3 = -(\mu - 1)^2\ (d_1 + d_2)\ (1 + d_1 d_2) \\
b_2 &= (\mu - 1)^2\ (1 + d_1^2 + d_2^2 + 2 d_1 d_2 + d_1^2 d_2^2) \\
d_1 &= \exp(-t_0 / T_1) \qquad\qquad d_2 = \exp(-t_0 / T_2) \\
\mu &= T_2 / T_1 \qquad\qquad\qquad\quad \lambda_1 = t_0 / T_1
\end{aligned}
\right\} \quad (2.12)
$$

For $\mu = 1$ (i.e. $T_1 = T_2$), we must take the limit of $\mu \rightarrow 1$.

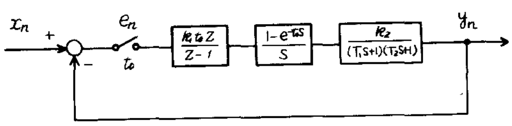


Fig. 2.8 Block diagram of system with 2nd-order element $k / (T_1 s+1)(T_2 s+1)$.

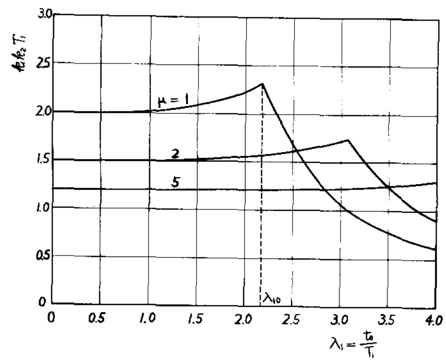


Fig. 2.9 Boundary of stability of system of Fig. 2.8.

The region of stability and the integrated-square error Q(0) are shown in Fig. 2.9 and Fig. 2.10 respectively. It is the remarkable fact that the region of stability increases as the sampling period becomes larger. This tendency ceases, however, at a certain value of the sampling period (i.e. $(\lambda_1)_0$ in Fig. 2.9).

The integrated square error (i.e. approximate quadratic control area) is nearly equal to that of the continuous system ($t_0 = 0$) where $(\lambda_1)_0 \geq \lambda_1 \geq 0$. As Q(0) is an excess approximation, the minimum value of the quadratic control area may become smaller than that of the continuous case even in the domain $\lambda_1 \geq (\lambda_1)_0$.

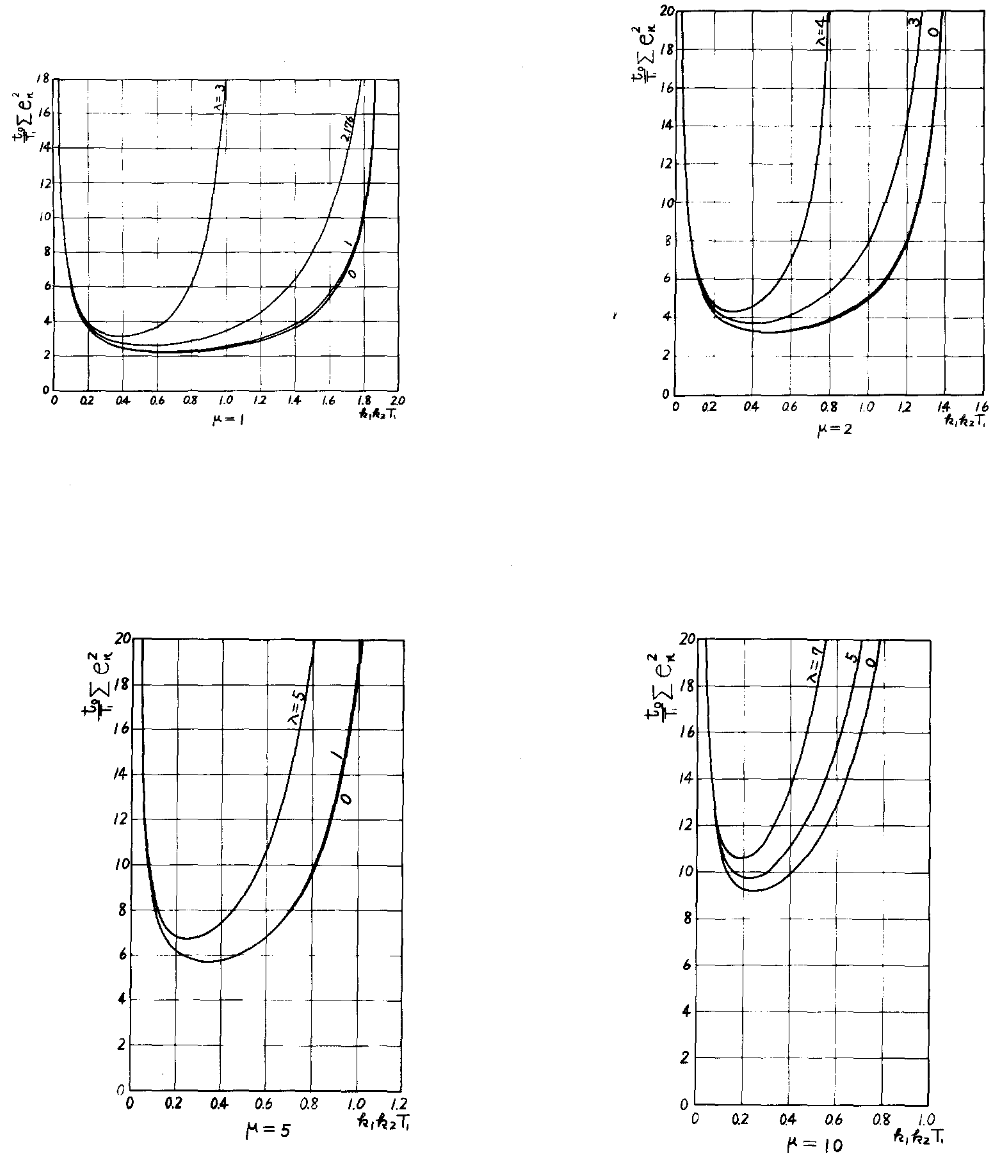


Fig. 2.10 Integrated-square error of system of Fig. 2.8.

From these facts, we may conclude as follows: the optimum value of the sampling period is $(\lambda_1)_0$ (i .e. $(t_0/T_1)_0$) where the gain margin becomes maximum (Fig. 2.11). The transient responses to unit step input are shown in Fig. 2.12. The case of $t_0 = 3$ is better than the continuous case.

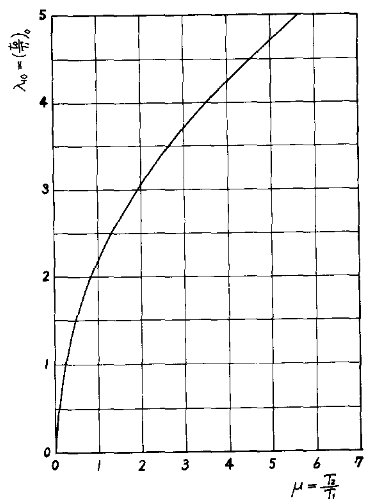


Fig. 2.11 Optimum sampling period of system of Fig. 2.8.

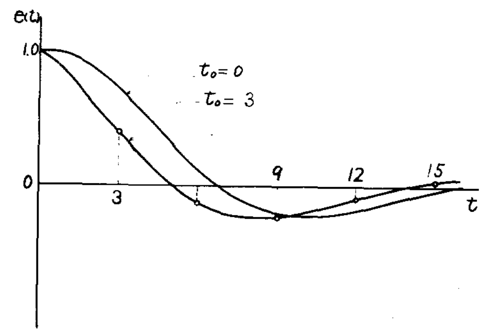


Fig. 2.12 Indicial responses of system of Fig. 2.8 for $t_0 = 0$ and $t_0 = 3$ ($k_1k_2 = 0.336$, $\mu = 2$).

2.4 Digital integral + 0th order hold + 2nd order element with oscillatory indicial response

We consider the system shown in Fig. 2.13. The transfer function of a controlled element is

$$\frac{k_2}{(TS+1)^2 + b^2} \tag{2.13}$$

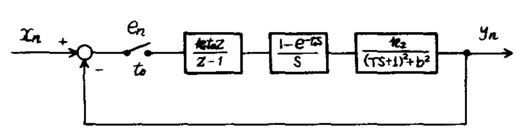


Fig. 2.13 Block diagram of system with 2nd-order element $k_2/(Ts+1)^2 + b^2$.

The boundary of stability of this system can be given by setting the denominator of I_3 to zero, and the integrated square error Q(0) is t_0 I_3. But a_i and b_i in I_3 are

$$\left.\begin{aligned}
a_0 &= 1\\
a_1 &= \frac{k_1k_2T\lambda}{1+b^2}\left(1-\cos\lambda b-\frac{d}{b}\sin\lambda b\right)-(2d\cos\lambda b+1)\\
a_2 &= \frac{dk_1k_2\lambda}{1+b^2}\left(d-\cos\lambda b+\frac{1}{b}\sin b\lambda\right)+d^2+2d\cos\lambda b\\
a_3 &= -d^2\\
b_0 &= b_4 = d^2\\
b_1 &= b_3 = -2d\,(1+d^2)\cos\lambda b\\
b_2 &= 1+4d^2\sin^2\lambda b+d^4\\
\lambda &= t_0/T\ ,\quad d = e^{-\lambda}
\end{aligned}\right\} \qquad (2.14)$$

The boundary of stabitity is shown in Fig. 2.14 and the integrated square error is shown in Fig. 2.15. From these figures, we can indicate same facts as mentioned in the preceeding section.

The optimum value of λ_0 where the region of stability becomes maximum as shown in Fig. 2.16.

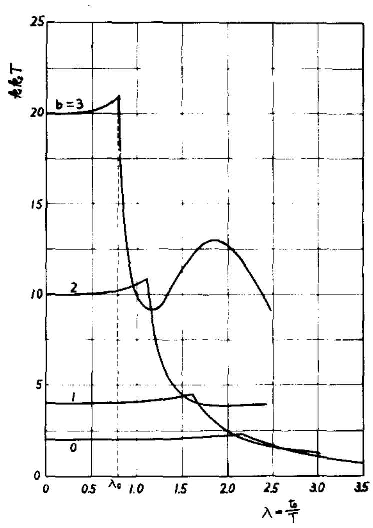


Fig. 2.14 Boundary of stability of system of Fig. 2.13.

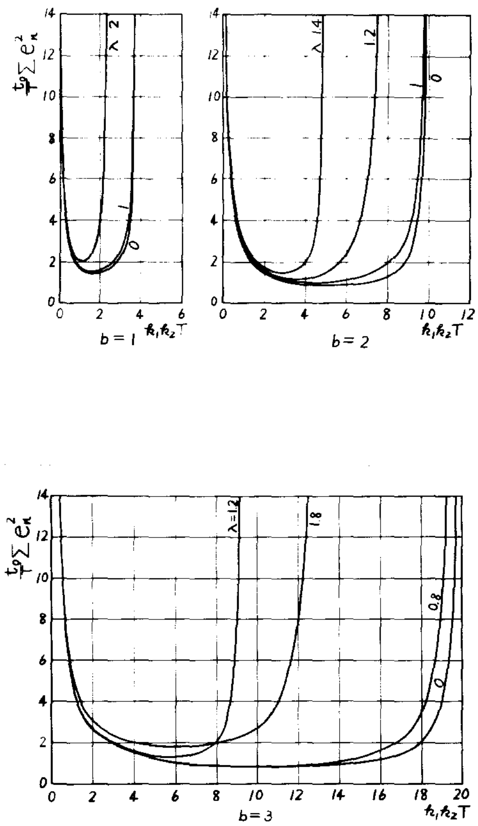


Fig. 2.15 Integrated-square error of system of Fig. 2.13.

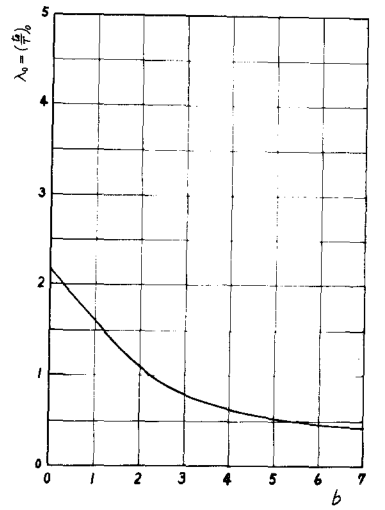


Fig. 2.16 Optimum sampling period of system of Fig. 2.13.

2.5 Conclusion

As the first step for the development of many point control systems, we investigated the influence of the sampling period upon the control performance. The reults may be summarized as follows: it is not always optimum to make the sampling period tend to zero.

In the system shown in Fig. 2.2, the continuous case ($t_0 = 0$) and the open case ($t_0 = \infty$) are both inferior, on the other hand, the control performance is the best when $t_0 \fallingdotseq 1.5T$ in the sense that it minimizes the quadratic control area. However, the gain margin is maximum at $t_0 = 0$.

On the contrary, in the system with the 2nd order controlled element as shown in Fig. 2.8 or Fig. 2.13, the gain margin is not maximum at $t_0 = 0$, but it is maximum at $t_0 = (\lambda_1)_0$ or $t_0 = \lambda_0$. That is, at a certain value of the sampling period other than zero, the degree of stability becomes the largest.

Therefore, we can expect that, when we adop many point control systems, we can control concurrently by one controller much more controlled elements than we expected without lowering the control quality.

Chapter 3

SYNTHESIS OF FINITE SETTLING TIME SYSTEMS BY THE METHOD OF MINIMIZING INTEGRATED SQUARE ERROR[2]

3.1 Introduction

The theory of finite settling time systems constitute an interesting field for study in the theory of sampled data control systems.

It is not always optimum to make a settling time the shortest.

So in this chapter, we study the systhesis of finite settling time systems by the method of minimizing integrated square error. In the sampled data control shown in Fig. 3.1, the Z transform of a system

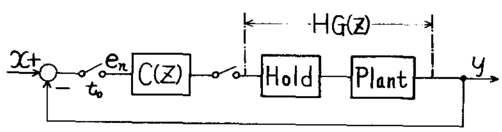


Fig. 3.1 Block diagram of sampled-data control system.

error is

$$E(z) = K(z)\cdot X(z) \tag{3.1}$$

where X(z) is the Z transform of an input x(t) and

$$K(z) = \frac{1}{1 + C(z)\ \ HG(z)} \tag{3.2}$$

In order to respond without steady state error to an integration type input of the order m: namely,

$$X(z) = \frac{F_m(z)}{(1 - z^{-1})^m} \tag{3.3}$$

where $F_m(z)$ is a polynomial in Z and $F(1) \neq 0$, K(z) must contain a factor $(1 - z^{-1})^m$

$$K(z) = (1 - z^{-1})^m \cdot (a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) \tag{3.4}$$

Substituting (3.3) and (3.4) in (3.1), we obtain the system error

$$E(z) = (a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) \cdot F_m(z) \tag{3.5}$$

If an input of order k is added to this system, the system error of this case becomes

$$E(z) = (1 - z^{-1})^{m-k} (a_0 + a_1 z^{-1} + \cdots\cdots + a_N z^{-N}) F_k(z) \tag{3.6}$$

The problem to minimize the integrated square error

$$Q = \sum_{n=0}^{\infty} e_n^2 = \frac{1}{2\pi j} \int_c E(z) \cdot E(z^{-1}) \frac{dz}{z} \tag{3.7}$$

where

$$E(z) = z^{-\nu}(b_0 + b_1 z^{-1} + \cdots + b_M z^{-M})(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) \tag{3.8}$$

will be treated in the following sections.

3.2 General procedure

The problem here is to evaluate a_0, $a_1, \ldots,$ and a_N which minimize (3,7), where b_0, $b_1, \ldots,$ and b_M are known as in (3.8), under the initial condition

$$a_0 - 1 = 0 \tag{3.9}$$

As

$$\sum_{\mu=0}^{N} a_\mu z^{-\mu} \sum_{\mu=0}^{N} a_\mu z^{\mu} = \sum_{n=-1}^{N} \left(\sum_{\mu=0}^{N-|n|} a_\mu a_{\mu+|n|} \right) z^{-n} \tag{3.10}$$

$$\sum_{\lambda=0}^{M} b_\lambda z^{-\lambda} \sum_{\lambda=0}^{M} b_\lambda z^{\lambda} = \sum_{m=-M}^{M} \left(\sum_{\mu=0}^{M-|m|} b_\lambda b_{\lambda+|m|} \right) z^{-m} \tag{3.11}$$

the integrated square error is given by

$$Q = \frac{1}{2\pi j} \int_C E(z) E\left(z^{-1}\right) \frac{dz}{z} = \sum_{\mu=0}^{N} a_\mu^2 \sum_{\lambda=0}^{M} b_\lambda^2 + 2 \sum_{n=1}^{\min(M \cdot N)} \left(\sum_{\mu=0}^{N-n} a_\mu a_{\mu+n} \right) \left(\sum_{\lambda=0}^{M-n} b_\lambda b_{\lambda+n} \right) \tag{3.12}$$

If we set

$$B_n = \sum_{\lambda=0}^{M-n} b_\lambda b_{\lambda+n} \tag{3.13}$$

(3.11) becomes as follows:

$$Q = B_0 \sum_{\mu=0}^{N} a_\mu^2 + 2 \sum_{n=1}^{\min(M \cdot N)} B_n \sum_{\mu=0}^{N-n} a_\mu a_{\mu+n} \tag{3.14}$$

Multiplying (3.9) by the Lagrange's multiplyer 2κ and adding it to (3.14), there ersults

$$B_0 \sum_{\mu=0}^{N} a_\mu^2 + 2 \sum_{n=0}^{\min(M \cdot N)} B_n \sum_{\mu=0}^{N-n} a_\mu a_{\mu+n} + 2\kappa\,(a_0 - 1) \tag{3.15}$$

Differentiating (3.15) and setting the derivatives equal to zero, we have

$$a_\mu B_0 + \sum_{n=1}^{\min(M \cdot N)} (a_{\mu+n} + a_{\mu-n}) B_n + \kappa \delta_\mu = 0 \tag{3.16}$$

$$(\mu = 0, 1, 2, \cdots\cdots N)$$

where

$$\left.\begin{aligned} \delta_\mu &= \begin{matrix} 1 & (\mu=0) \\ 0 & (\mu \neq 0) \end{matrix} \\ a_{\mu+n} &= 0 \quad (\mu+n > N) \\ a_{\mu-n} &= 0 \quad (\mu-n < 0) \end{aligned}\right\} \tag{3.17}$$

A set of linear equations (3.16) can be expressed in the matrix form as follows:

$$\begin{pmatrix} B_0 & B_1 & B_2 & \cdot & B_N & 1 \\ B_1 & B_0 & B_1 & \cdot & B_{N-1} & 0 \\ B_2 & B_1 & B_0 & \cdot & B_{N-2} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_N & B_{N-1} & B_{N-2} & \cdot & B_0 & 0 \\ 1 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ a_N \\ \kappa \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix} \tag{3.18}$$

where

$$B_k = 0 \qquad (k > \min(M,N)) \tag{3.19}$$

By solving (3.18), we can obtain desired values of a's. Therefore, the pulse transfer function is determined by (3.2) as follows:

$$C(z) = \frac{1 - K(z)}{HG(z) \quad K(z)} \tag{3.20}$$

3.3 Evaluation of Min. Q

The solution of (3.18) is given by Cramer's rule

$$a_\mu = \frac{D_{\mu+1}}{D} \qquad (\mu = 0, 1, \cdots\cdots, N) \tag{3.21}$$

where D is the determinant formed from coefficients of (3.18) and $D_{\mu+1}$ is that formed by replacing the μ + 1th row of D by the right hand side of (3.18) and $D_{\mu+1}$ is also the cofactor of $(N+2, \mu+1)$ element.

To evaluate the minimized integrated square error, we substitute (3.21) in (3.14)

$$\mathrm{Min}\, Q = \frac{1}{D}\left\{B_0 \sum_{\mu=0}^{N} a_\mu D_{\mu+1} + \sum_{n=1}^{\min(M,N)} B_n \sum_{\mu=0}^{N-n} (a_\mu D_{\mu+n+1} + a_{\mu+n} D_{\mu+1})\right\} \tag{3.22}$$

Arranging (3.22) in D's, there results

$$= \frac{1}{D} \cdot \sum_{k=0}^{N} D_{k+1} (\cdots\cdots + a_{k-1} B_1 + a_k B_0 + a_{k+1} B_1 + \cdots\cdots) \tag{3.23}$$

As D_{k+1} is the cofactor of (N+2, k+1) element of D, the right hand side of (3.23) becomes the determinant formed from D by replacing the N+2th column of D by

$$(\cdots\cdots + a_{k-1} B_1 + a_k B_0 + a_{k+1} B_1 + \cdots\cdots) \tag{3.24}$$

In this determinant, multiplying the 1st, 2nd,..., and N+1th column by $-a_0$, $-a_1$,.., and $-a_N$ respectively and adding to the N+2th column, all the elements in the N+2th column vanish except(N+2,N+2) element the value of which is

$$- a_0 \quad (= -1) \tag{3.25}$$

Therefore, we can obtain

$$\text{Min } Q = - \frac{D_{N+2}}{D} = - k\varkappa \tag{3.26}$$

This is the physical meaning of the Lagrange's multiplier $\varkappa$.

3.4 Example

We will consider, as an example, the case where the system error is given by

$$E(z) = k\, z^{-m} (1+dz^{-1})\,(a_0+a_1 z^{-1}+\cdots\cdots+a_N z^{-N}) \tag{3.27}$$

The factor kz^{-m} does not contribute to the determination of a's.

By using

$$\left.\begin{aligned} B_0 &= 1 + \alpha^2 \\ B_1 &= \alpha \end{aligned}\right\} \tag{3.28}$$

we will evaluate the determinants D and $D_{\mu+1}$.

$$D = \begin{vmatrix} B_0 & B_1 & 0 & \cdot & 0 & 0 & 1 \\ B_1 & B_0 & B_1 & \cdot & 0 & 0 & 0 \\ 0 & B_1 & B_0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & B_0 & B_1 & 0 \\ 0 & 0 & 0 & \cdot & B_1 & B_0 & 0 \\ 1 & 0 & 0 & \cdot & 0 & 0 & 0 \end{vmatrix} = - \overbrace{\begin{vmatrix} B_0 & B_1 & 0 & \cdot & 0 \\ B_1 & B_0 & B_1 & \cdot & 0 \\ 0 & B_1 & B_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & B_0 \end{vmatrix}}^{N} = - A_N \tag{3.29}$$

If we set D to $-A_N$ as (3.29), A_N satisfies the following recurrence relation

$$\left.\begin{aligned} & A_N - B_0 A_{N-1} + B_1^2\, A_{N-2} = 0 \\ \text{i.e.}\quad & A_N - (1+\alpha^2)A_{N-1} + \alpha^2 A_{N-2} = 0 \end{aligned}\right\} \tag{3.30}$$

Solving this recurrence relation under the following initial condition

$$\left.\begin{aligned} A_1 &= 1 + \alpha^2 \;\; (= B_0 \;) \\ A_2 &= 1 + \alpha^2 + \alpha^4 \; (= B_0^2 - B_1^2 \;) \end{aligned}\right\} , \tag{3.31}$$

there results

$$A_N = 1 + \alpha^2 + \alpha^4 + \cdots\cdots + \alpha^{2N} \tag{3.32}$$

On the other hand

$$D_{\mu+1} = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{\mu+1} & \overbrace{\qquad\qquad}^{N-\mu+1} \end{array} \\ \left| \begin{array}{ccccccc} B_0 & B_1 & \cdot & 0 & \cdot & 0 & 1 \\ B_1 & B_0 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & B_1 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & \cdot & B_0 & 0 \\ 1 & 0 & \cdot & 1 & \cdot & 0 & 0 \end{array} \right| \end{array} = (-1)^{\mu+1}\, B_1^{\mu}\, A_{N-\mu} \tag{3.33}$$

and

$$D_{N+2} = A_{N+1} \tag{3.34}$$

Therefore, we have

$$a_\mu = \frac{D_{\mu+1}}{D} = \frac{(-B_1)^{\mu} A_{N-\mu}}{A_N} = (-\alpha)^{\mu} \cdot \frac{1+\alpha^2+\cdots+\alpha^{2(N-\mu)}}{1+\alpha^2+\cdots+\alpha^{2N}} \tag{3.35}$$

and

$$\mathrm{Min}\,Q = -k^2 \frac{D_{N+2}}{D} = k^2 \frac{A_{N+1}}{A_N} = \frac{1+\alpha^2+\cdots\cdot+\alpha^{2N+2}}{1+\alpha^2+\cdots\cdot+\alpha^{2N}} \cdot k^2 \tag{3.36}$$

3.4.1 To an acceleration input

As

$$E(z) = K(z)X(z) = (1-z^{-1})^3 (a_0 + a_1 z^{-1} + \cdots\cdot + a_N z^{-N}) \frac{t_0^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$$

$$= \frac{t_0^2}{2} z^{-1}(1+z^{-1})(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) \tag{3.40}$$

setting $\alpha = 1$ in (3.33) and (3.34), we have

$$a_\mu = (-1)^{\mu} \frac{N-\mu+1}{N+1} \tag{3.38}$$

and

$$\mathrm{Min}\ Q = \frac{t_0^4}{4} \quad \frac{N+2}{N+1} \tag{3.39}$$

3.4.2 To make the system that has a finite settling time response to a ramp input and is optimum for a step input.

As

$$\begin{aligned} E(z) = K(z)X(z) &= (1-z^{-1})^2 (a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) \frac{1}{(1-z^{-1})} \\ &= (1-z^{-1})(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) \end{aligned} \tag{3.40}$$

setting $\alpha = -1$ in (3.33) and (3.34), there results

$$a_\mu = \frac{N-\mu+1}{N+1} \tag{3.41}$$

$$\mathrm{Min}\ Q = \frac{N+2}{N+1} \tag{3.42}$$

Chapter 4

STATISTICAL SYNTHESIS OF FINITE-SETTLING-TIME SYSTEMS

4.1 Introduction

The problem of finite settling time systems has attracted the interest of many investigators and contributions to this field have been made by many authors, but they did not treat the finite settling time system with a random noise. In this chapter, we study a method of determining the pulse transfer function of a controller of a sampled data control system with a stationary random noise which has been designed to respond to an integration-type input of order m :

$$\frac{F(Z)}{(1-Z^{-1})^{m}} \tag{4.1}$$

where F(z) is a polynomial in z^{-1} ($F(1) \neq 0$) without steady state error by the R.S.M.- Error Criterion.

4.2 General procedure

Let us consider the system in the presence of a stationary random noise v(t) as shown in Fig. 4.1. The autocorrelation function of a

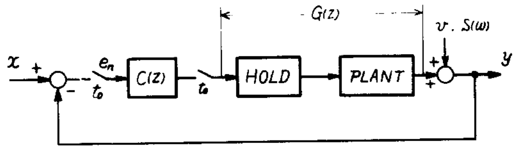


Fig. 4.1 Block diagram of sampled-data control system.

stationary and ergodic process may be obtained by an average of samples according to

$$\varphi(\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} v(nt_0)\, v(nt_0+\tau) \tag{4.2}$$

where t_0 is a sampling period. Setting $\varphi(nt_0) \equiv \varphi_n$, the pulse spectral density of the noise is defined by the following expression ;

$$S^*(\omega) = \varphi_0 + 2 \sum_{n=1}^{\infty} \varphi_n \cos n\omega t_0 \tag{4.3}$$

The mean square error of this system is evaluated by the following integral

$$\langle e_n^2 \rangle = \frac{t_0}{2\pi} \int_{-\frac{\pi}{t_0}}^{\frac{\pi}{t_0}} |K(e^{i\omega t_0})|^2 \cdot S^*(\omega)\, d\omega \tag{4.4}$$

where

$$K(Z) = \frac{1}{1+G(Z)C(Z)} \tag{4.5}$$

On the other hand, in order to obtain the finite settling time response, $K(z)$ must be a polynomial in z^{-1}

$$K(Z) = \sum_{\mu=0}^{n} a_\mu Z^{-\mu} \tag{4.6}$$

Substituting (4.3) and (4.6) in (4.4) and evaluating the integral, there results

$$\langle e_n^2 \rangle = \varphi_0 \sum_{\mu=0}^{n} a_\mu^2 + 2 \sum_{k=1}^{n} \varphi_k \sum_{\mu=0}^{n-k} a_\mu a_{\mu+k} \tag{4.7}$$

In order that this system responds to an integration-type input of order m without steady state error, $K(z)$ must contain a factor $(1-z^{-1})^m$.

Therefore, a's, coefficients of the polynomial $K(z)$, must satisfy the following m conditions

$$\sum_{\mu=\nu}^{n} \left\{ \begin{matrix} \mu \\ \nu \end{matrix} \right\} a_\mu = 0 \quad , \ (\nu = 0, 1, 2, \cdots\cdots, m-1)$$

where

$$\left.\begin{aligned} \left\{ {\mu \atop \nu} \right\} &\equiv \mu(\mu-1)\cdots(\mu-\nu+1) \quad (\mu>\nu>0) \\ \left\{ {\mu \atop \mu} \right\} &\equiv \mu! \\ \left\{ {\mu \atop 0} \right\} &\equiv 1 \end{aligned}\right\} \tag{4.8}$$

and an initial condition

$$a_0 - 1 = 0 \tag{4.9}$$

Now, our purpose is to determine a's (i.e. polynomial K(z)) which minimize $\langle e_n^2 \rangle$ in (4.7) under conditions (4.8) and (4.9).

Multiplying (4.8) and (4.9) by Lagrange's multipliers $\lambda_0, \lambda_1 \cdots\cdots \lambda_{m-1}$; κ and adding them to (4.7), there results

$$\varphi_0 \sum_{\mu=0}^{n} a_\mu^2 + 2\sum_{k=1}^{n} \varphi_k \sum_{\mu=0}^{n-k} a_\mu a_{\mu+k} + \kappa(a_0-1) + \sum_{\nu=0}^{m-1} \lambda_\nu \sum_{\mu=\nu}^{n} \left\{ {\mu \atop \nu} \right\} a_\mu \tag{4.10}$$

It is seen that the mean square error (4.10) is a function of the coefficients a's which must be so chosen as to minimize (4.10).

Differentiating and setting the derivatives equal to zero, we obtain a set of n+1 linear equations in a's : namely,

$$\left.\begin{aligned} &2\varphi_0 a_\mu + 2\sum_{k=1}^{n} \varphi_k (a_{\mu+k} + a_{\mu-k}) + \kappa\delta_\mu + \sum_{\nu=0}^{m-1} \lambda_\nu \left\{ {\mu \atop \nu} \right\} = 0 \\ &\qquad\qquad (\mu = 0, 1, 2, \cdots\cdots n) \\ &\text{where} \\ &\delta_\mu = \begin{cases} 1, & (\mu = 0) \\ 0, & (\mu \neq 0) \end{cases} \\ &a_{\mu-k} = 0, \quad (\mu-k<0) \\ &a_{\mu+k} = 0, \quad (\mu+k>n) \end{aligned}\right\} \tag{4.11}$$

From equations (4.8), (4.9) and (4.11), we can evaluate a's .

Expressing these n+m+2 equations in the matrix form, we have

$$
\left(\begin{array}{ccccc|cccccc}
2\varphi_0 & 2\varphi_1 & 2\varphi_2 & \cdot & 2\varphi_n & 1 & 1 & 0 & 0 & \cdot & 0 \\
2\varphi_1 & 2\varphi_0 & 2\varphi_1 & \cdot & 2\varphi_{n-1} & 0 & 1 & 1 & 0 & \cdot & 0 \\
2\varphi_2 & 2\varphi_1 & 2\varphi_0 & \cdot & 2\varphi_{n-2} & 0 & 1 & 2 & 2\cdot 1 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2\varphi_n & 2\varphi_{n-1} & 2\varphi_{n-2} & \cdot & 2\varphi_0 & 0 & 1 & n & n(n-1) & \cdot & \left\{ {n \atop m-1} \right\} \\
\hline
1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\
1 & 1 & 1 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 \\
0 & 1 & 2 & \cdot & n & 0 & 0 & 0 & 0 & \cdot & 0 \\
0 & 0 & 2\cdot 1 & \cdot & n(n-1) & 0 & 0 & 0 & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & \left\{ {n \atop m-1} \right\} & 0 & 0 & 0 & 0 & \cdot & 0
\end{array}\right)
\left(\begin{array}{c}
a_0 \\ a_1 \\ a_2 \\ \cdot \\ a_n \\ \hline \kappa \\ \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \cdot \\ \lambda_{m-1}
\end{array}\right)
=
\left(\begin{array}{c}
0 \\ 0 \\ 0 \\ \cdot \\ 0 \\ \hline 1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ 0
\end{array}\right)
\qquad (4.12)
$$

The solutions of these linear equations are given by the Cramer's rule : namely,

$$
a_\mu = \frac{\Delta_{\mu+1}}{\Delta} \quad , \quad (\mu = 0\,.\,1\,,\,2\,,\,\cdots\cdots n\,) \qquad (4.13)
$$

where

$$
\Delta = \left|\begin{array}{ccccc|cccccc}
\varphi_0 & \varphi_1 & \varphi_2 & \cdot & \varphi_n & 1 & 1 & 0 & 0 & \cdot & 0 \\
\varphi_1 & \varphi_0 & \varphi_1 & \cdot & \varphi_{n-1} & 0 & 1 & 1 & 0 & \cdot & 0 \\
\varphi_2 & \varphi_1 & \varphi_0 & \cdot & \varphi_{n-2} & 0 & 1 & 2 & 2\cdot 1 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\varphi_n & \varphi_{n-1} & \varphi_{n-2} & \cdot & \varphi_0 & 0 & 1 & n & n(n-1) & \cdot & \left\{ {n \atop m-1} \right\} \\
\hline
1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\
1 & 1 & 1 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 0 \\
0 & 1 & 2 & \cdot & n & 0 & 0 & 0 & 0 & \cdot & 0 \\
0 & 0 & 2\cdot 1 & \cdot & n(n-1) & 0 & 0 & 0 & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & \left\{ {n \atop m-1} \right\} & 0 & 0 & 0 & 0 & \cdot & 0
\end{array}\right| \qquad (4.14)
$$

and $\Delta_{\mu+1}$ is the determinant formed by replacing the μ +1th row of the determinant Δ by the row vector of the right hand side of (4.12).

Naturally $\Delta_1 = \Delta$.

Therefore, the pulse transfer function of the controller C(z) becomes as follows:

$$C(Z)=\frac{1}{G(Z)}\cdot\frac{1-K(Z)}{K(Z)}=-\frac{1}{G(Z)}\frac{\sum_{\mu=1}^{n}\Delta_{\mu+1}Z^{-\mu}}{\sum_{\mu=0}^{n}\Delta_{\mu+1}Z^{-\mu}} \tag{4.15}$$

Eq.(4.15) is the required result of this chapter.

In evaluating a's it is convenient to note that a's do not change their values if φ is replaced by $M\varphi$ (M is an arbitrary constant), because it only means that $\langle e_n^2\rangle \to M\langle e_n^2\rangle$ by the quation (4.7).

4.3 On the evaluation of minimized mean square error

To evaluate the minimized mean square error Min$\langle e_n^2\rangle$, substitution of (4.13) into (4.7) gives

$$\mathrm{Min}\langle e_n^2\rangle=\varphi_0\sum_{\mu=1}^{n} a_\mu^2+2\sum_{k=1}^{n}\varphi_k\sum_{\mu=1}^{n-k} a_\mu a_{\mu+k} \tag{4.16}$$

$$=\frac{1}{\Delta}\{\varphi_0\sum_{\mu=0}^{n} a_\mu\Delta_{\mu+1}+\sum_{k=1}^{n}\varphi_k\sum_{\mu=0}^{n-k}(a_\mu\Delta_{\mu+k+1}+a_{\mu+k}\Delta_{\mu+1})$$

Arranging this expression in Δ_μ, there results

$$\Delta\mathrm{Min}\langle e_n^2\rangle=\sum_{k=0}^{n}\Delta_{k+1}\left(\sum_{\mu=0}^{n} a_\mu\varphi_{|\mu-k|}\right) \tag{4.17}$$

As Δ_{k+2} is a cofactor of the (n+2,k+2) element of the determinant Δ, Δ Min $\langle e_n^2\rangle$ becomes the determinant formed by replacing the n+2th row of Δ by

$$\sum_{\mu=0}^{n} a_\mu\varphi_{|\mu-k|}, \quad (k=0,1,2,\cdots\cdots,n) \tag{4.18}$$

and 0. Multiplying the first, second,....., and n+1th column of this new determinant Δ Min $\langle e_n^2 \rangle$ by a_0, a_1,....., and a_n respectively and subtracting them from the n+2th column, all the elements in this n+2th column vanish by conditions (4.8) except the n+1th element. As this n+1th element is $-a_0$ (= -1), there results

$$\mathrm{Min} \langle e_n^2 \rangle = -\frac{\Delta_{n+2}}{\Delta} \tag{4.19}$$

By the notice mentioned at the end of the previous section, we obtain

$$\mathrm{Min} \langle e_n^2 \rangle = -\frac{1}{2} \kappa \tag{4.20}$$

This is the physical meaning of the Lagrange's multiplier κ .

4.4 Example

To illustrate the application of the general theory, it will be applied to the system with a stationary random noise having a finite bandwidth, including frequency components up to but not beyond a frequency of 2f rad/sec .

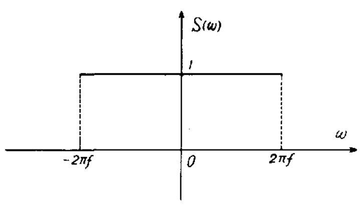


Fig. 4.2 Spectral density of noise.

Let its spectral density be as follows (Fig. 4.2) :

$$S(\omega) = \begin{cases} 1 , & |\omega| \leq 2\pi f \\ 0 , & |\omega| > 2\pi f \end{cases} \tag{4.21}$$

The autocorrelation function of this noise becomes

$$\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-2\pi f}^{2\pi f} e^{j\omega t}\, d\omega \tag{4.22}$$

$$= \frac{\sin 2\pi f t}{\pi t}$$

therefore,

$$\varphi_n = \frac{\sin 2n\pi f t_0}{2n\pi f t_0} \cdot \varphi_0 \quad , \qquad \varphi_0 = 2f \tag{4.23}$$

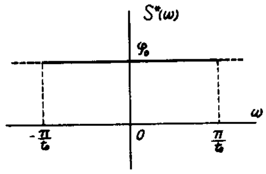
4.4.1 The case of a sampling period $t_0 = 1/2f$.

If $t_0 = 1/2f$, the autocorrelation coefficients become

$$\varphi_n = 0 \ , \ (n \neq 0) \quad : \ \varphi_0 = 2f \tag{4.24}$$

and the pulse spectral density becomes constant in $(-\pi/t_0, +\pi/t_0)$ as shown in Fig. 4.3 . In the evaluation of a_μ , we may set $\varphi_0 = 1$

Fig. 4.3 Pulse spectral density of noise ($t_0 = 1/2f$).



according to the notice mentioned at the end of the section 3.

Therefore, the determinant Δ in this case can be reduced to (m,m) determinant as follows (Appendix II-1) :

$$\Delta = (-1)^{m+1} \ |C_{p \cdot g}| \tag{4.25}$$

where

$$C_{p \cdot q} = \sum_{\mu=1}^{n} \{{}^{\mu}_{p-1}\}\{{}^{\mu}_{q-1}\} = C_{q \cdot p} \tag{4.26}$$

and especially

$$
\left.
\begin{aligned}
C_{11} &= \sum_{\mu=1}^{n} 1 = n \\
C_{12} &= \sum_{\mu=1}^{n} \mu = \frac{n\,(n+1)}{2} = C_{21} \\
C_{22} &= \sum_{\mu=1}^{n} \mu^2 = \frac{n\,(n+1)\ (2n+1)}{6} \\
C_{13} &= \sum_{\mu=2}^{n} \mu\,(\mu-1) = \frac{(n-1)\,n\,(n+1)}{3} = C_{31} \\
C_{23} &= \sum_{\mu=2}^{n} \mu^2\,(\mu-1) = \frac{(n-1)\,n\,(n+1)\ (3n+2)}{12} = C_{32} \\
C_{33} &= \sum_{\mu=2}^{n} \mu^2\,(\mu-1)^2 = \frac{(n-1)\,n\,(n+1)(3n^2-2)}{15}
\end{aligned}
\right\} \tag{4.27}
$$

Similarly $\triangle_{\mu+1}$ becomes (Appendix II-2)

$$\triangle_{\mu+1} = (-1)^m \left| C'_{p\cdot q} \right| \tag{4.28}$$

where $|C'_{p\cdot q}|$ is the determinant formed from $|C_{p\cdot q}|$ by replacing the first column by the following set of values

$$1 \ , \ \mu \ , \ \mu\,(\mu-1) \ , \ \ldots , \left\{ {\mu \atop m-1} \right\} \tag{4.29}$$

Finally

$$\triangle_{n+2} = (-1)^m \left| C''_{p\cdot q} \right| \tag{4.30}$$

where all elements of $|C''_{p\cdot q}|$ except C_{11} are equal to those of $|C_{p\cdot q}|$ (Appendix II-3) and

$$C''_{11} = C_{11} + 1 \tag{4.31}$$

Then we obtain

$$a_\mu = -\frac{|C'_{p\cdot q}|}{|C_{p\cdot q}|} \quad \text{and} \quad \mathrm{Min} \langle e_n^2 \rangle = \frac{|C''_{p\cdot q}|}{|C_{p\cdot q}|} \tag{4.32}$$

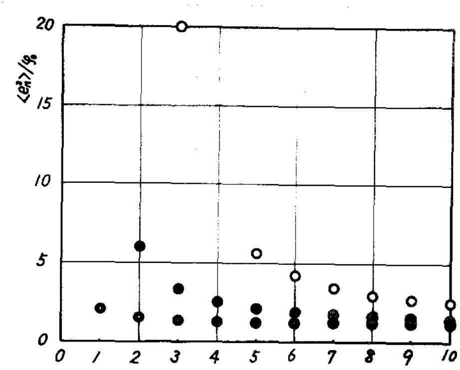
Derivations of (4.25), (4.28), and (4.30) are given in Appendixes ~~I, II~~ II-1, II-2, and ~~III~~ II-3. Table 4.1 shows some values of a's and Min$\langle e_n^2 \rangle$ When m = 1, 2, and 3 (step, ramp, and acceleration).

Table 4.1

input	a_μ $(\mu \neq 0)$	Min $\langle e_n^2 \rangle$
step	$-\frac{1}{n}$	$(1+\frac{1}{n})\varphi_0$
ramp	$\frac{2}{n(n-1)}\{3\mu-(2n+1)\}$	$\{1+\frac{2(2n+1)}{n(n-1)}\}\varphi_0$
acc.	$\frac{-3}{n(n-1)(n-2)}\{(3n^2+3n+2) -4(3n-1)\mu+10\mu(\mu-1)\}$	$\{1+\frac{3(3n^2+3n+2)}{n(n-1)(n-2)}\}\varphi_0$

The relation between Min $\langle e_n^2 \rangle$ and n (settling time) is shown in Fig. 4.4 .

Fig. 4.4 Minimized mean square error (case 1).



The sign ◎ shows the Min$\langle e_n^2 \rangle$ of the system which has been designed to respond to a step input without steady state error, ● to a ramp input, and ○ to an acceleration input. Of course, the smaller the Min $\langle e_n^2 \rangle$ and n are, the better the performance of the control system becomes. However, the Min $<e_n^2>$ decreases monotonously with the order 1/n as found in Table 4.1. On the other hand, for a fixed n, the Min $<e_n^2>$ increases according as the condition becomes severer in such a way as step, ramp, and acceleration (as m increses).

These results are reasonable.

Judging from Fig. 4.4 only, we had better not choose n=3 or 4 when we demand the finite settling time response to an acceleration input (○ , m=3). Similarly, we had better not choose n=2 to a ramp input (● , m=2). In order to make these facts clearer, let's examine the integrated square error $\sum_{n=0}^{\infty} e_n^2$ of the system designed as mentioned above for a step input, a ramp input, or an acceleration input. Fig. 4.5 shows integrated square errors for a step input of systems designed not only to respond to a step input (◎), a ramp input (●), and an acceleration input (○) without steady state error, but also to optimize the response to a random noise with the constant pulse spectral density as shown in Fig. 4.3 . Fig. 4.6 shows integrated square errors for a ramp input of such systems.

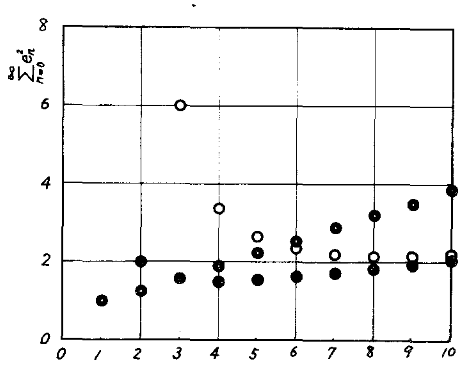


Fig. 4.5 Integrated-square error for step input (case 1).

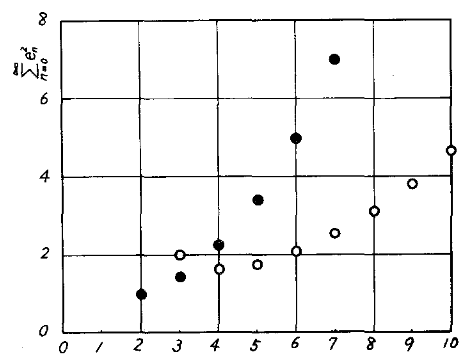


Fig. 4.6 Integrated-square error for ramp input (case 1).

Fig. 4.7 Integrated-square error
for acc. input (case 1).

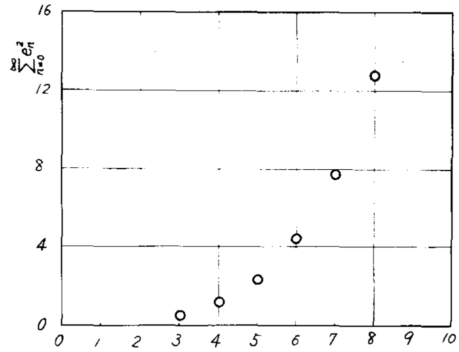


Fig. 4.7 shows the same for an acceleration input.

Now, we consider a number of systems having the capability of responding to an acceleration input without steady state error (◯).

According as n increases, Min $\langle e_n^2 \rangle$ decreases monotonously (by Fig. 4.4), and $\sum_{n=0}^{\infty} e_n^2$ for an acceleration input increases monotonously (by Fig. 4.7).

However, Fig. 4.5 shows that $\sum_{n=0}^{\infty} e_n^2$ for a step input takes the minimum values at n=8 and Fig. 4.6 shows that $\sum_{n=0}^{\infty} e_n^2$ for a ramp input takes the minimum value at n=4. From these facts, we can conclude that it is the best to choose the settling time as five or six sampling periods. The characteristics of these systems as a filter,

$$|K(e^{i\omega t_0})|^2 = \sum_{\mu=0}^{n} a_\mu^2 + 2\sum_{k=1}^{n}\sum_{\mu=0}^{n-k} a_\mu a_{\mu+k} \cos k\omega t_0 \qquad (4.33)$$

are given in Fig. 4.8 (for systems having the capability of responding to a step input without steady state error), Fig. 4.9 (for a ramp input), and Fig. 4.10 (for an acceleration input). As the pulse spectral density is constant, each system is designed to minimize the area enclosed by the corresponding curve and the horizontal axis under conditions (4.8) and (4.9).

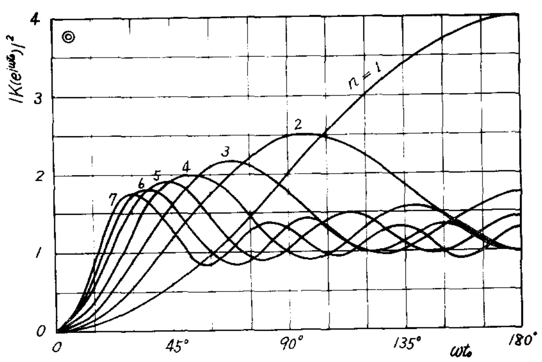


Fig. 4.8 Characteristics as a filter (m=1 ; case 1).

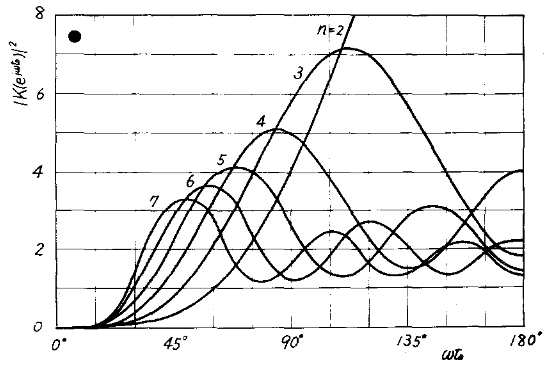


Fig. 4.9 Characteristics as a filter (m=2 ; case 1).

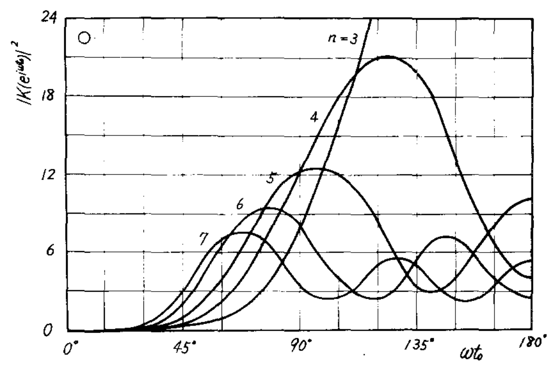


Fig.4.10 Characteristics as a filter (m=3 ; case 1).

4.4.2 The case of a sampling period $t_0 = 1/4f$

Putting $t_0 = 1/4f$ in (4.22), the autocorrelation coefficients of this case become

$$\varphi_n = \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\,\varphi_0 \;,\quad \varphi_0 = 2f \tag{4.34}$$

Therefore, the pulse spectral density becomes as follows :

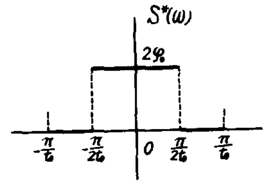
$$S^*(\omega) = \varphi_0\left\{1 + 2\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} \cdot \cos n\omega t_0\right\} \tag{4.35}$$

or

$$\left.\begin{aligned} S^*(\omega) &= 2\varphi_0 \quad \text{in } \left[-\frac{\pi}{2t_0}\,,\ \frac{\pi}{2t_0}\right] \\ &= 0 \qquad \text{Otherwise} \end{aligned}\right\} \tag{4.36}$$

as shown in Fig. 4.11 .

Fig. 4.11 Pulse spectral density of noise ($t_0 = 1/4f$).



Substituting (4.34) in (4.13) and (4.14) and evaluating a's, there results Table 4.2 when we demand that systems respond to a step input without steady state error (m = 1), Table 4.3 for systems responding to a ramp input without steady state error (m = 2), and Table 4.4 to an acceleration input (m = 3).

Table 4.2

For a step input (m = 1)						
n	a_0	a_1	a_2	a_3	a_4	a_5
1	1.	-1.				
2	1.	-1.3760	0.3760			
3	1.	-1.6935	1.5384	-0.8449		
4	1.	-2.1699	2.4057	-1.7996	0.5638	
5	1.	-2.6249	3.8414	-3.7097	2.2884	-0.7955

Table 4.3

for a ramp input (m = 2)						
n	a_0	a_1	a_2	a_3	a_4	a_5
2	1.	-2.	1.			
3	1.	-2.2632	1.5265	-0.2596		
4	1.	-2.5132	2.9337	-2.3277	0.9073	
5	1.	-3.0098	4.1246	-3.7838	2.1521	-0.4837

Table 4.4

for an acceleration input (m = 3)						
n	a_0	a_1	a_2	a_3	a_4	a_5
3	1.	-3.	3.	1.		
4	1.	-3.2102	3.6309	-1.6309	0.2103	

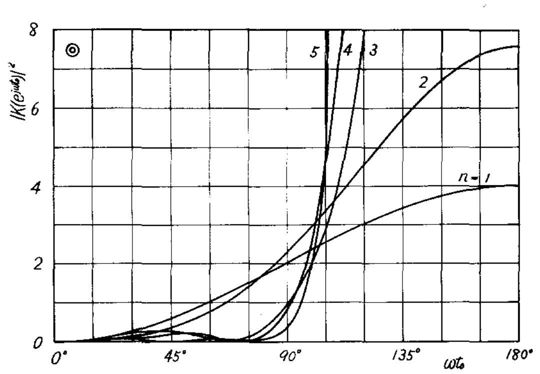


Fig. 4.12 Characteristics as a filter (m=1 ; case 2).

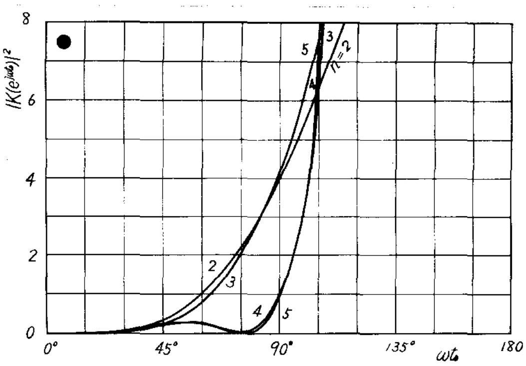


Fig. 4.13 Characteristics as a filter (m =2 ; case 2).

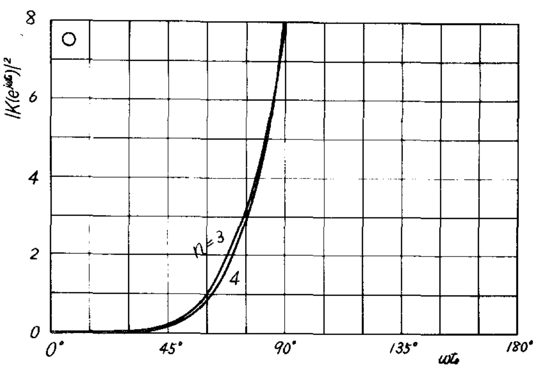


Fig. 4.14 Characteristics as a filter (m=3 ; case 2).

The characteristics as a filter $|K(e^{i\omega t_0})|^2$ are shown in Fig. 4.12 (m = 1), Fig. 4.13 (m = 2), and Fig. 4.14 (m = 3).

The remarkable feature of these curves is that they appoach zero in $90^\circ \geq \omega t_0 \geq -90^\circ$. This tendency is opposite to that of the pulse spectral density as shown in Fig. 4.11. As we designed so as to minimize the area enclosed by the curve in the interval $90^\circ \geq \omega t_0 \geq -90^\circ$ and the horizontal axis under conditions (4.8) and (4.9), this is reasonable.

The larger the n (settling time) or m (a sort of an input as a constraint) becomes, the more remarkable the tendency is.

At the ideal state $|K(e^{i\omega t_0})|^2 \to 0$ in $90^\circ \geq \omega t_0 \geq -90^\circ$, then $\langle e_n^2 \rangle \to 0$.

Five pairs of these curves — (n = 1 , n = 2),(n = 3 , n = 4) in Fig. 4.12, (n=2 , n=3),(n=4 , n=5) in Fig. 4.13, and (n=3 , n=4) in Fig. 4.14 — resemble one another, therefore, we had better choose the smaller n in each pair. Fig. 4.15 shows the minimized mean square errors, Min $<e_n^2>$, of this case ($t_0 = 1/4f$). we can recognize that five pairs mentiond also resemble one another in Fig. 4.15.

Figs 4.16, 4.17, and 4.18 show integrated square errors to a step input, a ramp input and an acceleration input respectively.

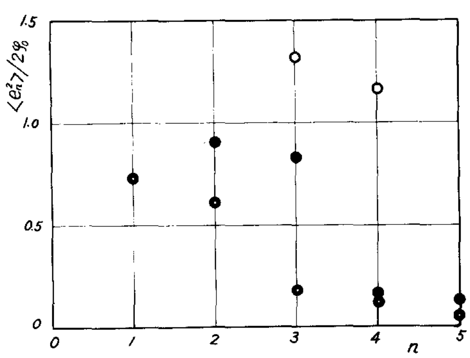


Fig. 4.15 Minimized-mean square error (case 2).

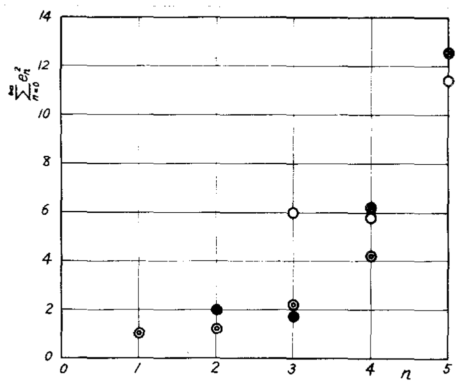


Fig. 4.16 Integrated-square error for step input (case 2).

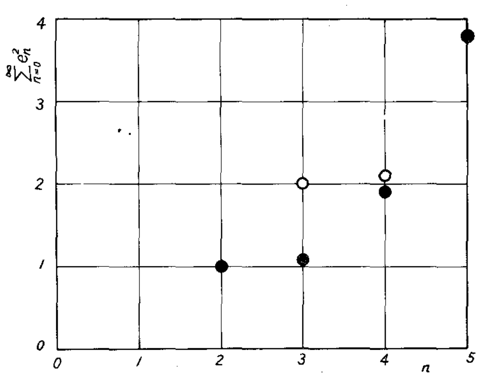


Fig. 4.17 Integrated-square error for ramp input (case 2).

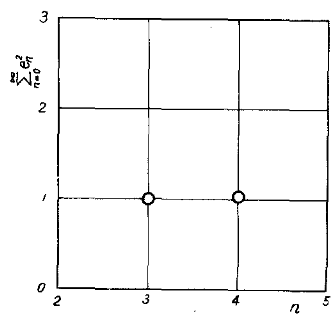


Fig. 4.18 Integrated-square error for acc. input (case 2).

Comparing the Min $<e_n^2>$ of this case with that of the case 1, we can recognize that if we choose a sampling period which is determined by the sampling theorem (case 1), the Min $<e_n^2>$ is always larger than the variance of noise (i.e. φ_0), and on the other hand, for a half of the sampling period determined by the theorem (case 2), Min $<e_n^2> \to 0$ as

$n \to 0$. Therefore, we had better choose a smaller sampling period than that which is determined by the theorem.

Fig. 4.19 shows the relation between minimized mean square errors and a sampling period.

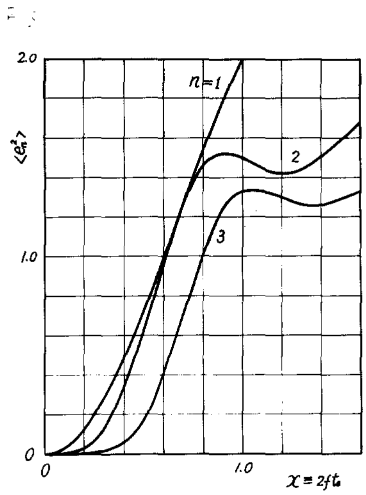


Fig. 4.19 Relation between minimized mean square error and sampling period.

4.5 Conclusion

Under the constraint that the system must respond to the integration-type input of the order m without steady state error, the pulse transfer function of the controller which minimize the mean square error of the control system to a stationary random noise is determined by (4.15).

From two examples of the noise having a finite bandwidth, we can conclude the following : if we reserve 2 or 3 freedoms without making the settling time the shortest and use them to adjust the system statistically as mentioned, then we can make small not only the mean square error for the noise, but also the integrated square error for the integration-type input.

Chapter 5

FUNDAMENTAL STUDY OF FINITE SETTLING TIME RESPONSE

5.1 Introduction

A synthesis of finite settling time systems has been studied by using the Z transform method or the advanced Z transform method.

However, in order to make the mechanism of such systems clear, we must investigate indicial responses of the controlled element in the time domain. This chapter will be devoted to the fundamental study of the finite settling time response.

5.2 The case of the second order controlled system: $\dfrac{1}{s(Ts+1)}$

Let us consider the case of the second order controlled element whose transfer function is

$$\frac{1}{s(Ts+1)} \quad . \tag{5.1}$$

The indicial response of this element is

$$g(t) = t - T(1-e^{-\frac{t}{T}}) \quad . \tag{5.2}$$

If we assume the zero-order hold element, this system can give finite settling time response to a step or a ramp input.

5.2.1 Step response synthesis

The response to three step signals of magnitudes k_0, k_1 and k_2, which are supplied at the instants $t = 0$, t_0 and $2t_0$, becomes

$$\begin{aligned} y(t) &= k_0 g(t) + k_1 g(t-t_0) + k_2 g(t-2t_0) \\ &= -T(k_0+k_1+k_2) - t_0(k_1+2k_2) \\ &\quad + (k_0+k_1+k_2)t + (k_0+mk_1+m^2k_2)\,e^{-\frac{t}{T}} \end{aligned} \tag{5.3}$$

In order that (5.3) is equal identically to unity, the following relations must be satisfied because 1, t and $e^{-\frac{t}{T}}$ are linearly independent.

$$k_1 + 2k_2 = -1/t_0$$

$$k_0 + k_1 + k_2 = 0 \tag{5.4}$$

$$k_0 + mk_1 + m^2 k_2 = 0$$

Solving these equations, there results

$$k_0 = 1/t_0(1-d)$$

$$k_1 = -(1+d)/t_0(1-d) \tag{5.5}$$

$$k_2 = d/t_0(1-d)$$

where $d = 1/m = e^{-\frac{t_0}{T}}$. Therefore, the system responds without steady state error to the manipulated signal shown in Fig. 5.1.

The pulse transfer function of a controller in an open loop as shown in Fig. 5.2 becomes

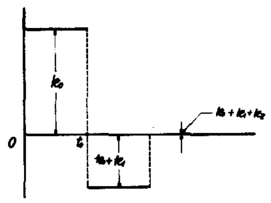


Fig. 5.1 Manipulated signal for illustrative example.

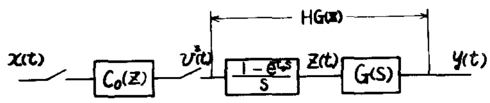


Fig. 5.2 Block diagram of sampled -data system with series compensator.

$$C_0(z) = k_0 + k_1 z^{-1} + k_2 z^{-2} = (1-z^{-1})(1-dz^{-1})/t_0(1-d) \tag{5.6}$$

This is the same result that is obtained by Jury-Schreoder's method.[3][4]

5.2.2 Ramp response synthesis

As we can not obtain a signal at t = 0 for a ramp input, we must supply three step signals of magnitudes k_1, k_2 and k_3 at $t = t_0$, $2t_0$ and $3t_0$ respectively. The response for $t \geq 3t_0$ becomes

$$
\begin{aligned}
y(t) &= k_1 g(t-t_0)+k_2 g(t-2t_0)+k_3 g(t-3t_0) \\
&= -T\ (k_1+k_2+k_3)-t_0(k_1+2k_2+3k_3) \\
&\quad + (k_1+k_2+k_3)t+T(mk_1+m^2k_2+m^3k_3)e^{-\frac{t}{T}} \qquad (5.7) \\
&\equiv t
\end{aligned}
$$

where $m = e^{\frac{t_0}{T}}$. Therefore, we obtain the following relations:

$$
\begin{aligned}
k_1+k_2+k_3 &= 1 \\
k_1+2k_2+3k_3 &= -T/t_0 \qquad (5.8) \\
mk_1+m^2k_2+m^3k_3 &= 0
\end{aligned}
$$

The solution of these equations becomes

$$
\begin{aligned}
k_1 &= \frac{1}{(1-d)^2}\left\{ (2+\frac{T}{t_0})\ (1-d)-d \right\} \\
k_2 &= \frac{-1}{(1-d)^2}\left\{ (1+\frac{T}{t_0})\ (1-d^2)-2d^2 \right\} \qquad (5.9) \\
k_3 &= \frac{d}{(1-d)^2}\left\{ (1+\frac{T}{t_0})\ (1-d)-d \right\}
\end{aligned}
$$

where $d = 1/m = e^{-\frac{t_0}{T}}$. Therefore, we have the pulse transfer function of the controller in the open-loop as follows:

$$
\begin{aligned}
C_0(z) &= \frac{(1-z^{-1})}{t_0}(k_1+k_2z^{-1}+k_3z^{-2}) \\
&= \frac{(1-z^{-1})\ (1-dz^{-1})}{t_0(1-d)^2}\left\{ (1-d)+\left[(1-2d)+\frac{t_0}{T}(1-d)\right]\ (1-z^{-1}) \right\} \qquad (5.10)
\end{aligned}
$$

5.3 Method by reverse-device

It is not neccessary for obtaining the finite settling time response to supply step signals at equal intervals of time t_0.

Another method of designing finite settling time systems was proposed by S. Hoshino(5). It is achieved by introducing a reverse-device but has some defects. His method can be radically improved by the method mentioned above. To illustrate the technique, the system, which is used in the preceding section, is considered. If three step signals of amplitudes K,-2K and K are supplied at $t = 0$, t_1 and t_0, the response of the system in $t \geq t_0$ becomes as follows:

$$\begin{aligned} y(t) &= Kg(t)-2Kg(t-t_1)+Kg(t-t_0) \\ &= K\ (2t_1-t_0)+KT(1-2e^{\frac{t_1}{T}}+e^{\frac{t_0}{T}})e^{-\frac{t}{T}} \end{aligned} \tag{5.11}$$

In order that (5.11) is equal identically to unity in $t \geq t_0$, the following conditions must be satisfied

$$\left.\begin{aligned} 2t_1 - t_0 &= \frac{1}{K} \\ 1 - 2e^{\frac{t_1}{T}} + e^{\frac{t_0}{T}} &= 0 \end{aligned}\right\} \tag{5.12}$$

This the same result that is given by Hoshino.

The feed-back system is constructed by introducing a reverse-device as shown in Fig. 5.3 where t_0 is a sampling period. The manipulated

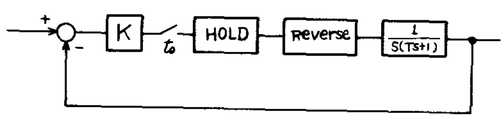


Fig. 5.3 Sampled-data system with reverse-device.

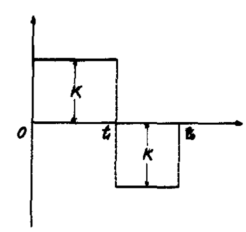


Fig. 5.4 Manipulated signal of system of Fig. 5.3.

signal is shown in Fig. 5.4, and the step signal is reversed at the instant $t = t_1$ ($t_0 > t_1 > 0$).

5.4 Settling time

If we supply manipulated step signals at equal intervals of time t_0, the settling time depends only upon a number of linearly independent functions which construct the indicial response of a controlled element.

Assume the number to be N, then the settling time becomes

$$\begin{array}{lll} & (N-1)t_0 & \text{for a step input} \\ \text{and} & Nt_0 & \text{for a higher order input} \end{array} \tag{5.13}$$

where t_0 is a sampling period. However if we choose the sampling period so as to reduce the number of linearly independent functions, we make the settling time shorter than (5.13).

For example, assume an indicial response to be

$$g(t) = 1 - e^{-t} \cos \frac{\pi}{2} \tag{5.14}$$

then

$$g(t-t_0)=1-e^{-t}\left(\cos\frac{\pi}{2} t_0 \cos\frac{\pi}{2} t+\sin\frac{\pi}{2} t_0 \sin\frac{\pi}{2} t\right). \tag{5.15}$$

If we choose the sampling period as $t_0 = 2$, (5.15) becomes

$$g(t) = 1 + e^{2} e^{-t} \cos\frac{\pi}{2} t . \tag{5.16}$$

Therefore, the response for two step signals of magnitudes k_0 and k_1 at the instants $t = 0$ and $t = 2$ becomes

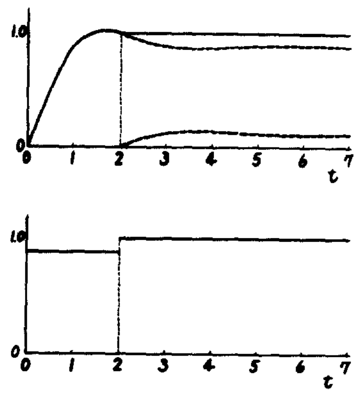
$$\begin{aligned} y(t) &= k_0\left(1-e^{-t} \cos\frac{\pi}{2} t\right)+k_1\left(1+e^2 e^{-t} \cos\frac{\pi}{2} t\right) \\ &= k_0+k_1-(k_0-e^2 k_1)e^{-t} \cos\frac{\pi}{2} t. \end{aligned} \tag{5.17}$$

If we choose k_0 and k_1 as

$$\begin{aligned} k_0 &= \frac{1}{1+d} \\ k_1 &= \frac{d}{1+d} \\ d &= e^{-2} \quad , \end{aligned} \tag{5.18}$$

we can settle the response completely to unity after $t = t_0 = 2$ (Fig. 5.5).

Fig. 5.5 Compensated response of system used in example.



5.5 Experiment

If a transfer function of a controlled element is

$$\frac{k}{(s+a)^2+b^2}, \tag{5.19}$$

we can obtain the finite settling response by supplying three step signal of the following magnitude k_i at the instants $t = 0$, t_0 and $2t_0$,

$$\begin{aligned}
k_0 &= \frac{a^2+b^2}{k}\,\frac{e^{2at_0}}{e^{2at_0}-2e^{at_0}\cos bt_0+1} \\
k_1 &= \frac{a^2+b^2}{k}\,\frac{-e^{at_0}\cos bt_0}{e^{2at_0}-2e^{at_0}\cos bt_0+1} \\
k_2 &= \frac{a^2+b^2}{k}\,\frac{1}{e^{2at_0}-2e^{at_0}\cos bt_0}
\end{aligned} \tag{5.20}$$

The results of an experiment by an analoge computer are shown in Fig. 5.6 and Fig. 5.7. Fig. 5.6 shows the indicial response of (5.19) and Fig. 5.7 shows the compensated response of this systems, where $a = 1/4$, $b = 2$, and $k = 1$ in (5.19).

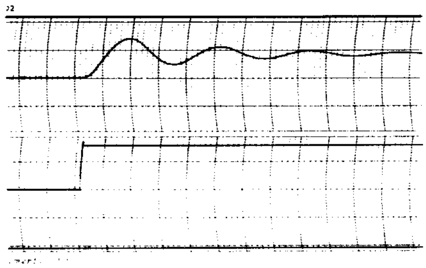


Fig. 5.6 Indicial response of system of (5.19).

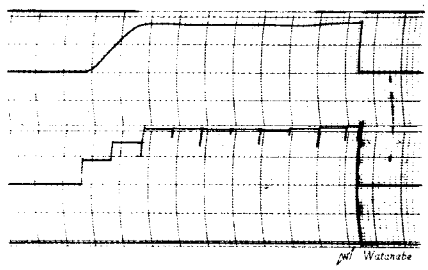


Fig. 5.7 Compensated response of system of (5.19).

5.6 Generalization of theory

We can generalize the above theory as follows.

Assume an indicial response of a controlled system to be given by a linear conbination of N linearly independent functions $f_i\,(t)$

$$g\,(t) = \sum_{i=1}^{N} a_i f_i\,(t) \tag{5.21}$$

and that the number of linearly independent functions (i.e. N) is invariable or its increase is finite by a transfer of the origin of time,

$$g\,(t - t_\mu) = \sum_{i=1}^{N} a_i^\mu f_i\,(t) \tag{5.22}$$

where a_i^μ is a function of t_μ , then we can settle the response to the desired value by supplying step signals N times.

The response of this system to step signals of magnitude k_1 , k_2 , ... and k_N, which are supplied at the instants $t = t_1$, t_2 ,... and t_N respectively, becomes as follows:

$$\sum_{\mu=1}^{N} k_\mu g\,(t - t_\mu) = \sum_{\mu=1}^{N} k_\mu \sum_{i=1}^{N} a_i^\mu f_i\,(t) = \sum_{i=1}^{N} \{ \sum_{\mu=1}^{N} a_i^\mu k_\mu \} f_i\,(t) \tag{5.23}$$

If the setting signal is given by the linear combination of the same functions that construct the indicial response of the system: namely,

$$\sum_{i=1}^{N} b_i f_i\,(t) \tag{5.24}$$

then we can obtain the finite settling time response as follows.

Setting (5.23) identically equal to (5.24), there results

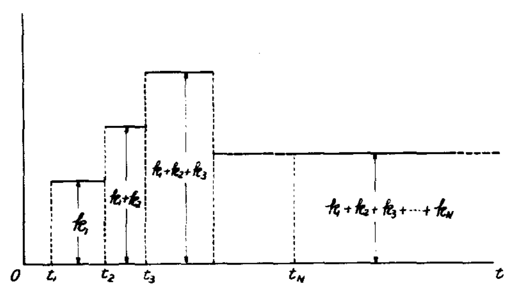
$$\sum_{i=1}^{N}\{\sum_{\mu=1}^{N} a_i^{\mu} k_{\mu} - b_i\} f_i(t) = 0 \tag{5.25}$$

As the functions $f_i(t)$ are linearly independent, we have the following relations

$$\sum_{\mu=1}^{N} a_i^{\mu} k_{\mu} = b_i \qquad (i = 1, 2, \cdots, N) \tag{5.26}$$

Solving (5.26), we have the required manipulated signal as shown in Fig. 5.8. The settling time is t_N .

Fig. 5.8 Manipulated signal.



5.6.1 Step response synthesis

Assume that

$$t_{\mu} = (\mu - 1)t_0 \tag{5.27}$$

where t_0 is the sampling period, and the pulse transfer function of the controller in the open loop as shown in Fig. 5.2 becomes

$$C_0(z) = \sum_{\mu=1}^{N} k_{\mu} z^{-\mu+1} \tag{5.28}$$

5.6.2 Ramp response synthesis

Assume that

$$t_{\mu} = \mu t_0 , \tag{5.29}$$

then

$$C_0(z) = \frac{(1-z^{-1})}{t_0} \sum_{\mu=1}^{N} k_\mu z^{-\mu+1} \tag{5.30}$$

The controller in a closed loop is given from $C_0(z)$ by the following relation

$$C(z) = \frac{C_0(z)}{1 - G(z) \quad C_0(z)} \tag{5.31}$$

5.7 Conclusion

The finite settling time response means to extract only the desired signal out of an indicial response by supplying the same type inputs several times to a controlled system.

The finite settling time systems can be designed in the following case, if we employ the zero-order hold element.

(1) The indicial response is constructed by a linear combination of a finite number of linearly independent functions.

(2) The number is invariable or its increase is finite by a transfer of the origin of time.

(3) The setting signal must be given by a linear combination of the same functions that construct the indicial response, as an old proverb says 'Nai sode wa furenu. (Out of nothing nothing comes.)'

Then the settling time is determined by the number of the above functions and the sampling period.

Chapter 6

EVALUATING METHOD OF TRANSIENT RESPONSE OF NON-LINEAR SAMPLED DATA CONTROL SYSTEMS

6.1 Introduction

It is the purpose of this chapter to present the evaluating method of the transient response of sampled data control systems with non-linear elements.

6.2 General procedure

In the sampled data control system with a non-linear element : $V_n = f(e_n, e_{n-1}, \ldots e_{n-l})$ as shown in Fig. 6.1 , let e_n be the value of

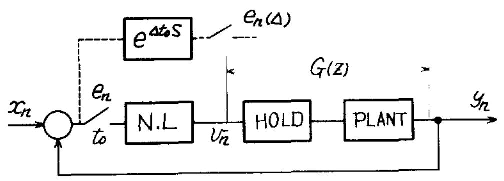


Fig. 6.1 Block diagram of sampled data system with non-linear element.

the control error e(t) at sampling instants, then it can be expressed as follows :

$$e_n = x_n - G(z) f(e_n, e_{n-1}, \ldots\ldots e_{n-l}) \tag{6.1}$$

where x_n is an input and G(z) is the pulse transfer function of the plant including the data hold element. If G(z) is a ratio of polynomials in Z^{-1} of the form

$$G(z) = \frac{b_m z^{-m} + \ldots . + b_q z^{-q}}{1 + a_1 z^{-1} + \ldots . + a_p z^{-p}} \tag{6.2}$$

then, substituting (6.2) in (6.1), there results :

$$e_n = X_n - \frac{b_m z^{-m} + \ldots + b_q z^{-q}}{1 + a_1 z^{-1} + \ldots + b_p z^{-p}} \cdot f(e_n, e_{n-1}, \ldots, e_{n-l}) \tag{6.3}$$

Multiplying both sides of (6.3) by the denominator of G(z) and putting $z^{-k}X_n = X_{n-k}, Z^{-k}e_n = e_{n-k}$ etc. , there results the following recurrence relation :

$$\begin{aligned} e_n = & (X_n + a_1 X_{n-1} + \ldots + a_p X_{n-p}) - (a_1 e_{n-1} + \ldots + a_p e_{n-p}) \\ & - (b_m f(e_{n-m}, \ldots, e_{n-l-m}) + \ldots + b_q f(e_{n-q}, \ldots, e_{n-l-q})) \end{aligned} \tag{6.4}$$

Therefore, under suitable initial conditions, we can calculate all values of e_n succesively by (6.4) .

The above mentioned method describes the behavier of the system at sampling instant only. On the other hand, by employing the modified or advanced Z transformation method, we can obtain the information between sampling instants. With the fictitious time advance $\exp(\Delta t_0 s)$ inserted as shown in Fig. 6.1 , the value of the control error becomes a function of Δ . Therefore, the following recurrence relation must be added to (6.1)

$$e_n(\Delta) = X_n(\Delta) - G(Z, \Delta) f(e_n, e_{n-1}, \ldots e_{n-l}) \tag{6.5}$$

where $G(z, \Delta)$ is the advanced pulse transfer fuction of the plant with the data hold of the form

$$G(Z, \Delta) = \frac{b_m(\Delta) Z^{-m} + \ldots + b_p(\Delta) Z^{-q}}{1 + a_1 Z^{-1} + \ldots + a_p Z^{-p}} \tag{6.6}$$

If $\Delta \to 0$, then $G(z, \Delta) \longrightarrow G(z)$. Substituting (6.6) in (6.5), the recurrence relation in this case can be written as follows :

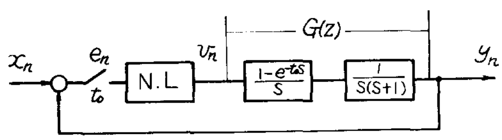
$$\begin{aligned} e_n(\Delta) = & (X_n + a_1 X_{n-1} + \ldots + a_p X_{n-p}) - (a_1 e_{n-1}(\Delta) + \ldots + a_p e_{n-p}(\Delta)) \\ & - (b_m(\Delta) f(e_{n-m}, \ldots, e_{n-l-m}) + \ldots + b_q(\Delta) f(e_{n-q}, \ldots, e_{n-l-q})) \end{aligned} \tag{6.7}$$

Since e_n (= e_n (0)) or $f(e_n \cdots e_{n-l})$ may be calculated (6.4), the value of the continuous control error is explored at any sampling intervals, if Δ is taken as a number ranging between zero and unity.

Example

To illustrate the technique, the sampled data control system shown in Fig. 6.2 will be used as an example. The pulse transfer

Fig. 6.2 Block diagram for illustrative example.



function of the plant including the zero-order hold element of the system shown in Fig. 6.2 becomes

$$G(Z) = \frac{(t_0 - 1 + d)\, Z^{-1} + (1 - d - t_0 d)\, Z^{-2}}{(1 - Z^{-1})\,(1 - dZ^{-1})} \tag{6.8}$$

where d = exp ($-t_0$) and t_0 is the sampling period.

The recurrence relation corresponding to (6.4) becomes as follows:

$$\begin{aligned} e_n - (1+d)\, e_{n-1} + d e_{n-2} &= X_n - (1+d)\, X_{n-1} + dX_{n-2} \\ &\quad - (t_0 - 1 + d) f(e_{n-1}) - (1 - d - t_0 d) f(e_{n-2}) \end{aligned} \tag{6.9}$$

If t_0 = 0.1, then

$$\begin{aligned} e_n &= 1.90484 e_{n-1} - 0.90484 e_{n-2} + (X_n - 1.90484 X_{n-1} + 0.90484 X_{n-2}) \\ &\quad - 0.00484 f(e_{n-1}) - 0.00468 f(e_{n-2}) \end{aligned} \tag{6.10}$$

As non-linear elements, let us assume those as shown in Fig. 6.3, and the responses of the control error to step inputs become as shown in Fig. 6.4 and Fig. 6.5 .

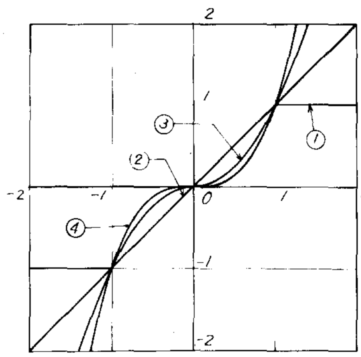


Fig. 6.3 Non-linear elements: 1) saturation, 2) linear, 3) quadratic and 4) cubic.

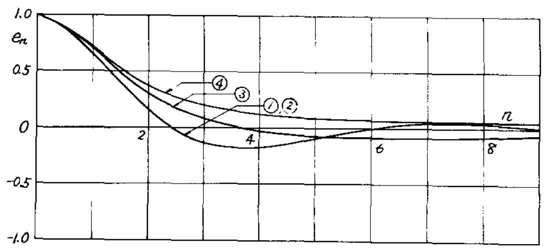


Fig. 6.4 Transient responses to unit step input.

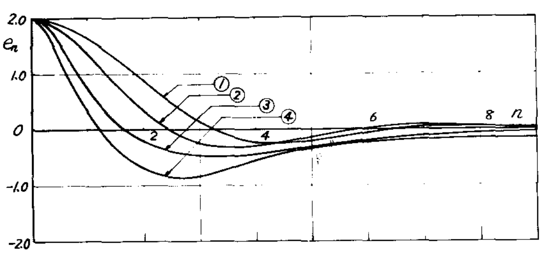


Fig. 6.5 Transient responses to step input of magnitude 2.

6.3 Application to many point control system with reays

Let us consider the many point control system with relays shown in Fig. 6.6 and assume that every relay takes only two states: unit positive output and unit negative output, and that for all relays a common digital computer determines their switching points.

For the system described by the difference equation, or the recurrence formula of order N, the optimum switcing line would be given

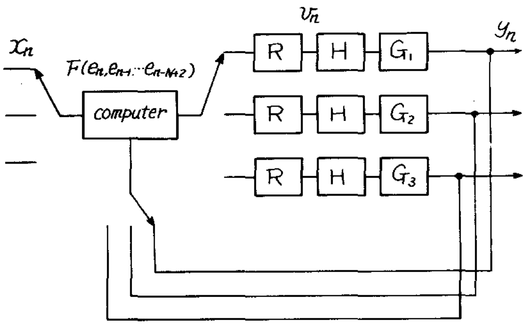


Fig. 6.6 Block diagram of many point control system with relays.

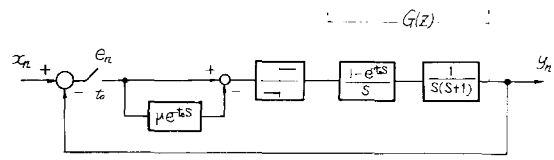


Fig. 6.7 Block diagram of system with linear switching line.

by functions of N-1 sampled data, e_n, e_{n-1}, ... and e_{n-N+2}

$$F_i\,(e_n, e_{n-1}, \dots e_{n-N+2}) = 0 \quad (i = 1, 2, \dots) \tag{6.11}$$

As an example, the same system as discussed in the previous section will be used. Let all controlled systems G(s)'s be equal to 1/s(s+1), then the many point control system as shown in Fig. 6.6 can be replaced by a single sampled data control system as shown in Fig. 6.7 .

In case of linear switching, $e_n - \mu\, e_{n-1} = 0$, where μ is a suitably chosen constant, the error response of this control system to unit step input can be calculated by the following recurrence relation

$$e_n = 1.90484e_{n-1} - 0.90484e_{n-2} - 0.00484V_{n-1} - 0.00468V_{n-2}$$

$$V_n = \mathrm{sign}\,(e_n - \mu\, e_{n-1}) \tag{6.12}$$

where the initial condition is $e_0 = 1$ and $e_1 = 0.99516$.

Although this differnce equation can easily be solved analytically, it is more practical to employ the procedure mentioned below to determine the values of V_n (= 1 or -1). As the switching line becomes a straight line with tangent μ passing through the origin of the (e_n , e_{n-1}) plane, V_n changes its sign from 1 to -1 or from -1 to 1 at the first point across the switching line when we plot (e_n , e_{n-1}) determined by the recurrence relation (6.12). The responses of this system for μ = 0, 0.75 and 0.9 are given in Figs. 6.8 (a),(b), and (c) respectively.

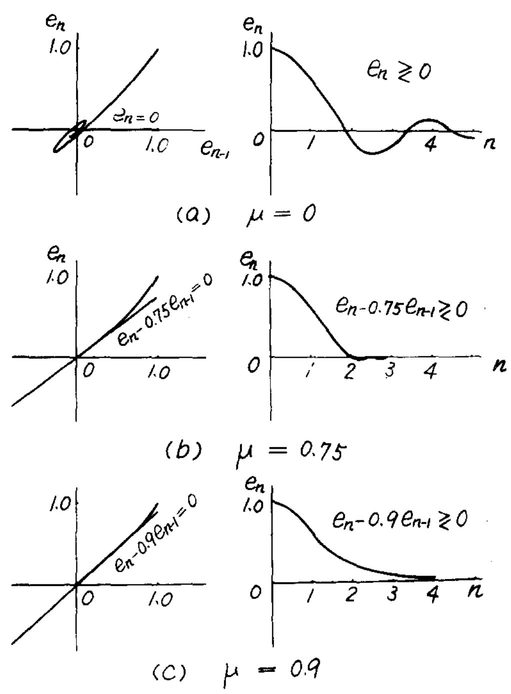


Fig. 6.8 Transients of system shown in Fig. 6.7.

On the other hand, the so-called optimum switching line corresponding to that of the continuous system can be obtained as the solution of the difference equation (6.12) ($V_{n-1} = V_{n-2}$ = 1 or -1) passing through the origin. The difference equation describing the behavier of the system shown in Fig. 6.7 for any step input can be written as

$$e_n-(1+d)\,e_{n-1}+d\,e_{n-1}=\pm A, \qquad (A = \text{const.}) \qquad (6.13)$$

The general solution of (6.13) becomes

$$e_n = C_1 + C_2 d^n \pm \frac{nA}{1-d} \tag{6.14}$$

where C_1 and C_2 are arbitrary constants. Eliminating n from e_n and e_{n-1}, there results

$$e_n - e_{n-1} = \pm \frac{A}{1-d} - C_2 \frac{1-d}{d} \cdot d^{\pm \frac{1}{A}\{e_n - de_{n-1} - C_1(1-d) \mp \frac{dA}{1-d}\}} \tag{6.15}$$

Determining C_1 and C_2 so that this solution curve may pass through the origin of the (e_n, e_{n-1}) plane, (6.15) becomes

$$e_n - e_{n-1} = \pm \frac{1}{1-d}\{1 - d^{\pm \frac{e_n - de_{n-1}}{A}}\} \tag{6.16}$$

As A is $-t_0(1-d)$, if $t_0 = 0.1$, the switching line becomes

$$e_n - e_{n-1} = \pm 0.1\{1 - \exp(\pm \frac{e_n - 0.90404 e_{n-1})}{0.9516}\} \tag{6.17}$$

or

$$e_n^* = \pm\{1 - \exp[\pm(e_n + 0.9086 e_n^*)]\}$$

$$e_n^* = \frac{e_n - e_{n-1}}{t_0} \tag{6.18}$$

These two curves are shown in Fig. 6.9 and Fig. 6.10. However, this switching line is not necessarily optimum, because the switching does not occur at the instant the trajectory crosses the switching line on the (e_n , e_{n-1}) plane, but, owing to the discrete detection of the error signal, it occurs at the next sampling instant. Therefore, the linear switching is more practical in respect of simplicity.

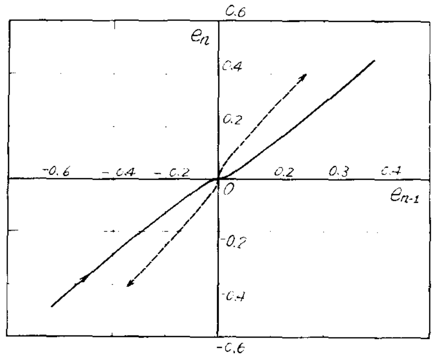


Fig. 6.9 Optimum switching line in (e_n , e_{n-1}) plane.

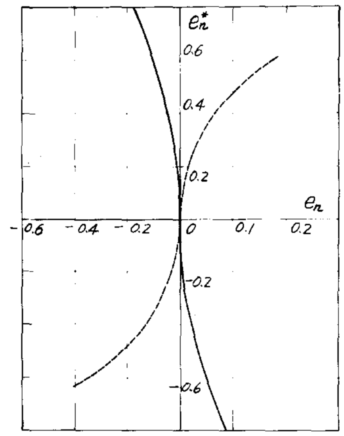


Fig. 6.10 Optimum switching line in (e_n^* , e_n) plane.

6.4 Conclusion

The transient response of sampled data control systems with non-linear elements can be calculated exactly, if there are a sampler and a hold element before and after each non-linear element. Otherwise, fictitious ones must be inserted for each non-linear element, but in this case, of course, we obtain the response only approximately.

This fact implies that this method can be also applied to calculate approximately the transient response of the conventional continuous non-linear control systems. In this case, as the physical realizability of the fictitious hold element need not be considered, we can improve the accuracy of the approximation by using the higher order hold element.

In the control system with relays as mentioned in the section 6.3, it is not economical to use the digital computer for determining the switching point only. This form of the control is not practical unless we use a part of the function of the digital computer equipped for the so-called computing control, profit control etc.

Chapter 7

TRIAL MANUFACTURE OF OPTIMUM NON-LINEAR CONTROLLER USING SAMPLED DATA

7.1 Control circuit

This chapter is concerned with the optimum non-linear controller using sampled data. This type of the controller has been already proposed by Prof. Y. Sawaragi and others.(6) We made it especially for process control.

The principle of this controller is shown in Fig. 7.1. Sampler A closes long enough for a signal to remain in the memory of a potentiometer.

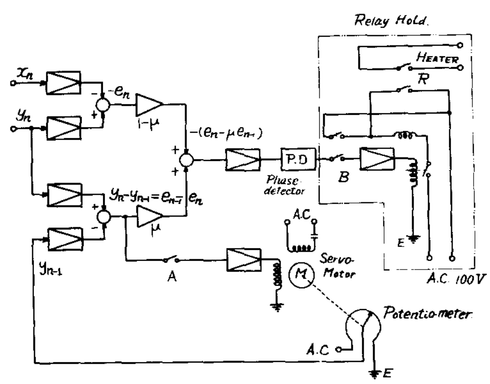


Fig. 7.1 Block diagram of optimum non-linear controller.

Sampler B operates one sampling period after the sampler A. A operates immediately after B was opened. These operations are repeated, so we obtain the signal at the previous sampling instant from the potentiometer whenever B closes. Relay R in a manipulated part operates by discriminating the sign of

$$e_n - \mu e_{n-1} \tag{7.1}$$

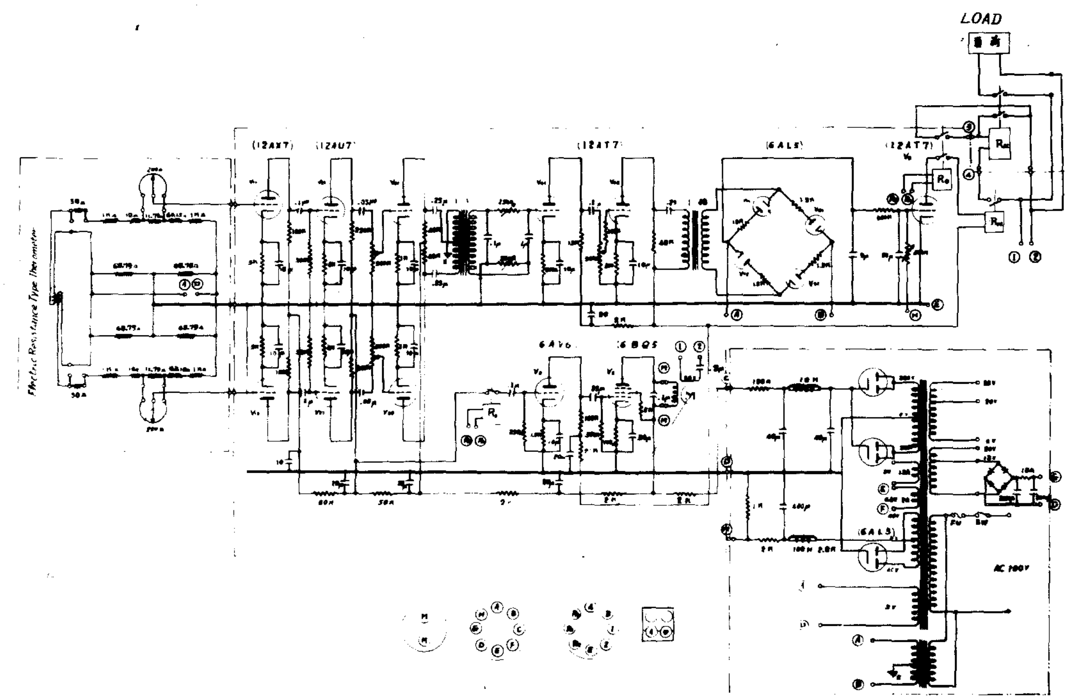


Fig. 7.2 Over-all circuit of optimum non-linear controller

by a phase detector as shown in Fig. 7.1, where e_n is the present sample and e_{n-1} is the previous one. Fig. 7.2 shows the circuit of this controller.

The transients of inner temperature of the water tank as shown in

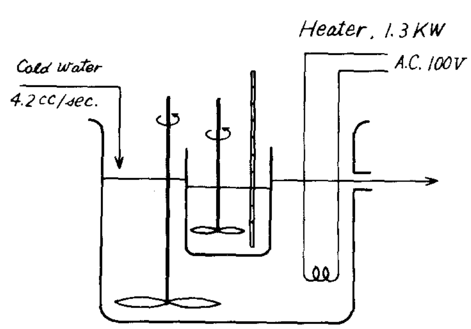


Fig. 7.3 Controlled system (water tank), outer tank : 4.45 litre, inner tank : 0.6 litre.

Fig. 7.3. are shown in Figs. 7.4 (a),(b), and (c). If we choose the value of μ properly, we have good response which has no overshoot and has small hunting.

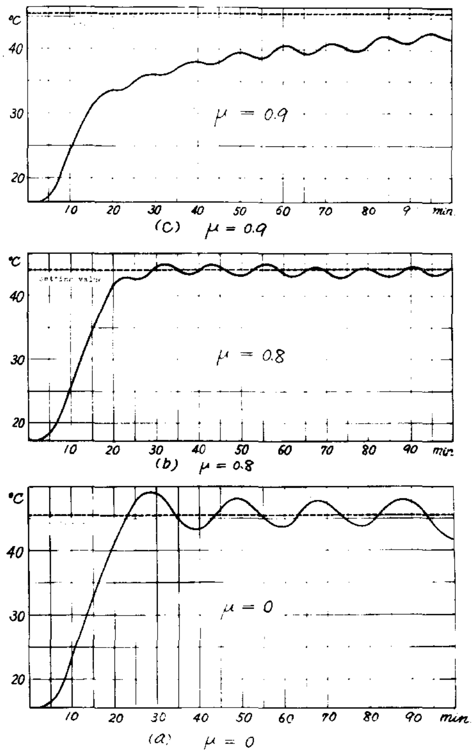


Fig. 7.4 Comparison of transients for threevalues of μ .

7.2 Evaluation of optimum switching line

The transfer function of the controlled system shown in Fig. 7.4 is

$$G(z) = \frac{ke^{-sL}}{(T_1 S+1)\ (T_2 S+1)}$$

$$k = 42\cdot 3^{\circ}C$$

$$L = 2.0\ \text{min.} \qquad (7.2)$$

$$T_1 = 4.8\,\text{min.}$$

$$T_2 = 17.9\,\text{min.}$$

so we have the Z transform of it as follows:

$$HG(z) = \frac{a z^{-m-1} + b Z^{-m-2}}{(1-d_1 Z^{-1})(1-d_2 Z^{-2})}$$

$$d_1 = e^{-\frac{t_0}{T_1}}, \qquad d_2 = e^{-\frac{t_0}{T_2}}, \qquad a = 1 - d_2 + \frac{T_1(d_1 - d_2)}{T_2 - T_1} \tag{7.3}$$

$$m = \frac{L}{t_0}, \qquad b = -d_1(1-d_2) + \frac{T_1(d_2 - d_1)}{T_2 - T_1},$$

Therefore, we obtain the recurrence relation which describes the response of the system as

$$\begin{aligned} e_n - (d_1 + d_2) e_{n-1} + d_1 d_2 e_{n-2} \\ = (1-d_1)(1-d_2) - aV_{n-m-1} - bV_{n-m-2} \end{aligned} \tag{7.4}$$

$$V_n = \mathrm{sign}\{ e_n - \mu e_{n-1} \}$$

The optimum switching line of this case is given by calculating (7.4) successively backward in such a way as $e_0 = e_{-1} = 0, e_{-2}, e_{-3}, \ldots\ldots$ (Fig. 7.5), when we set $V_{n-m-1} = V_{n-m-2} = +1$ or -1 in (7.4)

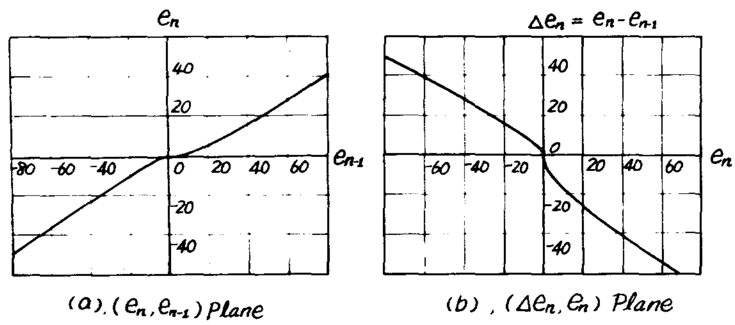


Fig. 7.5 Optimum swiching line.

APPENDIXES

Appendix I-1 Analytical Test for Stability[7]

A sampled data control system is stable if all the roots of the system characteristic equation f(z) = 0 lie within the unit circle.

For the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0 \tag{A.1}$$

we set

$$D_n = \left|\begin{array}{ccccc|ccccc}
a_0 & a_1 & \cdot & \cdot & \cdot & a_n & & & & \\
 & a_0 & \cdot & \cdot & \cdot & a_{n-1} & a_n & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & a_0 & a_1 & \cdot & \cdot & \cdot & a_n \\
\hline
a_n & a_{n-1} & \cdot & \cdot & \cdot & a_0 & & & & \\
 & a_n & \cdot & \cdot & \cdot & a_1 & a_0 & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & a_n & a_{n-1} & \cdot & \cdot & \cdot & a_0
\end{array}\right| \tag{A.2}$$

then we make D_{n-1} by taking off 2 columns and 2 rows with number n and 2n from D_n. D_{n-2} is formed from D_{n-1} in the same way......and finally

$$D_2 = \left|\begin{array}{cc|cc}
a_0 & a_1 & a_n & \\
 & a_0 & a_{n-1} & a_n \\
\hline
a_n & a_{n-1} & a_0 & \\
 & a_n & a_1 & a_0
\end{array}\right|, \quad D_1 = \left|\begin{array}{cc}
a_0 & a_n \\
a_n & a_0
\end{array}\right| \tag{A.3}$$

The stability criterion can be formulated as

$$D_k > 0 , (k = 1, 2, \dots, n) \tag{A.4}$$

The above (2n,2n) determinant D_n can be reduced to the following (n,n) determinant

$$D_n = \begin{vmatrix} d_{1,n} & d_{1,n-1} & \cdot & d_{1,1} \\ d_{2n} & d_{2,n-1} & \cdot & d_{2,1} \\ \cdot & \cdot & \cdot & \cdot \\ d_{n,n} & d_{n,n-1} & \cdot & d_{n1} \end{vmatrix} \tag{A.5}$$

where

$$d_{p,q} = \sum_{\lambda=0}^{p-1} \begin{vmatrix} a_\lambda & a_\mu \\ a_{n-\lambda} & a_{n-\mu} \end{vmatrix} , \quad (\lambda+\mu = p+q-1) \tag{A.6}$$

and $a_\mu = 0$ for $\mu > n$.

If we set

$$\begin{vmatrix} a_\lambda & a_\mu \\ a_{n-\lambda} & a_{n-\mu} \end{vmatrix} = (\lambda , \mu) \tag{A.7}$$

(A.6) becomes as follows :

$$d_{p,q} = (0 , p+q-1) + (1 , p+q-2) + \cdots + (p-1 , q) \tag{A.8}$$

If we pay attention to

$$(h , h) = 0 , (h , k) + (k , h) = 0 \tag{A.9}$$

the following relations can easily be shown

$$d_{p,q} = d_{q,p} \tag{A.10}$$

and from (A.8)

$$d_{p,q} = d_{n-p+1,n-q+1} = d_{n-q+1,n-p+1} \tag{A.11}$$

Therefore, the determinant (A.5) is simmetrical about two diagonals.

Thus the stability criterion can also be formulated as :

$$\text{All the principal minors of (A.5)} > 0 \tag{A.12}$$

Examples

For a quadratic polynomial

$$f(z) = a_0 z^2 + a_1 z + a_2 \tag{A.13}$$

the following determinants are obtained :

$$d_{12} = d_{21} = (0\,,2) = a_0^2 - a_2^2$$

$$d_{11} = d_{22} = (0\,,1) = a_1\,(a_0 - a_2)$$

$$D_1 = d_{12} = a_0^2 - a_2^2$$

$$D_2 = \begin{vmatrix} d_{12} & d_{11} \\ d_{22} & d_{21} \end{vmatrix} = (a_0 - a_2)^2\,(a_0 + a_1 + a_2)\,(a_0 - a_1 + a_2)$$

Then the stability conditions are

$$\begin{aligned} a_0^2 - a_2^2 &> 0 \\ (a_0 + a_1 + a_2)\,(a_0 - a_1 + a_2) &> 0 \end{aligned} \tag{A.14}$$

Similarly, for a third order system,

$$f\,(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3 = 0 \tag{A.15}$$

the following determinants are obtained :

$$\begin{aligned} d_{13} &= (0,3) = a_0^2 - a_3^2 \\ d_{12} &= (0,2) = a_0 a_1 - a_2 a_3 \\ d_{11} &= (0,1) = a_0 a_2 - a_1 a_3 \\ d_{22} &= (0,3) + (1,2) = a_0^2 - a_3^2 + a_1^2 - a_2^2 \end{aligned}$$

$$D_1 = d_{13} = a_0^2 - a_3^2$$

$$D_2 = \begin{vmatrix} d_{13} & d_{12} \\ d_{23} & d_{22} \end{vmatrix} = \begin{vmatrix} d_{13} & d_{12} \\ d_{12} & d_{22} \end{vmatrix}$$

$$= (a_0^2 + a_0 a_2 - a_1 a_3 - a_3^2)(a_0^2 - a_0 a_2 + a_1 a_3 - a_3^2)$$

$$D_3 = \begin{vmatrix} d_{13} & d_{12} & d_{11} \\ d_{23} & d_{22} & d_{21} \\ d_{33} & d_{32} & d_{31} \end{vmatrix} = \begin{vmatrix} d_{13} & d_{12} & d_{11} \\ d_{12} & d_{22} & d_{12} \\ d_{11} & d_{12} & d_{13} \end{vmatrix}$$

$$= (a_0^2 - a_0 a_2 + a_1 a_3 - a_3^2)^2 (a_0 + a_1 + a_2 + a_3)(a_0 - a_1 + a_2 - a_3)$$

Therefore, the stability conditions are

$$a_0^2-a_3^2 > 0$$

$$(a_0^2+a_0a_2-a_1a_3-a_3^2)\ (a_0^2-a_0a_2+a_1a_3-a_3^2) > 0 \qquad \text{(A.16)}$$

$$(a_0+a_1+a_2+a_3)\ (a_0-a_1+a_2-a_3) > 0$$

Appendix I-2 On the Boundary of Stability

The Schur's determinant D_n of (A.2) is nothing but the resultant formed from the following two polynomials :

$$\begin{aligned} f(z) &= a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n \\ f^*(z) &= a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0 \end{aligned} \qquad \text{(A.17)}$$

namely, $D_n = 0$ is the necessary and sufficient condition in which $f(z) = 0$ and $f^*(z) = 0$ have commom roots.

On the other hand, if we assume $f(z) = 0$ to be the system characteristic equation, $f(z) = 0$ and $f^*(z) = 0$ have commom roots at the boundary of stability. This fact can be proved as in the following : if $f(z) = 0$, the characteristic equation with real coefficients, has a root $\alpha = re^{i\theta}$, it has also its complex conjugate root $\bar{\alpha} = re^{-i\theta}$. On that occasion, $f^*(z) = 0$ has two roots : $\frac{1}{\alpha} = \frac{1}{r}e^{-i\theta}$ and $\frac{1}{\bar{\alpha}} = \frac{1}{r}e^{i\theta}$. As α (a root of $f(z) = 0$) and $1/\bar{\alpha}$ (a root of $f^*(z) = 0$) have the same argument θ, these two roots coincide on the unit circle (i.e. at the boundary of stability).

Thus $D_n = 0$ is the necessary condition for the boundary of stability, but not the sufficient one.

Appendix II-1 Derivation of (4.25)

Let E_n be (n,n) unit matrix and O_n be (n,n) zero matrix, then

$$\Delta = \begin{vmatrix} E_{n+1} & \Lambda^t \\ \Lambda & O_{m+1} \end{vmatrix} = - \begin{vmatrix} E_n & \Lambda_2^t \\ \Lambda_2 & O_m \end{vmatrix} \qquad \text{(A.1)}$$

where Λ is the $(m+1, n+1)$ matrix as follows :

$$
\Lambda \quad = \quad \left[\begin{array}{c|cccccc}
1 & 0 & 0 & 0 & \cdot & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & \cdot & 1 & 1 \\
0 & 1 & 2 & 3 & \cdot & n-1 & n \\
0 & 0 & 2.1 & 3.2 & \cdot & (n-1)(n-2) & n(n-1) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \left\{{n-1 \atop m-1}\right\} & \left\{{n \atop m-1}\right\}
\end{array}\right] \qquad (A.2)
$$

and Λ^t is a transpose matrix of Λ , Λ_1 is the matrix with the first column taken off Λ , and Λ_2 is the matrix with the first row taken off Λ_1 .

Multiplying $\triangle$ by the following determinant

$$
\triangle^* = \begin{vmatrix} E_n & O \\ -\Lambda_2 & E_m \end{vmatrix} = 1 \qquad (A.3)
$$

there results

$$
\triangle = \triangle^* \triangle = -\begin{vmatrix} E_n & O \\ -\Lambda_2 & E_m \end{vmatrix} \begin{vmatrix} E_n & \Lambda_2^t \\ \Lambda_2 & E_m \end{vmatrix} = - \begin{vmatrix} E_n & \Lambda^t \\ O & -\Lambda_2\Lambda_2^t \end{vmatrix}
$$

$$
\therefore \ \triangle = (-1)^{m+1} |\Lambda_2 \Lambda_2^t| = (-1)^{m+1} |C_{p,q}| \qquad (A.4)
$$

Appendix II-2 Derivation of (4.28)

Let $E_n^{(\mu)}$ be the unit matrix E_n in which the element (μ,μ) is replaced by 0 and $\Lambda^{(\mu)}$ be the matrix Λ in which the μ th row is replaced by the following row vector

$$
(1, 0, 0, \ldots\ldots, 0)^t \qquad (A.5)
$$

then $\triangle_{\mu+1}$ can be expressed as folloes :

$$
\triangle_{\mu+1} = \begin{vmatrix} E_{n+1}^{(\mu+1)} & \Lambda^t \\ \Lambda^{(\mu+1)} & O_{m+1} \end{vmatrix} \qquad (A.6)
$$

By the same procedure as mentioned above, $\triangle_{\mu+1}$ becomes

$$\triangle_{\mu+1} = \begin{vmatrix} E_{n+1} & O \\ -\Lambda & E_{m+1} \end{vmatrix} \begin{vmatrix} E_{n+1}^{(\mu+1)} & \Lambda^t \\ \Lambda^{(\mu+1)} & O_{m+1} \end{vmatrix} = \begin{vmatrix} E_{n+1}^{(\mu+1)} & \Lambda^t \\ -\Lambda E_{n+1}^{(\mu+1)} + E_{m+1}\Lambda^{(\mu+1)} , & -\Lambda\Lambda^t \end{vmatrix} \tag{A.7}$$

where $-\Lambda E_{n+1}^{(\mu+1)} + E_{m+1}\Lambda^{(\mu+1)}$ is the matrix whose elements are all zero except the element (1, 1) which is unity.

Exchanging the μ +1th column with the n+2th column, there results

$$\triangle_{\mu+1} = -\begin{vmatrix} E_{n+1} & (\Lambda^t)' \\ O & (-\Lambda\Lambda^t)' \end{vmatrix} = -\left| (-\Lambda\Lambda^t)' \right| \tag{A.8}$$

where $(-\Lambda\Lambda^t)'$ is the matrix formed by replacing the first column of $-\Lambda\Lambda^t$ by the μ +1th column of Λ^t, i.e.

$$O, 1, \mu, \mu(\mu-1), \cdots, \{ {}^{\mu}_{m-1} \} \tag{A.9}$$

As all the elements of the first row of $(-\Lambda\Lambda^t)'$ are zero except the second element which is unity, we obtain

$$\triangle_{\mu+1} = (-1)\, |C'_{p,q}| \tag{A.10}$$

Appendix II-3 Derivation of (4.30) :

$$\triangle_{n+2} = \begin{vmatrix} E_{n+1} & \Lambda_1^t \\ \Lambda_1 & O_m \end{vmatrix} = |-\Lambda_1\Lambda_1^t| = (-1)^m |C''_{p,q}| \tag{A.11}$$

Acknowledgements

I recall it to my mind with thanks that I have been able to be engaged in the study of sampled data control systems these five years in the Sawaragi Laboratory of Kyoto University.

I am most gratefull to my respected Prof. Y. Sawaragi who has always encouraged me with generous support. His constant criticism and encouragement were invaluable. To him go my foremost thanks.

I would also like to express my sincere gratitude to Assist. Prof. H. Akashi for his valiable help with my preparation of this English paper, and to my superiors of the laboratory for their constant assistances and criticisms. The development of this manuscript was aided considerably by the assistances of many friends. Let me thank Messrs. J. Tanaka, S. Kimoto, A. Maebuchi, T. Hiroshige, A. Watanabe, and S. Suzuki.

REFERENCES

(1) Barker, R.H. : The Pulse Transfer Function and Its Application to Sampling Servo Systems; Proc. of I.E.E. Vol. 99, Part IV, 1952 p307.

(2) Ragazzini, J.R. and Franklin, G.F. : Sampled-Data Control Systems; McGraw-Hill Book Co. Inc. 1958, chap. 7.

(3) Jury, E.I. and Schroeder, W. : Discrete Compensation of Sampled-Data and Continuous Control Systems; Trans. A.I.E.E. Vol. 75, pt. II, 1956, pp. 317-325.

(4) Jury, E.I. : Sampled-Data Control Systems; John Wiley and Sons, Inc. 1958, chap. 5.

(5) Hayashi, S. and Hoshino, S. : Design of Sampled-Data control Systems using Reverse-Device (a tentative title); Preprint at the 25th meeting of the Japan Association of Automatic Control Engineers, 1959.

(6) Sawaragi, Y. , Akashi, H. and Okada, T. : Optimum Non-linear Control Using Sampled Data : Journal of the Japan Association of Automatic Control Engineers, Vol. 3, No. 4 1959 , pp. 217-222.

(7) Takagi, T. : Lecture on Algebra (a tentative title) ; Kyoritsu Book Co. 1948, chap. 10.