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Kyoto University
On blow-analytic equivalence

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This is a resume for the talk, with the title above, at 29 November 2007 at RIMS workshop. This is a joint work with Laurentiu Paunescu.

Motivated by the classification problem of analytic function germs, T.-C. Kuo ([31]) introduced the notions of blow-analytic maps and blow-analytic equivalence. We start the article explaining this motivation to define blow-analytic equivalence.

He discovered a finite classification theorem for analytic function germs with isolated singularities and also shows some important triviality theorems. We are going to report several facts known now about the blow-analytic triviality and invariants.

We then discuss Lipschitz property of blow-analytic maps and show blow-analytic homeomorphism can be far from Lipschitz map. We also discuss exotic pathologies on a blow-analytic homeomorphism: this is illustrated by the examples in §7. We then introduce a strengthened notion, called blow-analytic isomorphism, and discuss the behavior of their jacobians.

In §8, we present a version of the Inverse Mapping Theorem for blow-analytic isomorphisms.

1. Motivations

The notion of blow-analytic equivalence arises from attempts to classify analytic function germs. One is tempted to use the following equivalence relation.

**Definition 1.1.** Let $k = 0, 1, 2, \ldots, \infty, \omega$. We say that two analytic function-germs $f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ are $C^k$- equivalent if there is a $C^k$-diffeomorphism-germ $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ so that $f = g \circ h$.

However, the following example, due to H. Whitney, shows that the $C^1$-equivalence is already too fine for the classification purpose.

**Example 1.2 ([41]).** Consider the functions $f_t : \mathbb{R}^2, 0 \to \mathbb{R}, 0 < t < 1$, defined by $f_t(x, y) = xy(y - x)(y - tx)$. Then $f_t$ is $C^1$-equivalent to $f_{t'}$, if and only if $t = t'$.

As for the $C^0$-equivalence, the functions $(x, y) \mapsto x^2 + y^{2k+1}$, $k \geq 1$, for instance, are $C^0$-equivalent to the regular function $(x, y) \mapsto y$. Hence it seems hopeless to expect a decent classification theory.

Now we consider the blowing-up $\pi : M \to \mathbb{R}^2$ at 0. This map is illustrated by the following picture.
The anti-podal points of the inner circle of the annulus in the middle figure are identified to obtain the Möbius strip in the left figure. Collapsing the inner circle to a point, yields a mapping from the Möbius strip to the disk at the right. This is called the blowing-up of the disk at its centre point. One can introduce local coordinates on the Möbius strip and then the above mapping can be expressed as a real analytic map, as follows. Let \( M = \{(x,y) \times [\xi : \eta] \in D^2 \times P^1 : x\eta = y\xi \} \), where \( D^2 \) is a 2-dimensional disk and \( P^1 \) is the real projective line. The restriction of the projection \( (x,y) \times [\xi : \eta] \rightarrow (x,y) \) to \( M \) is the desired \( \pi \). For the functions \( f_i \) in Example 1.2, all \( f_i \circ \pi \) are \( C^\omega \)-equivalent to each other ([31]).

2. Definition of blow-analytic map

2.1. A naive introduction.

**Definition 2.1** (Blowing-up). Let \( U \) be a disk in \( \mathbb{R}^n \) with analytic coordinates \( x_1, \ldots, x_n \), and let \( C \subset U \) be the locus \( x_1 = \cdots = x_k = 0 \). Let \( [\xi_1 : \cdots : \xi_k] \) be homogeneous coordinates of the real projective space \( P^{k-1} \) and let \( \tilde{U} \subset U \times P^{k-1} \) be the nonsingular manifold defined by

\[
\tilde{U} = \{(x_1, \ldots, x_n) \times [\xi_1, \ldots, \xi_k] : x_i\xi_j = x_j\xi_i, \ 1 \leq i, j \leq k\}.
\]

The projection \( \pi : \tilde{U} \rightarrow U \) on the first factor is clearly an isomorphism away from \( C \). The manifold \( \tilde{U} \), together with the map \( \pi : \tilde{U} \rightarrow U \) is called the blowing-up with nonsingular center \( C \). It is well-known that the blowing-up \( \pi : \tilde{U} \rightarrow U \) is independent of the coordinates chosen in \( U \). This allows us to globalize the definition. Let \( M \) be a real analytic manifold of dimension \( n \) and \( C \) a submanifold of codimension \( k \). Let \( \{U_\alpha\} \) be a collection of disks in \( M \) covering \( C \) such that in each disc \( U_\alpha \) the submanifold \( C \cap U_\alpha \) may be given as the locus \( (x_1 = \cdots = x_k = 0) \), and let \( \pi_\alpha : \tilde{U}_\alpha \rightarrow U_\alpha \) be the blowing-up with center \( C \cap U_\alpha \). We then have isomorphisms

\[
\pi_{\alpha\beta} : \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \rightarrow \pi_\beta^{-1}(U_\alpha \cap U_\beta),
\]

and we can patch together \( \tilde{U}_\alpha \) to form a manifold \( \tilde{U} = \bigcup_{\alpha\beta} \tilde{U}_\alpha \) with map \( \pi : \tilde{U} \rightarrow \bigcup U_\alpha \). Since \( \pi \) is an isomorphism away from \( C \), we can take \( \tilde{M} = \tilde{U} \cup_\pi (M - C); \tilde{M} \), together with the map \( \pi : \tilde{M} \rightarrow M \) extending \( \pi \) on \( \tilde{U} \) and the identity on \( M - C \), is called the blowing-up of \( M \) with center \( C \). We call \( E = \pi^{-1}(C) \) the exceptional divisor of the blowing-up \( \pi \).

Let \( M \) be a real analytic manifold. Take a function \( f \) defined on \( M \) except possibly on some nowhere dense subset of \( M \). We often denote this function by \( f : M \rightarrow \mathbb{R} \) and say that \( f \) is defined almost everywhere.
**Definition 2.2.** Let $\pi: \widetilde{M} \to M$ be a locally finite composition of blowing-ups with nonsingular centers. We say that $f: M \to \mathbb{R}$ is blow-analytic via $\pi$ if $f \circ \pi$ has an analytic extension on $\widetilde{M}$. We say that $f$ is blow-analytic if there is $\pi: \widetilde{M} \to M$, a locally finite composition of blowing-ups with nonsingular centers, so that $f$ is blow-analytic via $\pi$.

Many functions, used as counterexamples in Calculus, are blow-analytic. Some of them are as follows.

**Example 2.3.** (i) $f(x, y) = \frac{xy}{x^2 + y^2}, (x, y) \neq (0, 0)$. This function $f$ is not continuously extendable at the origin. It is clearly blow-analytic via the blowing-up at the origin.

(ii) $f(x, y) = \frac{x^2y}{x^4 + y^2}, (x, y) \neq (0, 0)$. This function is not continuously extendable at the origin, although all directional derivatives exist, if we define $f(0, 0) = 0$. This function $f$ is also blow-analytic.

(iii) $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, (x, y) \neq (0, 0)$. This function is continuously extendable at the origin, but the second order derivatives depend on the order of differentiation:

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

This function $f$ is also blow-analytic via the blowing-up at the origin.

**Example 2.4 ([1]).** Another typical example of blow-analytic function is $f(x, y) = \sqrt[3]{x^4 + y^4}$. The zero set of $z^3 + (x^2 + y^2)z + x^3$ is also the graph of a blow-analytic function $z = g(x, y)$.

The notion of blow-analytic map between real analytic manifolds is defined using local coordinates.

**Definition 2.5.** Let $X$, $Y$ be real analytic manifolds. We say that $f: X \to Y$ is a blow-analytic homeomorphism (bah, for short) if $f$ is a homeomorphism and that both $f$ and $f^{-1}$ are blow-analytic.

**Definition 2.6.** Let $f, g: \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be analytic functions. We say that $f$ and $g$ are blow-analytically equivalent if there is a blow-analytic homeomorphism $h: \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ so that $f = g \circ h$.

Note that $h$ preserves the zero sets of $f$ and $g$. The equivalence relation determined by the above relation on the set of analytic function-germs $\mathbb{R}^n, 0 \to \mathbb{R}, 0$ will be called the blow-analytic equivalence.

**Example 2.7.** (i) Consider the map $f: \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ defined by

$$f(x, y) \mapsto \frac{1}{x^2 + y^2}(x^3, y^3).$$

The map $f$ is continuously extendable at the origin and blow-analytic. The extension is a homeomorphism. But the inverse is not blow-analytic. In fact, $f^{-1}$ is given by

$$(X, Y) \mapsto (X^\frac{2}{3} + Y^\frac{2}{3})(X, Y).$$
(ii) Consider the map $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ defined by
\[(x, y) \mapsto (x^2 + y^2)(x, y)\].
The map $f$ is analytic and a homeomorphism. But the inverse is not blow-analytic.
In fact, $f^{-1}$ is given by
\[(X, Y) \mapsto (X^2 + Y^2)^{-1/3}(X, Y)\].

**Problem 2.8.** Classify the analytic function-germs by blow-analytic equivalence.

2.2. Real v.s. complex.

**Remark 2.9.** Let $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ be a blow-analytic homeomorphism. Let $\pi_i : M_i \to \mathbb{R}^n, i = 1, 2$, be compositions of blowing-ups with nonsingular centers so that $h \circ \pi_1$ and $h^{-1} \circ \pi_2$ are analytic. It is natural to expect that, by repeating blowing-ups of $M_i$ at nonsingular centers, if necessary, there will be an analytic isomorphism $H$ between $\tilde{M}_1$ and $\tilde{M}_2$ which induces $h$. In other words, we expect to have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{M}_1 & \xrightarrow{H} & \tilde{M}_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
\mathbb{R}^n & \xrightarrow{h} & \mathbb{R}^n
\end{array}
\]

Unfortunately, it is not known whether this is true or not.

Let $\mu : M \to N$ be a proper analytic map between real analytic manifolds. It is known that there are complexifications $M^*$ and $N^*$ of $M$, $N$, respectively, and a holomorphic map-germ $\mu^* : M^* \to N^*$, $N$ so that $\mu^*|M = \mu$. (See [23], page 208.)

In complex analytic geometry, a holomorphic map which is bimeromorphic is often called a modification. Let $M^*$, $N^*$ be complex analytic manifolds with anti-holomorphic involutions $\sigma_M$, $\sigma_N$. We denote the fixed point sets of $\sigma_M$, $\sigma_N$ by $M$, $N$, respectively. Let $\pi^* : M^* \to N^*$ be a proper modification so that $\sigma_N \circ \pi^* = \pi^* \circ \sigma_M$. We take its real part (restriction to $M$) and denote it by $\pi : M \to N$. In this paper, we call such a modification a complex modification.

In the setup in Remark 2.9, we can take the fiber product of $h \circ \pi_1$ and $\pi_2$ (or $\pi_1$ and $h^{-1} \circ \pi_2$) and obtain the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\pi_1} & \mathbb{R}^n \\
\downarrow & & \downarrow h \\
M_1 & & \mathbb{R}^n \\
\downarrow \pi_2 & & \downarrow h \\
M_2 & \xrightarrow{\mu} & \mathbb{R}^n
\end{array}
\]

But we do not know whether $M$ has a complexification so that the composed maps $M \to M_i \to \mathbb{R}^n, i = 1, 2$, are complex modifications, even though one can take proper complexifications of $\pi_i, i = 1, 2$. One can say that these compositions are real modifications in the following sense. We say $\mu : M \to N$ is a real modification, if one can take a representative of a complexification $\mu^*$ which is an isomorphism everywhere except on a nowhere dense subset of a neighbourhood of $M$ in $M^*$. Clearly a complex modification is a real modification. But it is not clear whether,
or not, a real modification is a complex modification, that is, isomorphic to the real part of a complex proper modification.

**Example 2.10.** The following map is an analytic isomorphism, hence a real modification,

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto x + \frac{1}{2(1 + x^2)}.$$ 

But the homeomorphism $\mathbb{R} \to \mathbb{R}, \ x \mapsto x^3$, is not a real modification.

3. Triviality theorem

Let $I$ be an interval in $\mathbb{R}$, which contains the origin $0$. Let $F : (\mathbb{R}^n, 0) \times I \to \mathbb{R}, 0$ be an analytic function-germ. We consider the family $f_t : \mathbb{R}^n, 0 \to \mathbb{R}, 0, t \in I$, defined by $f_t(x) = F(x, t)$.

**Definition 3.1 ( Blow-analytic triviality).** Let $\pi : M, E \to \mathbb{R}^n, 0$ be a proper analytic modification. We say $f_t, t \in I$, is blow-analytically trivial via $\pi$ if there are a $t$-level preserving homeomorphism $h : (\mathbb{R}^n, 0) \times I \to (\mathbb{R}^n, 0) \times I$ and a $t$-level preserving analytic isomorphism $H : (M, E) \times I \to (M, E) \times I$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
(M, E) \times I & \xrightarrow{\pi \times \text{id}} & (\mathbb{R}^n, 0) \times I \\
\downarrow H & & \downarrow F \\
(M, E) \times I & \xrightarrow{\pi \times \text{id}} & (\mathbb{R}^n, 0) \times I \\
\end{array}
$$

where $F_0 : (\mathbb{R}^n, 0) \times I \to \mathbb{R}, 0$ is the map defined by $(x, t) \mapsto f_0(x)$.

In all the cases we are interested in, $\pi : M \to \mathbb{R}^n$ is the real part of a complex proper modification $\pi^* : M^* \to \mathbb{C}^n$ defined over reals.

Consider the Taylor expansion of $f_t(x) = F(x, t)$ at $0$ in $\mathbb{R}^n$:

$$f_t(x) = \sum_{\nu} c_{\nu}(t)x^\nu, \quad \text{where} \quad x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n).$$

We set $H_j(x, t) = \sum_{|\nu| = j} c_{\nu}(t)x^\nu$ where $|\nu| = \nu_1 + \cdots + \nu_n$, and assume that $k$ is the smallest number so that $H_k(x, t)$ is not identically equal to $0$.

**Theorem 3.2 ([30]).** If $H_k(x, t)$ has an isolated singularity in $\mathbb{R}^n$ for any $t \in I$, then $f_t, t \in I$, is blow-analytically trivial via the blowing-up at the origin.

Let $w = (w_1, \ldots, w_n)$ be an $n$-tuple of positive integers. We set

$$H_j^{(w)} = \sum_{\nu \in \mathbb{Z}^n, |\nu| = w} c_{\nu}(t)x^\nu \quad \text{where} \quad |\nu| = w_1\nu_1 + \cdots + w_n\nu_n,$$

and assume that $k$ is the smallest number so that $H_k^{(w)}$ is not identically equal to $0$.

**Theorem 3.3 ([14]).** If $H_k^{(w)}(x, t)$ has an isolated singularity in $\mathbb{R}^n$ for any $t \in I$, then $f_t, t \in I$, is blow-analytically trivial via a toric modification.

See §1.5 in [36], §5 in [6], [16]. about toric modifications. See [37] for a generalization of this theorem.
Example 3.4 ([4]). Consider the family $f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$, $t > -15^{1/7}(7/2)^{4/5}/3$. This function is a weighted homogeneous polynomial with weight $(1, 2, 3)$ and weighted degree 15. This family satisfies the assumption of Theorem 3.3 and hence $f_t$ is blow-analytically trivial. An important fact is that this family is not bilipschitz trivial near $t = 0$. See S. Koike ([28]) for a proof.

It is expected that the blow-analytic equivalence should not have moduli. Indeed T.-C. Kuo proved the following: If an analytic function $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ defines an isolated singularity, then the number of blow-analytic equivalence classes nearby $f$ is finite. A more precise statement is the following.

Theorem 3.5 ([31]). Let $P$ be a subanalytic set and let $F : (\mathbb{R}^n, 0) \times P \to \mathbb{R}, 0$ be an analytic function. If the functions $f_t : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ defined by $x \mapsto F(x, t)$ have an isolated singularity for all $t \in P$, then there is a subanalytic filtration

$$P = P_0 \supset P_1 \supset \cdots \supset P_n \supset P_{n+1} = \emptyset, \quad \dim P_i > \dim P_{i+1},$$

such that $f_t$ and $f_{t'}$ are blow-analytically equivalent for $t, t'$ belonging to the same connected component of $P_i - P_{i+1}$.

K. Kurdyka ([32]) introduced the notion of arc-analytic map. We recall some fundamental facts here.

Definition 3.6 (Arc-analytic map). Let $X$ and $Y$ be real analytic manifolds. We say that a map $f : X \to Y$ is arc-analytic (a.a. for short) if $f \circ \alpha$ is analytic for any analytic map $\alpha : \mathbb{R}, 0 \to X$.

Theorem 3.7 ([1]). Let $f : U \to \mathbb{R}$ be an arc-analytic function and $U$ be an open subset of $\mathbb{R}^n$. If there are analytic functions $G_i(x), i = 0, \ldots, p, \text{ so that }\]

$$G_0(x)f(x)^n + G_1(x)f(x)^{p-1} + \cdots + G_{p-1}(x)f(x) + G_p(x) \equiv 0,$$

then $f$ is blow-analytic.

Corollary 3.8. An arc-analytic function with semi-algebraic graph is blow-analytic.

Example 3.9 ([1]). The function $f(x, y) = x^3e^{x^3/(x^4+y^4)}$ is blow-analytic. But there are no non-zero analytic functions vanishing on its graph.

Definition 3.10. Let $X$ and $Y$ be real analytic manifolds. We say that a map $f : X \to Y$ is locally blow-analytic if there is a locally finite family of analytic maps $\{\psi_i : M_i \to X\}$ with the following properties:

- $\psi_i$ are compositions of finitely many local blowing-ups with nonsingular centers,
- there are compact subsets $K_i$ of $M_i$ with $\bigcup_i \psi_i(K_i) = X$, and
- $f \circ \psi_i$ are analytic.

Theorem 3.11 ([1]). An arc-analytic function $f : U \to \mathbb{R}$ with subanalytic graph is locally blow-analytic.

See also [40] for another proof of this theorem.

Question 3.12. Is a locally blow-analytic function $f : U \to \mathbb{R}$ blow-analytic?

When $\dim U = 2$, the answer is "yes", since local blowing-ups can be glued together to yield blowing-ups.
4. Arc lifting property

A remarkable property of blowing-up is the arc lifting property.

**Definition 4.1 (Arc lifting property).** Let $I$ be an open interval in $\mathbb{R}$. Let $X$ and $Y$ be real analytic manifolds. We say that a map $f : X \rightarrow Y$ has the arc lifting property (alp. for short) if for any analytic map $\alpha : I \rightarrow Y$ there is an analytic map $\tilde{\alpha} : I \rightarrow X$ so that $f \circ \tilde{\alpha} = \alpha$.

```
\begin{array}{ccc}
  & X & \\
  \alpha & \downarrow f & \tilde{\alpha} \\
  I & \downarrow \alpha & Y
\end{array}
```

The blowing-up $\pi : \overline{M} \rightarrow M$ with a nonsingular center has the alp.

The blowing-up with an ideal center has the alp. because it is dominated by a composition of blowing-ups with nonsingular centers.

**Example 4.2.** Let $f : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0$ be the map-germ defined by

$$(x, y) \mapsto \left( x, \frac{y(y^2 - x^2)}{x^2 + y^2} \right)$$

This map can be extended continuously at 0. Let $\pi : M \rightarrow \mathbb{R}^2$ be the blowing-up at the origin. Consider the map

$$F : M \rightarrow M, \quad (x, y) \times [\xi : \eta] \mapsto f(x, y) \times [\xi(\xi^2 + \eta^2) : \eta(\eta^2 - \xi^2)].$$

Here we use the same notation as that at the end of §1. It is easy to see that $\pi \circ F = f \circ \pi$. Since the image of the set of regular points of $F$ by $F$ is $M$, $f$ has the arc lifting property. Since the jacobian of $f$ is $\frac{-x^4 + 4x^2y^2 + y^4}{(x^2 + y^2)^2}$, which is zero along $x^2 - (2 + \sqrt{5})y^2 = 0, (x, y) \neq 0$, the lifting is not global.

5. Blow-analytic invariants

5.1. Singular set.

**Theorem 5.1 ([39]).** Let $f, g : \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0$ be two analytic function germs, and let $\Sigma_f$ and $\Sigma_g$ denote their singular sets. If there is a blow-analytic homeomorphism $h : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0$ with $f = g \circ h$, then $h(\Sigma_f) = \Sigma_g$. (That is $h$ preserves the singular set.)

However, a blow-analytic equivalence of analytic functions does not, in general, preserve their singular loci, as the following example shows.

**Example 5.2.** Let $f_t(x, y) = x^4 + 2t^2x^2y^2 + y^4 + x^5, t \in \mathbb{R}$. By Theorem 3.2, this family is blow-analytically trivial. Nevertheless, the dimension $\dim_{\mathbb{R}} R\{x, y\}/(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y})$ changes at $t = 1$. 
5.2. **Numerical invariant.** Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be an analytic function and let \( \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \) be an analytic map. If \( f \circ \alpha \) is not identically zero, then there is a positive integer \( k \) so that

\[
f \circ \alpha(t) = ct^k + \text{higher order terms}, \quad c \neq 0.
\]

We call \( k \) the order of \( f \) along \( \alpha \) and denote it by \( \text{ord}_\alpha(f) \). Define \( \text{ord}_\alpha(f) = \infty \) when \( f \circ \alpha \) is identically zero. We define \( A(f) \) by

\[
A(f) := \{ \text{ord}_\alpha(f) : \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \text{ analytic} \}.
\]

**Theorem 5.3.** If two analytic function germs \( f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) are blow-analytically equivalent, then \( A(f) = A(g) \).

**Remark 5.4.** Let \( \text{mult}_0(f) \) denote the multiplicity of \( f \) at 0, i.e., the degree of the initial polynomial of \( f \). It is easy to show that \( \text{mult}_0(f) = \min A(f) \). As a consequence, the multiplicity is a blow-analytic invariant of analytic function germs. So, this theorem should be compared with Zariski’s multiplicity conjecture: If two holomorphic functions \( f, g : \mathbb{C}^n, 0 \to \mathbb{C}, 0 \) are topologically equivalent (\( C^0 \)-equivalent or \( C^0 \)-\( V \)-equivalent), then \( \text{mult}_0(f) = \text{mult}_0(g) \). This is still open. It is clear that the definition of \( A(f) \) makes sense for a holomorphic function \( f \) and it is interesting to ask the following question: Is \( A(f) \) a topological invariant for holomorphic functions \( f \)?

**Example 5.5.** Let \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( f : \mathbb{K}^n, 0 \to \mathbb{K}, 0 \) be the analytic function defined by \( f(x_1, \ldots, x_n) = x_1^{m_1} \cdots x_n^{m_n} \). Then

\[
A(f) = \left( \sum_{i \in I} m_i \mathbb{N} \right) \cup \{ \infty \}.
\]

Let \( f : \mathbb{K}^n, 0 \to \mathbb{K}, 0 \) be an analytic function. Let \( \pi : M, E \to \mathbb{K}^n, 0, E = \pi^{-1}(0) \), denote a real modification. e.g., a composition of finitely many blowing-ups with nonsingular centers. We assume that \( f \circ \pi \) is normal crossing, that is, \( f \circ \pi \) can be locally expressed as a product of powers of a number of local coordinates. Let \( (f \circ \pi)_0 = \sum_{j \in J} m_j E_j \) denote the irreducible decomposition of the zero locus of \( f \circ \pi \) and \( C \) denote the set of subsets \( I \) of \( J \) with \( E^*_I \subset E \) where \( E^*_I = E^*_I \cap E, \ E^*_I = \bigcap_{i \in I} E_i - \bigcup_{j \in J \not\in I} E_j \).

The following formula is stated in [25], Theorem I.

**Theorem 5.6.** \( A(f) = \bigcup_{I \in C} A_I(f) \) where \( A_I(f) = (\sum_{i \in I} m_i \mathbb{N}) \cup \{ \infty \} \).

Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be a real analytic function. We set

\[
A^\pm(f) = \{ \text{ord}_\alpha(f) : \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \text{ analytic and } \pm f \circ \alpha(t) \geq 0 \text{ near 0} \}
\]

The proof of Theorem 5.3 shows \( A^\pm(f) = A^\pm(g) \) if \( f \) and \( g \) are blow-analytically equivalent. In a way similar to the proof of Theorem 5.6, we obtain the following

**Theorem 5.7.** \( A^\pm(f) = \bigcup_{I \in C^\pm} A_I(f) \) where \( C^\pm \) denotes the set of \( I \in C \) so that \( E_I^* \) intersects with the closure of \( \{ y \in M : \pm f \circ \pi(y) > 0 \} \).
5.3. Zeta functions. Recently S. Koike and A. Parusiński ([27]) have introduced
zeta functions for the blow-analytic equivalence. In their paper ([27]), they call their
zeta functions the 'motivic type invariants', since their zeta functions can be derived
from zeta functions whose coefficients are motives. G. Fichou ([10]) generalizes their
invariants using the virtual Poincaré polynomial. Since these are very interesting
invariants, we review their results in this section. See also [35] for the virtual Betti
numbers.

Let $C$ be a category whose objects are a class of subsets of the Euclidean spaces
with some good properties. We consider an invariant $\beta : C \to R$, where $R$ is a
commutative ring, with the following properties.

- $\beta(X) = \beta(X - Y) + \beta(Y)$ if $Y$ is a closed subset in $X$.
- $\beta(X \times Y) = \beta(X)\beta(Y)$.

When $C$ is the category of subanalytic subsets in Euclidean spaces which have fi-
nite homologies, the $Z/2Z$-Euler characteristic $\beta$ with compact supports has these
properties.

We say a semi-algebraic set $A$ in a compact nonsingular real algebraic manifold
$M$ is a $\text{AS}$-subset if for any analytic map $\alpha : (-\epsilon, \epsilon) \to M$, $\epsilon > 0$, with $\alpha(0, \epsilon) \subset A$,
there is a positive number $\epsilon'$ so that $\alpha(-\epsilon', 0) \subset A$. See [33] for more information
about $\text{AS}$-subsets.

**Theorem 5.8** ([10]). Let $\text{AS}$ denote the set of all semi-algebraic $\text{AS}$-subsets in compact
compact nonsingular real algebraic manifolds. There is an invariant $\beta : \text{AS} \to \mathbb{Z}[u, u^{-1}]
with the above properties which satisfies the following:

$$\beta(X) = \sum_k (\dim H_k(X, \mathbb{Z}/2\mathbb{Z})) u^k$$

when $X$ is compact and nonsingular. Moreover, if two $\text{AS}$-sets $X, Y$ are Nash
(i.e., semi-algebraically and analytically) equivalent, then $\beta(X) = \beta(Y)$.

Notice the following: $\beta(\emptyset) = 0$, $\beta(P^n) = 1 + u + u^2 + \cdots + u^n$, $\beta(R^n) = u^n$.

**Example 5.9.** It is not true that $\beta(X) = k\beta(Y)$ when there is an unbranched
$k$-fold covering $X \to Y$. Consider the double covering $S^1 \to P^1$ and observe that
$\beta(S^1) = \beta(P^1) = u + 1$.

We consider the space of polynomial arcs of order $k$:

$$L_k := \{\alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 : \text{polynomial of degree } k\} = \mathbb{R}^n_k.$$

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function. The following spaces are algebraically
constructible

$$A_k(f) := \{\alpha \in L_k : \text{ord}(f \circ \alpha) = k\} \quad A^\pm_k(f) := \{\alpha \in L_k : f \circ \alpha = \pm t^k + \cdots\}.$$

Notice that if $f$ and $g$ are analytically equivalent, then $A_k(f)$ (resp. $A^\pm_k(f)$) and
$A_k(g)$ (resp. $A^\pm_k(g)$) are actually isomorphic as algebraic constructible sets. Define
Zeta functions by the following formulas.

$$Z_f(t) := \sum_{k \geq 1} \beta(A_k(f)) \left(\frac{t}{u^n}\right)^k \quad Z^\pm_f(t) := \sum_{k \geq 1} \beta(A^\pm_k(f)) \left(\frac{t}{u^n}\right)^k$$

where $u = -1$ when $\beta$ is the $\mathbb{Z}/2\mathbb{Z}$-Euler characteristic with compact supports ([27]),
or $u$ is an indeterminate when $\beta$ is the virtual Poincaré polynomial ([10]).
Let $\pi : M, E \rightarrow \mathbb{R}^n, 0, E = \pi^{-1}(0)$, be a proper analytic modification so that $f \circ \pi$, det$(d\pi)$ are in normal crossing and that $\pi$ is an isomorphism over $\mathbb{R}^n - f^{-1}(0)$. We assume that $\pi^{-1}(0)$ is a normal crossing divisor. We use the notation defined in the paragraph after Example 5.5. We consider the irreducible decompositions of the zero loci of $f \circ \pi$ and det$(d\pi)$, the jacobian determinant of $\pi$:

$$(f \circ \pi)_0 = \sum_{j \in J} m_j E_j, \quad (\det(d\pi))_0 = \sum_{j \in J} (\nu_j - 1) E_j.$$ 

The following formula is often called the Denef-Loeser formula.

**Theorem 5.10** ([27], [10]). Setting $\phi(\lambda) = \lambda/(1 - \lambda) = \lambda + \lambda^2 + \lambda^3 + \cdots$, we have

$$Z_f(t) = \sum_{l \neq 0} \beta(E^*_l)(u - 1)^{|I|} \prod_{i \in I} \phi\left(\frac{t_m}{u^m}\right).$$

**Remark 5.11.** When $\beta$ is the virtual Poincaré polynomial we need to assume that $f$ is a polynomial and that $\pi$ is algebraic (since we do not know that $E^*_l$ is semi-algebraic).

It is also possible to obtain a formula for $Z^\pm(t)$ similar to Theorem 5.10. To do this, we introduce somenotation. We define $A^\pm_k(f, E^*_l)$ by

$$A^\pm_k(f, E^*_l) := p_k(\pi_*^{-1}(A^\pm_k(f)) \cap \mathcal{L}(M, E^*_l)) = \bigcup_{j: (m_j, j) = k} p_k(A^\pm_{k,j}(f; E^*_l)),$$

where $A^\pm_{k,j}(f, E^*_l) := \{\gamma \in \pi_*^{-1}(A^\pm_k(f)) \cap \mathcal{L}(M, E^*_l) : \text{ord}_i E_i = j_i\}$. Let $p \in E^*_l$ and let $U$ be a coordinate neighbourhood at $p$. Using the local coordinates $y = (y_1, \ldots, y_n) : U \rightarrow \mathbb{R}^n$ with $E^*_l = \{y_i = 0, i \in I, y_i \neq 0, i \notin I\}$, we can express $f \circ \pi$ as follows:

$$f \circ \pi(y) = u(y) \prod_{i \in I} y_i^{m_i}, \quad \text{where } u(y) \text{ is a unit.}$$

We set $y = (y_i)_{i \in I}$ and define

$$\tilde{E}^\pm_U(t) = \{ (p, y) \in (E^*_l \cap U) \times \mathbb{R}^{|I|} : u(p) \prod_{i \in I} y_i^{m_i} = \pm 1 \}.$$

The sets $\tilde{E}^\pm_U$ can be patched together and we obtain a set $\tilde{E}^\pm$. We denote by $m_I$ the greatest common divisor of $m_i$, $i \in I$, and define

$$\tilde{E}^\pm_U = \{ (p, w) \in (E^*_l \cap U) \times \mathbb{R} : u(p) w^m = \pm 1 \}.$$

The sets $\tilde{E}^\pm_U$ can be patched together and we obtain a set $\tilde{E}^\pm$. Setting $\tilde{\beta}^\pm_I = \beta(\tilde{E}^\pm)$, we obtain

$$Z^\pm_I(t) = \sum_{l} \tilde{\beta}^\pm_I (u - 1)^{|I| - 1} \prod_{i \in I} \phi\left(\frac{t_m}{u^m}\right).$$

**Theorem 5.12** ([27]). Let $f, g : \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0$ be two analytic functions and let $\beta$ be the $\mathbb{Z}/2\mathbb{Z}$-Euler characteristic with compact supports. Assume that there are real modifications $\pi_i : M_i \rightarrow \mathbb{R}^n$, $i = 1, 2$, so that $\pi_1$ (resp. $\pi_2$) is an isomorphism except possibly over the zero set of $f$ (resp. $g$). If there is an analytic isomorphism $(M_1, \pi_1^{-1}(0)) \rightarrow (M_2, \pi_2^{-1}(0))$ which induces a blow-analytic homeomorphism $h : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0$ with $f = g \circ h$, then $Z_f(l) = Z_g(l)$. $Z^\pm_I(t) = Z^\pm_g(t)$. 
Similarly we obtain the following

**Theorem 5.13 ([10]).** Let \( f, g : \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0 \) be two polynomial functions and let \( \beta \) be the virtual Poincaré polynomial. Assume that there are algebraic modifications \( \pi_i : M_i \rightarrow \mathbb{R}^n \), \( i = 1, 2 \), whose critical loci are normal crossings. We assume that \( \pi_1 \) (resp. \( \pi_2 \)) is an isomorphism except over the zero set of \( f \) (resp. \( g \)). If there is an analytic isomorphism \( (M_1, \pi_1^{-1}(0)) \rightarrow (M_2, \pi_2^{-1}(0)) \) which induces a blow-analytic isomorphism \( h : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) with \( f = g \circ h \), then \( Z_f(t) = Z_g(t) \), \( Z_f^+ = Z_g^+ \).

See Definition 7.2 below for the notion of blow-analytic isomorphism.

## 6. Lipschitz maps

An interesting class of maps which are not differentiable is the class of Lipschitz maps. We start with some basics.

Let \( U \) be a convex open subset of \( \mathbb{R}^n \). A map \( f : U \rightarrow \mathbb{R}^p \) is said to be **Lipschitz** if there is a positive constant \( K \) so that

\[
|f(x) - f(x')| \leq K|x - x'| \quad \forall x, x' \in U.
\]

Recall that Rademacher's theorem ([15, Theorem 4.1.1]), states that a function which is Lipschitz on an open subset of \( \mathbb{R}^n \) is differentiable almost everywhere (in the sense of Lebesgue measure) on that set. This allows us to introduce the following definition.

**Definition 6.1** (Generalized Jacobian). The generalized Jacobian \( \partial f(0) \) of \( f \) at 0 is the convex hull of all matrices obtained as limits of sequences of the Jacobi matrices of \( f \) at \( x_i \) where \( x_i \rightarrow 0, x_i \notin Z \). Here \( Z \) denotes the set of points at which \( f \) fails to be differentiable.

**Theorem 6.2** ([5]). Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be a Lipschitz map-germ. If \( \partial f(0) \) does not contain singular matrices, then \( f \) has a Lipschitz inverse.

In this section, we are interested in blow-analytic maps satisfying the Lipschitz condition.

Let \( U \) be a convex open subset of \( \mathbb{R}^n \) and let \( f : U \rightarrow \mathbb{R} \) be a continuous function with subanalytic graph. Then there is an nowhere dense closed subanalytic subset \( Z \) so that \( f \) is analytic on \( U - Z \).

**Lemma 6.3.** The function \( f \) is Lipschitz if and only if all partial derivatives of \( f \) are bounded on \( U - Z \).

**Theorem 6.4** ([13]). Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be an arc-analytic map with subanalytic graph. If \( f \) is bilipschitz, i.e., there are positive constants \( c_1, c_2 \) so that

\[
c_1|y - y'| \leq |f(y) - f(y')| \leq c_2|y - y'|;
\]

then \( f^{-1} \) is arc-analytic.

Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be a homeomorphism which is blow-analytic and Lipschitz. The theorem asserts that the inverse \( f^{-1} \) is blow-analytic, if \( f^{-1} \) is Lipschitz.

**Corollary 6.5.** Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be an arc-analytic map with semi-algebraic graph. If \( f \) is bilipschitz, then \( f^{-1} \) is blow-analytic.
Theorem 6.6 ([13]). Let $F : \mathbb{R}^m \times \mathbb{R}^n, 0 \to \mathbb{R}^n, (x, y) \mapsto F(x, y)$, be an arc-analytic map with subanalytic graph. If there are positive constants $c_1, c_2$ so that
\begin{equation}
(1) \quad c_1 |y - y'| \leq |F(x, y) - F(x, y')| \leq c_2 |y - y'|,
\end{equation}
then there is an arc-analytic and subanalytic map $\tau : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ such that
\begin{equation}
(2) \quad \{ F(x, y) = 0 \} = \{ y = \tau(x) \}.
\end{equation}

Remark 6.7. Let $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbb{R}, 0 \to \mathbb{R}^n, 0$ be an analytic map. Let $\text{ord}(\alpha)$ denote $\min \{ \text{ord}(\alpha_1), \ldots, \text{ord}(\alpha_n) \}$. If an arc-analytic map $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is Lipschitz, then $\text{ord}(f \circ \alpha) \geq \text{ord}(\alpha)$. If the map $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is bilipschitz, then $\text{ord}(f \circ \alpha) = \text{ord}(\alpha)$. In particular, the image of a nonsingular curve by an arc-analytic bilipschitz map $\mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is a nonsingular curve.

Question 6.8. Does there exist a blow-analytic map (or an arc-analytic map with subanalytic graph) $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ with the following properties?

- there is a positive constant $c$ so that $c |y - y'| \leq |f(y) - f(y')| \quad \forall y, y' \in \mathbb{R}^n, 0$;
- $f$ is not Lipschitz.

7. Blow-analytic isomorphism and analytic arcs

A blow-analytic homeomorphism can be quite far from a bilipschitz homeomorphism.

Theorem 7.1 ([26]). For any unbranched curve $C \subset \mathbb{R}^2, 0$, there is a blow-analytic homeomorphism $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ such that $h(C)$ is nonsingular.

Theorem 7.1 motivates us to strengthen the conditions imposed to the definition of blow-analytic homeomorphisms.

Definition 7.2. We say that a map $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is a blow-analytic isomorphism (bai for short) if there are two neighbourhoods $U, U'$ of $0$ in $\mathbb{R}^n$ so that the following conditions are satisfied.

- there are complex modifications $\pi : M \to U, \pi' : M' \to U'$, and an analytic isomorphism $F : (M, E) \to (M', E')$ of analytic spaces, where $E$ and $E'$ denote the critical loci of $\pi$ and $\pi'$ respectively.
- $f$ is a homeomorphism and $\pi'_0 F = f_0 \pi$.

A blow-analytic isomorphism is clearly a blow-analytic homeomorphism. But the converse is not true. For example, the blow-analytic homeomorphism in Example 7.1 is not a bai. In fact, the critical locus of the composites of horizontal arrows are normal crossing, and we have a correspondence between their irreducible components, but they have different multiplicities.

Let $\pi : M \to \mathbb{R}^n$ be a complex modification whose critical locus is a normal crossing divisor. We consider an analytic vector $\xi$ on $M$ which is tangent to each irreducible component of the critical locus. By integrating $\xi$, we obtain an analytic isomorphism of $M$. If it induces a homeomorphism of $\mathbb{R}^n$ near 0, this is a blow-analytic isomorphism. Thus, in all triviality theorems stated before, we can replace bah by bai.
Definition 7.3. Let \( \pi : M \to U \) be a composition of blowing-ups with nonsingular centers. A blow-analytic function \( P : U \to \mathbb{R} \) is said to be a **blow-analytic unit** (bau for short) via \( \pi \) if \( P \circ \pi \) extends to an analytic unit (i.e., an analytic function which is nowhere vanishing). \( P \) is said to be a blow-analytic unit (bau for short) if there is \( \pi : M \to U \) such that \( P \) is a bau via \( \pi \).

Theorem 7.4. If \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) is a blow-analytic isomorphism, then the Jacobian determinant \( \text{det}(df) \) is a blow-analytic unit.

Let \( w_1, \ldots, w_n \) be real numbers. We consider the map

\[
(3) \quad f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0, \quad x = (x_1, \ldots, x_n) \mapsto (x_1 P(x)^{w_1}, \ldots, x_n P(x)^{w_n}),
\]

where \( P : \mathbb{R}^n, 0 \to \mathbb{R} \) is a bounded blow-analytic function.

Theorem 7.5. Let \( P \) be a non-negative blow-analytic function via some toric modification \( \pi : M \to \mathbb{R}^n \). If \( P + \sum_{i=1}^{n} w_i x_i \frac{\partial P}{\partial x_i} \) is a blow-analytic unit via the modification \( \pi \), and if \( P \) and \( \sum_{i=1}^{n} w_i x_i \frac{\partial P}{\partial x_i} \) are continuously extendable on \( \mathbb{R}^n - 0, 0 \), then the map \( f \) defined by (3) is a blow-analytic isomorphism.

Example 7.6. The map

\[
f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0), \quad (x, y) \mapsto (x P^3, y P^2), \quad P = \frac{x^4 + 2y^6}{x^4 + y^6},
\]
is a blow-analytic isomorphism.

Consider the map

\[
f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0, \quad x = (x_1, \ldots, x_n) \mapsto (x_1 + Q(x_2, \ldots, x_n), x_2, \ldots, x_n),
\]

where \( Q : \mathbb{R}^{n-1}, 0 \to \mathbb{R} \) is a blow-analytic function. Since the map \( (x_1, \ldots, x_n) \mapsto (x_1 - Q(x_2, \ldots, x_n), x_2, \ldots, x_n) \) is the inverse of \( f \), \( f \) is a homeomorphism.

Theorem 7.7. If \( Q \) is blow-analytic, then \( f \) is a blow-analytic isomorphism.

Example 7.8 ([38]). Consider a blow-analytic map \( f : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0 \) defined by

\[
(x, y, z) \mapsto \left( x, y, z + \frac{2x^5 y}{x^6 + y^4} \right).
\]

This is a blow-analytic isomorphism by Theorem 7.7. Let \( \alpha : \mathbb{R}, 0 \to \mathbb{R}^3, 0 \) be the map defined by \( t \to (t^2, t^3, 0) \). Observe that \( f \circ \alpha(t) = (t^2, t^3, t) \). This means that the blow-analytic isomorphism \( f \) sends a singular curve, the image of \( \alpha \), to a regular curve.

We say that an analytic map \( \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \) is irreducible if \( \alpha \) cannot be written as \( \alpha = \beta \circ \psi \), where \( \beta : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \) and \( \psi : \mathbb{R}, 0 \to \mathbb{R}, 0 \), are analytic and \( \psi'(0) = 0 \).

Theorem 7.9. Let \( \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0, n \geq 3, \) be an irreducible analytic map. Then there is a blow-analytic isomorphism \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) such that \( f \circ \alpha \) is a regular map.
8. Jacobian of blow-analytic map

Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be a blow-analytic map. It is interesting to investigate what we can conclude when we assume that \( \det(df) \) is a blow-analytic unit. For example, is such a \( f \) a blow-analytic isomorphism?

**Example 8.1.** We identify \( \mathbb{R}^2 \) with \( C \) by the map \((x, y) \mapsto z = x + \sqrt{-1}y \). Let \( k \) be a positive integer. Consider the continuous blow-analytic map

\[
f : C, 0 \to C, 0, \quad z \mapsto z^{k+1}/z^k = z^{2k+1}/|z|^{2k}.
\]

Looking at the restriction to a small circle \(|z| = \varepsilon\), the mapping degree of \( f \) is \( 2k+1 \). In particular, \( f \) is not a homeomorphism. Since

\[
\det(df) = \left| \begin{array}{cc}
(k+1)z^k/z^k & -kz^{k+1}/z^{k+1} \\
-kz^{k+1}/z^{k+1} & (k+1)z^k/z^k
\end{array} \right| = (k+1)^2 - k^2 = 2k+1, \quad z \neq 0,
\]

det(\( df \)) is a blow-analytic unit. We also have that \( f \) is Lipschitz, by Lemma 6.3.

Let \( M \to C \) denote the blowing-up at the origin. Since the map \( f \) is induced by an unbranched covering \( M \to M \) of degree \( 2k+1 \), \( f \) has the arc lifting property.

Example 8.1 shows that a blow-analytic map \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) may not be a homeomorphism, even though \( \det(df) \) is a blow-analytic unit. However, this kind of phenomenon is not possible in higher codimensional cases.

**Proposition 8.2.** Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be a blow-analytic map so that \( \det(df) \) is a blow-analytic unit. If there is a subset \( C \) of \( \mathbb{R}^n, 0 \), of codimension \( \geq 3 \), so that \( f|_{\mathbb{R}^n-C} \) is analytic, then \( f \) is a homeomorphism.

It is an open question whether \( f \) is a bai or not.

We have a version of the inverse mapping theorem via toric modification, which is the following.

**Theorem 8.3.** Let \( h = (h_1, \ldots, h_n) : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be a continuous blow-analytic map via a toric modification. If \( \frac{\partial h_1}{\partial x_1} \), \( \frac{\partial (h_1, h_2)}{\partial (x_1, x_2)}, \ldots, \frac{\partial (h_1, \ldots, h_n)}{\partial (x_1, \ldots, x_n)} \) are blow-analytic units and they are continuously extendable on \( \mathbb{R}^n - 0, 0 \), then \( h \) is a blow-analytic isomorphism.

If the map \( h = (h_1, \ldots, h_n) : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) satisfies the assumption of Theorem 8.3 after permutations of \( x_1, \ldots, x_n \) and \( h_1, \ldots, h_n \), then \( h \) is a blow-analytic isomorphism, by Theorem 8.3.

This is the corrected version of Theorem 6.1 in [12].

Lastly, we have three more theorems.

**Theorem 8.4.** Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be a blow-analytic map so that \( \det(df) \) is a blow-analytic unit. If there are nonsingular subanalytic subsets \( C, C' \) so that \( f \) is blow-analytic via the blowing up with center \( C \) and that \( f(C) = C' \), then codim \( C = \text{codim} \, C' \) and \( f \) has the arc lifting property. Moreover, there is an analytic map \( \tilde{f} : M \to M' \) such that \( \tilde{f} \) is locally an isomorphism and that \( \pi' \circ \tilde{f} = f \circ \pi \). Where \( \pi : M \to \mathbb{R}^n \) is the blowing-up at \( C \) and \( \pi' : M' \to \mathbb{R}^n \) is the blowing-up at \( C' \).

**Theorem 8.5.** Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) be a blow-analytic map. If \( \det(df) \) is a blow-analytic unit, then \( f \) is finite.
Theorem 8.6. Consider a blow-analytic map \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) defined by
\[
(x_1, \ldots, x_n) \mapsto (x_1 p_1(x), \ldots, x_n p_n(x)),
\]
where \( p_i \) are blow-analytic units.
If \( f \) is blow-analytic via a toric modification and \( \text{det}(df) \) is a blow-analytic unit, then \( f \) is a blow-analytic isomorphism.

References