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Kyoto University
On blow-analytic equivalence

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This is a resume for the talk, with the title above, at 29 November 2007 at RIMS workshop. This is a joint work with Laurentiu Paunescu.

Motivated by the classification problem of analytic function germs, T.-C. Kuo ([31]) introduced the notions of blow-analytic maps and blow-analytic equivalence. We start the article explaining this motivation to define blow-analytic equivalence.

He discovered a finite classification theorem for analytic function germs with isolated singularities and also shows some important triviality theorems. We are going to report several facts known now about the blow-analytic triviality and invariants.

We then discuss Lipschitz property of blow-analytic maps and show blow-analytic homeomorphism can be far from Lipschitz map. We also discuss exotic pathologies on a blow-analytic homeomorphism: this is illustrated by the examples in §7. We then introduce a strengthened notion, called blow-analytic isomorphism, and discuss the behavior of their jacobians.

In §8, we present a version of the Inverse Mapping Theorem for blow-analytic isomorphisms.

1. Motivations

The notion of blow-analytic equivalence arises from attempts to classify analytic function germs. One is tempted to use the following equivalence relation.

Definition 1.1. Let \( k = 0, 1, 2, \ldots, \infty, \omega \). We say that two analytic function-germs \( f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) are \( C^k \)-equivalent if there is a \( C^k \)-diffeomorphism-germ \( h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) so that \( f = g \circ h \).

However, the following example, due to H. Whitney, shows that the \( C^1 \)-equivalence is already too fine for the classification purpose.

Example 1.2 ([41]). Consider the functions \( f_t : \mathbb{R}^2, 0 \to \mathbb{R}, 0 < t < 1 \), defined by \( f_t(x, y) = xy(y - x)(y - tx) \). Then \( f_t \) is \( C^1 \)-equivalent to \( f_{t'} \), if and only if \( t = t' \).

As for the \( C^0 \)-equivalence, the functions \( (x, y) \mapsto x^2 + y^{2k+1}, k \geq 1 \), for instance, are \( C^0 \)-equivalent to the regular function \( (x, y) \mapsto y \). Hence it seems hopeless to expect a decent classification theory.

Now we consider the blowing-up \( \pi : M \to \mathbb{R}^2 \) at 0. This map is illustrated by the following picture.
The anti-podal points of the inner circle of the annulus in the middle figure are identified to obtain the Möbius strip in the left figure. Collapsing the inner circle to a point, yields a mapping from the Möbius strip to the disk at the right. This is called the blowing-up of the disk at its centre point. One can introduce local coordinates on the Möbius strip and then the above mapping can be expressed as a real analytic map, as follows. Let $M = \{(x,y) \times [\xi : \eta] \in D^2 \times P^1 : x\eta = y\xi\}$, where $D^2$ is a 2-dimensional disk and $P^1$ is the real projective line. The restriction of the projection $(x,y) \times [\xi : \eta] \mapsto (x,y)$ to $M$ is the desired projection. For the functions $f_i$ in Example 1.2, all $f_i \circ \pi$ are $C^\infty$-equivalent to each other ([31]).

2. Definition of blow-analytic map

2.1. A naive introduction.

**Definition 2.1** (Blowing-up). Let $U$ be a disk in $\mathbb{R}^n$ with analytic coordinates $x_1, \ldots, x_n$, and let $C \subset U$ be the locus $x_1 = \cdots = x_k = 0$. Let $[\xi_1 : \cdots : \xi_k]$ be homogeneous coordinates of the real projective space $P^{k-1}$ and let $\overline{U} \subset U \times P^{k-1}$ be the nonsingular manifold defined by

$$\overline{U} = \{(x_1, \ldots, x_n) \times [\xi_1, \ldots, \xi_k] : x_i\xi_j = x_j\xi_i, \ 1 \leq i, j \leq k\}.$$ 

The projection $\pi : \overline{U} \rightarrow U$ on the first factor is clearly an isomorphism away from $C$. The manifold $\overline{U}$, together with the map $\pi : \overline{U} \rightarrow U$ is called the blowing-up with nonsingular center $C$. It is well-known that the blowing-up $\pi : \overline{U} \rightarrow U$ is independent of the coordinates chosen in $U$. This allows us to globalize the definition. Let $M$ be a real analytic manifold of dimension $n$ and $C$ a submanifold of codimension $k$. Let $\{U_\alpha\}$ be a collection of disks in $M$ covering $C$ such that in each disc $U_\alpha$ the submanifold $C \cap U_\alpha$ may be given as the locus $(x_1 = \cdots = x_k = 0)$, and let $\pi_\alpha : \tilde{U}_\alpha \rightarrow U_\alpha$ be the blowing-up with center $C \cap U_\alpha$. We then have isomorphisms

$$\pi_{\alpha\beta} : \pi^{-1}_\alpha(U_\alpha \cap U_\beta) \rightarrow \pi^{-1}_\beta(U_\alpha \cap U_\beta),$$

and we can patch together $\tilde{U}_\alpha$ to form a manifold $\tilde{U} = \bigcup_{\alpha, \beta} \tilde{U}_\alpha$ with map $\pi : \tilde{U} \rightarrow \bigcup U_\alpha$. Since $\pi$ is an isomorphism away from $C$, we can take $\tilde{M} = \tilde{U} \cup_\pi (M - C)$; $\tilde{M}$, together with the map $\pi : \tilde{M} \rightarrow M$ extending $\pi$ on $\tilde{U}$ and the identity on $M - C$, is called the blowing-up of $M$ with center $C$. We call $E = \pi^{-1}(C)$ the exceptional divisor of the blowing-up $\pi$.

Let $M$ be a real analytic manifold. Take a function $f$ defined on $M$ except possibly on some nowhere dense subset of $M$. We often denote this function by $f : M \dashrightarrow \mathbb{R}$ and say that $f$ is defined almost everywhere.
Definition 2.2. Let $\pi : \widetilde{M} \rightarrow M$ be a locally finite composition of blowing-ups with nonsingular centers. We say that $f : M \rightarrow \mathbb{R}$ is blow-analytic via $\pi$ if $f \circ \pi$ has an analytic extension on $\widetilde{M}$. We say that $f$ is blow-analytic if there is $\pi : \widetilde{M} \rightarrow M$, a locally finite composition of blowing-ups with nonsingular centers, so that $f$ is blow-analytic via $\pi$.

Many functions, used as counterexamples in Calculus, are blow-analytic. Some of them are as follows.

Example 2.3. (i) $f(x, y) = \frac{xy}{x^{2} + y^{2}}$, $(x, y) \neq (0, 0)$. This function $f$ is not continuously extendable at the origin. It is clearly blow-analytic via the blowing-up at the origin.

(ii) $f(x, y) = \frac{x^{2}y}{x^{4} + y^{2}}$, $(x, y) \neq (0, 0)$. This function is not continuously extendable at the origin, although all directional derivatives exist, if we define $f(0, 0) = 0$. This function $f$ is also blow-analytic.

(iii) $f(x, y) = \frac{xy(x^{2} - y^{2})}{x^{2} + y^{2}}$, $(x, y) \neq (0, 0)$. This function is continuously extendable at the origin, but the second order derivatives depend on the order of differentiation:

$$\frac{\partial^{2} f}{\partial x \partial y} (0, 0) \neq \frac{\partial^{2} f}{\partial y \partial x} (0, 0).$$

This function $f$ is also blow-analytic via the blowing-up at the origin.

Example 2.4 ([1]). Another typical example of blow-analytic function is $f(x, y) = \sqrt[3]{x^{4} + y^{4}}$. The zero set of $z^{3} + (x^{2} + y^{2})z + x^{3}$ is also the graph of a blow-analytic function $z = g(x, y)$.

The notion of blow-analytic map between real analytic manifolds is defined using local coordinates.

Definition 2.5. Let $X$, $Y$ be real analytic manifolds. We say that $f : X \rightarrow Y$ is a blow-analytic homeomorphism (bah, for short) if $f$ is a homeomorphism and that both $f$ and $f^{-1}$ are blow-analytic.

Definition 2.6. Let $f, g : \mathbb{R}^{n}, 0 \rightarrow \mathbb{R}, 0$ be analytic functions. We say that $f$ and $g$ are blow-analytically equivalent if there is a blow-analytic homeomorphism $h : \mathbb{R}^{n}, 0 \rightarrow \mathbb{R}^{n}, 0$ so that $f = g \circ h$.

Note that $h$ preserves the zero sets of $f$ and $g$. The equivalence relation determined by the above relation on the set of analytic function-germs $\mathbb{R}^{n}, 0 \rightarrow \mathbb{R}, 0$ will be called the blow-analytic equivalence.

Example 2.7. (i) Consider the map $f : \mathbb{R}^{2}, 0 \rightarrow \mathbb{R}^{2}, 0$ defined by

$$(x, y) \mapsto \frac{1}{x^{2} + y^{2}}(x^{3}, y^{3}).$$

The map $f$ is continuously extendable at the origin and blow-analytic. The extension is a homeomorphism. But the inverse is not blow-analytic. In fact, $f^{-1}$ is given by

$$(X, Y) \mapsto (X^{\frac{2}{3}} + Y^{\frac{2}{3}})(X, Y).$$
(ii) Consider the map $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ defined by

$$(x, y) \mapsto (x^2 + y^2)(x, y).$$

The map $f$ is analytic and a homeomorphism. But the inverse is not blow-analytic. In fact, $f^{-1}$ is given by

$$(X, Y) \mapsto (X^2 + Y^2)^{-1/3}(X, Y).$$

**Problem 2.8.** Classify the analytic function-germs by blow-analytic equivalence.

**2.2. Real v.s. complex.**

**Remark 2.9.** Let $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ be a blow-analytic homeomorphism. Let $\pi_i : M_i \to \mathbb{R}^n, i = 1, 2$, be compositions of blowing-ups with nonsingular centers so that $h \circ \pi_1$ and $h^{-1} \circ \pi_2$ are analytic. It is natural to expect that, by repeating blowing-ups of $M_i$ at nonsingular centers, if necessary, there will be an analytic isomorphism $H$ between $\tilde{M}_1$ and $\tilde{M}_2$ which induces $h$. In other words, we expect to have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{M}_1 & \xrightarrow{H} & \tilde{M}_2 \\
\pi_1 & \downarrow & \pi_2 \\
\mathbb{R}^n & \xrightarrow{h} & \mathbb{R}^n
\end{array}$$

Unfortunately, it is not known whether this is true or not.

Let $\mu : M \to N$ be a proper analytic map between real analytic manifolds. It is known that there are complexifications $M^*$ and $N^*$ of $M$, $N$, respectively, and a holomorphic map-germ $\mu^* : M^*, M \to N^*, N$ so that $\mu^*|M = \mu$. (See [23], page 208.)

In complex analytic geometry, a holomorphic map which is bimeromorphic is often called a **modification**. Let $M^*, N^*$ be complex analytic manifolds with anti-holomorphic involutions $\sigma_M, \sigma_N$. We denote the fixed point sets of $\sigma_M, \sigma_N$ by $M, N$, respectively. Let $\pi^* : M^* \to N^*$ be a proper modification so that $\sigma_N \circ \pi^* = \pi^* \circ \sigma_M$. We take its real part (restriction to $M$) and denote it by $\pi : M \to N$. In this paper, we call such a modification a **complex modification**.

In the setup in Remark 2.9, we can take the fiber product of $h \circ \pi_1$ and $\pi_2$ (or $\pi_1$ and $h^{-1} \circ \pi_2$) and obtain the following diagram:

$$\begin{array}{ccc}
M_1 & \xrightarrow{\pi_1} & \mathbb{R}^n \\
\downarrow & & \downarrow h \\
M_2 & \xrightarrow{\pi_2} & \mathbb{R}^n
\end{array}$$

But we do not know whether $M$ has a complexification so that the composed maps $M \to M_i \to \mathbb{R}^n, i = 1, 2$, are complex modifications, even though one can take proper complexifications of $\pi_i, i = 1, 2$. One can say that these compositions are real modifications in the following sense. We say $\mu : M \to N$ is a **real modification**, if one can take a representative of a complexification $\mu^*$ which is an isomorphism everywhere except on a nowhere dense subset of a neighbourhood of $M$ in $M^*$. Clearly a complex modification is a real modification. But it is not clear whether,
or not, a real modification is a complex modification, that is, isomorphic to the real part of a complex proper modification.

**Example 2.10.** The following map is an analytic isomorphism, hence a real modification,

$$
R \to R, \quad x \mapsto x + \frac{1}{2(1 + x^2)}.
$$

But the homeomorphism $R \to R, x \mapsto x^3$, is not a real modification.

3. **Triviality theorem**

Let $I$ be an interval in $R$, which contains the origin $0$. Let $F : (R^n, 0) \times I \to R, 0$ be an analytic function-germ. We consider the family $f_t : R^n, 0 \to R, 0, t \in I$, defined by $f_t(x) = F(x, t)$.

**Definition 3.1** (Blow-analytic triviality). Let $\pi : M, E \to R^n, 0$ be a proper analytic modification. We say $f_t, t \in I$, is blow-analytically trivial via $\pi$ if there are a $t$-level preserving homeomorphism $h : (R^n, 0) \times I \to (R^n, 0) \times I$ and a $t$-level preserving analytic isomorphism $H : (M, E) \times I \to (M, E) \times I$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
(M, E) \times I & \xrightarrow{\pi \times \text{id}} & (R^n, 0) \times I \\
\downarrow H & & \downarrow F_0 \\
(M, E) & \xrightarrow{\pi \times \text{id}} & (R^n, 0) \times I \\
\end{array}
$$

where $F_0 : (R^n, 0) \times I \to R, 0$ is the map defined by $(x, t) \mapsto f_0(x)$.

In all the cases we are interested in, $\pi : M \to R^n$ is the real part of a complex proper modification $\pi^* : M^* \to C^n$ defined over reals.

Consider the Taylor expansion of $f_t(x) = F(x, t)$ at 0 in $R^n$:

$$f_t(x) = \sum_{\nu} c_{\nu}(t)x^{\nu}, \quad \text{where} \quad x^{\nu} = x_1^{\nu_1} \cdots x_n^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n).$$

We set $H_j(x, t) = \sum_{\nu, |\nu| = j} c_{\nu}(t)x^{\nu}$ where $|\nu| = \nu_1 + \cdots + \nu_n$, and assume that $k$ is the smallest number so that $H_k(x, t)$ is not identically equal to 0.

**Theorem 3.2** ([30]). If $H_k(x, t)$ has an isolated singularity in $R^n$ for any $t \in I$, then $f_t, t \in I$, is blow-analytically trivial via the blowing-up at the origin.

Let $w = (w_1, \ldots, w_n)$ be an n-tuple of positive integers. We set

$$H_j^{(w)} = \sum_{\nu, |\nu|_w = j} c_{\nu}(t)x^{\nu} \quad \text{where} \quad |\nu|_w = w_1\nu_1 + \cdots + w_n\nu_n,$$

and assume that $k$ is the smallest number so that $H_k^{(w)}$ is not identically equal to 0.

**Theorem 3.3** ([14]). If $H_k^{(w)}(x, t)$ has an isolated singularity in $R^n$ for any $t \in I$, then $f_t, t \in I$, is blow-analytically trivial via a toric modification.

See §1.5 in [36], §5 in [6], [16], about toric modifications. See [37] for a generalization of this theorem.
Example 3.4 ([4]). Consider the family \( f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}, \) \( t > -15^{1/7}(7/2)^{4/5}/3. \) This function is a weighted homogeneous polynomial with weight (1, 2, 3) and weighted degree 15. This family satisfies the assumption of Theorem 3.3 and hence \( f_t \) is blow-analytically trivial. An important fact is that this family is not bilipschitz trivial near \( t = 0. \) See S. Koike ([28]) for a proof.

It is expected that the blow-analytic equivalence should not have moduli. Indeed T.-C. Kuo proved the following: If an analytic function \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) defines an isolated singularity, then the number of blow-analytic equivalence classes nearby \( f \) is finite. A more precise statement is the following.

Theorem 3.5 ([31]). Let \( P \) be a subanalytic set and let \( F : (\mathbb{R}^n, 0) \times P \to \mathbb{R}, 0 \) be an analytic function. If the functions \( f_t : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) defined by \( x \mapsto F(x, t) \) have an isolated singularity for all \( t \in P, \) then there is a subanalytic filtration
\[
P = P_0 \supset P_1 \supset \cdots \supset P_N \supset P_{N+1} = \emptyset, \quad \dim P_i > \dim P_{i+1},
\]
such that \( f_t \) and \( f_{t'} \) are blow-analytically equivalent for \( t, t' \) belonging to the same connected component of \( P_i - P_{i+1}. \)

K. Kurdyka ([32]) introduced the notion of arc-analytic map. We recall some fundamental facts here.

Definition 3.6 (Arc-analytic map). Let \( X \) and \( Y \) be real analytic manifolds. We say that a map \( f : X \to Y \) is arc-analytic (a.a. for short) if \( f \circ \alpha \) is analytic for any analytic map \( \alpha : \mathbb{R}, 0 \to X. \)

Theorem 3.7 ([1]). Let \( f : U \to \mathbb{R} \) be an arc-analytic function and \( U \) be an open subset of \( \mathbb{R}^n. \) If there are analytic functions \( G_i(x), i = 0, \ldots, p, \) so that
\[
G_0(x)f(x)^p + G_1(x)f(x)^{p-1} + \cdots + G_{p-1}(x)f(x) + G_p(x) \equiv 0,
\]
then \( f \) is blow-analytic.

Corollary 3.8. An arc-analytic function with semi-algebraic graph is blow-analytic.

Example 3.9 ([1]). The function \( f(x, y) = x^3e^{x^2/(x^4+y^2)} \) is blow-analytic. But there are no non-zero analytic functions vanishing on its graph.

Definition 3.10. Let \( X \) and \( Y \) be real analytic manifolds. We say that a map \( f : X \to Y \) is locally blow-analytic if there is a locally finite family of analytic maps \( \{ \psi_i : M_i \to X \} \) with the following properties:
- \( \psi_i \) are compositions of finitely many local blowing-ups with nonsingular centers,
- there are compact subsets \( K_i \) of \( M_i \) with \( \bigcup_i \psi_i(K_i) = X, \) and
- \( f \circ \psi_i \) are analytic.

Theorem 3.11 ([1]). An arc-analytic function \( f : U \to \mathbb{R} \) with subanalytic graph is locally blow-analytic.

See also [40] for another proof of this theorem.

Question 3.12. Is a locally blow-analytic function \( f : U \to \mathbb{R} \) blow-analytic?

When \( \dim U = 2, \) the answer is "yes", since local blowing-ups can be glued together to yield blowing-ups.
4. Arc lifting property

A remarkable property of blowing-up is the arc lifting property.

**Definition 4.1 (Arc lifting property).** Let $I$ be an open interval in $\mathbb{R}$. Let $X$ and $Y$ be real analytic manifolds. We say that a map $f : X \to Y$ has *the arc lifting property* (alp. for short) if for any analytic map $\alpha : I \to Y$ there is an analytic map $\tilde{\alpha} : I \to X$ so that $f \circ \tilde{\alpha} = \alpha$.

The blowing-up $\pi : \overline{M} \to M$ with a nonsingular center has the alp.

The blowing-up with an ideal center has the alp. because it is dominated by a composition of blowing-ups with nonsingular centers.

**Example 4.2.** Let $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ be the map-germ defined by

$$(x, y) \mapsto \left( x, \frac{y(y^2 - x^2)}{x^2 + y^2} \right)$$

This map can be extended continuously at 0. Let $\pi : M \to \mathbb{R}^2$ be the blowing-up at the origin. Consider the map

$$F : M \to M, \quad (x, y) \times [\xi : \eta] \mapsto f(x, y) \times [\xi(\xi^2 + \eta^2) : \eta(\eta^2 - \xi^2)].$$

Here we use the same notation as that at the end of §1. It is easy to see that $\pi \circ F = f \circ \pi$. Since the image of the set of regular points of $F$ by $F$ is $M$, $f$ has the arc lifting property. Since the jacobian of $f$ is $\frac{-x^4 + 4x^2y^2 + y^4}{(x^2 + y^2)^2}$, which is zero along $x^2 - (2 + \sqrt{5})y^2 = 0$, $(x, y) \neq 0$, the lifting is not global.

5. Blow-analytic invariants

5.1. Singular set.

**Theorem 5.1 ([39]).** Let $f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be two analytic function germs, and let $\Sigma_f$ and $\Sigma_g$ denote their singular sets. If there is a blow-analytic homeomorphism $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ with $f = g \circ h$, then $h(\Sigma_f) = \Sigma_g$. (That is $h$ preserves the singular set.)

However, a blow-analytic equivalence of analytic functions does not, in general, preserve their singular loci, as the following example shows.

**Example 5.2.** Let $f_t(x, y) = x^4 + 2t^2x^2y^2 + y^4 + x^5, t \in \mathbb{R}$. By Theorem 3.2, this family is blow-analytically trivial. Nevertheless, the dimension $\dim_{\mathbb{R}} \mathbb{R}(x, y)/\langle \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y} \rangle$ changes at $t = 1$. 
5.2. Numerical invariant. Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be an analytic function and let \( \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \) be an analytic map. If \( f \circ \alpha \) is not identically zero, then there is a positive integer \( k \) so that

\[
f \circ \alpha(t) = ct^k + \text{higher order terms}, \quad c \neq 0.
\]

We call \( k \) the order of \( f \) along \( \alpha \) and denote it by \( \text{ord}_\alpha(f) \). Define \( \text{ord}_\alpha(f) = \infty \) when \( f \circ \alpha \) is identically zero. We define \( A(f) \) by

\[
A(f) := \{ \text{ord}_\alpha(f) : \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \text{ analytic} \}.
\]

**Theorem 5.3.** If two analytic function germs \( f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) are blow-analytically equivalent, then \( A(f) = A(g) \).

**Remark 5.4.** Let \( \text{mult}_0(f) \) denote the multiplicity of \( f \) at 0, i.e., the degree of the initial polynomial of \( f \). It is easy to show that \( \text{mult}_0(f) = \min A(f) \). As a consequence, the multiplicity is a blow-analytic invariant of analytic function germs. So, this theorem should be compared with Zariski’s multiplicity conjecture: If two holomorphic functions \( f, g : \mathbb{C}^n, 0 \to \mathbb{C}, 0 \) are topologically equivalent (\( C^0 \)-equivalent or \( C^0\)-V-equivalent), then \( \text{mult}_0(f) = \text{mult}_0(g) \). This is still open. It is clear that the definition of \( A(f) \) makes sense for a holomorphic function \( f \) and it is interesting to ask the following question: Is \( A(f) \) a topological invariant for holomorphic functions \( f \)?

**Example 5.5.** Let \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( f : \mathbb{K}^n, 0 \to K, 0 \) be the analytic function defined by \( f(x_1, \ldots, x_n) = x_1^{m_1} \cdots x_n^{m_n} \). Then

\[
A(f) = \left( \sum_{i \in I} m_iN \right) \cup \{ \infty \}.
\]

Let \( f : \mathbb{K}^n, 0 \to K, 0 \) be an analytic function. Let \( \pi : M, E \to \mathbb{K}^n, 0, E = \pi^{-1}(0) \), denote a real modification. e.g., a composition of finitely many blowing-ups with nonsingular centers. We assume that \( f \circ \pi \) is normal crossing, that is, \( f \circ \pi \) can be locally expressed as a product of powers of a number of local coordinates. Let \( (f \circ \pi)_0 = \sum_{j \in J} m_j E_j \) denote the irreducible decomposition of the zero locus of \( f \circ \pi \) and \( C \) denote the set of subsets \( I \) of \( J \) with \( E_I \subset E \) where \( E_I = E_I^0 \cap E \), \( E_I^0 = \cap_{i \in I} E_i - \cup_{j \in J - I} E_j \).

The following formula is stated in [25], Theorem I.

**Theorem 5.6.** \( A(f) = \bigcup_{I \in C} A_I(f) \) where \( A_I(f) = (\sum_{i \in I} m_iN) \cup \{ \infty \} \).

Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be a real analytic function. We set

\[
A^\pm(f) = \{ \text{ord}_\alpha(f) : \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 \text{ analytic and } \pm f \circ \alpha(t) \geq 0 \text{ near } 0 \}
\]

The proof of Theorem 5.3 shows \( A^\pm(f) = A^\pm(g) \) if \( f \) and \( g \) are blow-analytically equivalent. In a way similar to the proof of Theorem 5.6, we obtain the following

**Theorem 5.7.** \( A^\pm(f) = \bigcup_{I \in C^\pm} A_I(f) \) where \( C^\pm \) denotes the set of \( I \in C \) so that \( E_I^0 \) intersects with the closure of \( \{ y \in M : \pm f \circ \pi(y) > 0 \} \).
5.3. Zeta functions. Recently S. Koike and A. Parusiński ([27]) have introduced zeta functions for the blow-analytic equivalence. In their paper ([27]), they call their zeta functions the 'motivic type invariants', since their zeta functions can be derived from zeta functions whose coefficients are motives. G. Fichou ([10]) generalizes their invariants using the virtual Poincaré polynomial. Since these are very interesting invariants, we review their results in this section. See also [35] for the virtual Betti numbers.

Let $C$ be a category whose objects are a class of subsets of the Euclidean spaces with some good properties. We consider an invariant $\beta : C \to R$, where $R$ is a commutative ring, with the following properties.

- $\beta(X) = \beta(X - Y) + \beta(Y)$ if $Y$ is a closed subset in $X$.
- $\beta(X \times Y) = \beta(X)\beta(Y)$.

When $C$ is the category of subanalytic subsets in Euclidean spaces which have finite homologies, the $\mathbb{Z}/2\mathbb{Z}$-Euler characteristic $\beta$ with compact supports has these properties.

We say a semi-algebraic set $A$ in a compact nonsingular real algebraic manifold $M$ is a $\mathcal{AS}$-subset if for any analytic map $\alpha : (-\epsilon, \epsilon) \to M$, $\epsilon > 0$, with $\alpha(0, \epsilon) \subset A$, there is a positive number $\epsilon'$ so that $\alpha(-\epsilon', 0) \subset A$. See [33] for more information about $\mathcal{AS}$-subsets.

**Theorem 5.8 ([10]).** Let $\mathcal{AS}$ denote the set of all semi-algebraic $\mathcal{AS}$-subsets in compact nonsingular real algebraic manifolds. There is an invariant $\beta : \mathcal{AS} \to \mathbb{Z}[u, u^{-1}]$ with the above properties which satisfies the following:

$$3(X) = \sum_k (\dim H_k(X, \mathbb{Z}/2\mathbb{Z})) u^k$$

when $X$ is compact and nonsingular. Moreover, if two $\mathcal{AS}$-sets $X, Y$ are Nash (i.e., semi-algebraically and analytically) equivalent, then $\beta(X) = \beta(Y)$.

Notice the following: $\beta(\emptyset) = 0$, $\beta(P^n) = 1 + u + u^2 + \cdots + u^n$, $\beta(\mathbb{R}^n) = u^n$.

**Example 5.9.** It is not true that $\beta(X) = k\beta(Y)$ when there is an unbranched $k$-fold covering $X \to Y$. Consider the double covering $S^1 \to P^1$ and observe that $\beta(S^1) = \beta(P^1) = u + 1$.

We consider the space of polynomial arcs of order $k$:

$$L_k := \{ \alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0 : \text{polynomial of degree } k \} = \mathbb{R}^{nk}.$$ 

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}$, be an analytic function. The following spaces are algebraically constructible

$$A_k(f) := \{ \alpha \in L_k : \text{ord}(f \circ \alpha) = k \} \quad A_k^\pm(f) := \{ \alpha \in L_k : f \circ \alpha = \pm t^k + \cdots \}.$$ 

Notice that if $f$ and $g$ are analytically equivalent, then $A_k(f)$ (resp. $A_k^\pm(f)$) and $A_k(g)$ (resp. $A_k^\pm(g)$) are actually isomorphic as algebraic constructible sets. Define Zeta functions by the following formulas.

$$Z_f(t) := \sum_{k \geq 1} \beta(A_k(f)) \left(\frac{t}{u^n}\right)^k \quad Z_f^\pm(t) := \sum_{k \geq 1} \beta(A_k^\pm(f)) \left(\frac{t}{u^n}\right)^k$$

where $u = -1$ when $\beta$ is the $\mathbb{Z}/2\mathbb{Z}$-Euler characteristic with compact supports ([27]), or $u$ is an indeterminate when $\beta$ is the virtual Poincaré polynomial ([10]).
Let \( \pi : M, E \to \mathbb{R}^n, 0, E = \pi^{-1}(0) \), be a proper analytic modification so that \( f \circ \pi \), \( \det(d\pi) \) are in normal crossing and that \( \pi \) is an isomorphism over \( \mathbb{R}^n - f^{-1}(0) \). We assume that \( \pi^{-1}(0) \) is a normal crossing divisor. We use the notation defined in the paragraph after Example 5.5. We consider the irreducible decompositions of the zero loci of \( f \circ \pi \) and \( \det(d\pi) \), the jacobian determinant of \( \pi \):

\[
(f \circ \pi)_0 = \sum_{j \in J} m_j E_j, \quad (\det(d\pi))_0 = \sum (\nu_j - 1)E_j.
\]

The following formula is often called the Denef-Loeser formula.

**Theorem 5.10** ([27], [10]). Setting \( \phi(\lambda) = \lambda/(1 - \lambda) = \lambda + \lambda^2 + \lambda^3 + \cdots \), we have

\[
Z_f(t) = \sum_{l \neq \emptyset} \beta(E_I^+) (u - 1)^{|I|} \prod_{i \in I} \phi\left( \frac{t^{m_i}}{u^{n_i}} \right).
\]

**Remark 5.11.** When \( \beta \) is the virtual Poincaré polynomial we need to assume that \( f \) is a polynomial and that \( \pi \) is algebraic (since we do not know that \( E_I^+ \) is semi-algebraic).

It is also possible to obtain a formula for \( Z_f^\pm(t) \) similar to Theorem 5.10. To do this, we introduce somenotation. We define \( A_k^\pm(f, E_I^+) \) by

\[
A_k^\pm(f, E_I^+) := \pi_k(\pi_*^{-1}(A_k^\pm(f)) \cap L(M, E_I^+)) = \bigsqcup \pi_k(A_{k,j}^\pm(f, E_I^+))
\]

where \( A_{k,j}^\pm(f, E_I^+) := \{ \gamma \in \pi_*^{-1}(A_k^\pm(f)) \cap L(M, E_I^+) : \text{ord}_E, E_i = j_i \} \). Let \( p \in E_I^+ \) and let \( U \) be a coordinate neighbourhood at \( p \). Using the local coordinates \( y = (y_1, \ldots, y_n) : U \to \mathbb{R}^n \) with \( E_I^+ = \{ y_i = 0, i \in I, y_i \neq 0, i \not\in I \} \), we can express \( f \circ \pi \) as follows:

\[
f \circ \pi(y) = u(y) \prod_{i \in I} y_i^{m_i}, \quad \text{where } u(y) \text{ is a unit.}
\]

We set \( y_I = (y_i)_{i \in I} \) and define

\[
E_{I}^\pm|_U = \left\{ (p, y_I) \in (E_I^+ \cap U) \times \mathbb{R}^{|I|} : u(p) \prod_{i \in I} y_i^{m_i} = \pm 1 \right\}
\]

The sets \( E_{I}^\pm|_U \) can be patched together and we obtain a set \( \hat{E}_{I}^\pm \). We denote by \( m_I \) the greatest common divisor of \( m_i, i \in I \), and define

\[
\hat{E}_{I}^\pm|_U = \left\{ (p, w) \in (E_I^+ \cap U) \times \mathbb{R} : u(p)w^{m_I} = \pm 1 \right\}.
\]

The sets \( \hat{E}_{I}^\pm|_U \) can be patched together and we obtain a set \( \hat{E}_{I}^\pm \). Setting \( \beta_I^\pm = 3(\hat{E}_{I}^\pm) \), we obtain

\[
Z_f^\pm(t) = \sum_{l} \beta_I^\pm (u - 1)^{|I|-1} \prod_{i \in I} \phi\left( \frac{t^{m_i}}{u^{n_i}} \right).
\]

**Theorem 5.12** ([27]). Let \( f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be two analytic functions and let \( \beta \) be the \( \mathbb{Z}/2\mathbb{Z} \)-Euler characteristic with compact supports. Assume that there are real modifications \( \pi_i : \mathbb{R}/4 \to \mathbb{R}^n, i = 1, 2 \), so that \( \pi_1 \) (resp. \( \pi_2 \)) is an isomorphism except possibly over the zero set of \( f \) (resp. \( g \)). If there is an analytic isomorphism \( (M_1, \pi_1^{-1}(0)) \to (M_2, \pi_2^{-1}(0)) \) which induces a blow-analytic homeomorphism \( h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) with \( f = g \circ h \), then \( Z_f(l) = Z_g(l) \). \( Z_f^\pm(t) = Z_g^\pm(t) \).
Similarly we obtain the following

**Theorem 5.13 ([10]).** Let \( f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0 \) be two polynomial functions and let \( \beta \) be the virtual Poincaré polynomial. Assume that there are algebraic modifications \( \pi_i : M_i \to \mathbb{R}^n, i = 1, 2 \), whose critical loci are normal crossings. We assume that \( \pi_1 \) (resp. \( \pi_2 \)) is an isomorphism except over the zero set of \( f \) (resp. \( g \)). If there is an analytic isomorphism \( (M_1, \pi_1^{-1}(0)) \to (M_2, \pi_2^{-1}(0)) \) which induces a blow-analytic isomorphism \( h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) with \( f = g \circ h \), then \( Z_f(t) = Z_g(t), Z_f^+(t) = Z_g^+(t) \).

See Definition 7.2 below for the notion of blow-analytic isomorphism.

6. Lipschitz maps

An interesting class of maps which are not differentiable is the class of Lipschitz maps. We start with some basics.

Let \( U \) be a convex open subset of \( \mathbb{R}^n \). A map \( f : U \to \mathbb{R}^p \) is said to be Lipschitz if there is a positive constant \( K \) so that
\[
|f(x) - f(x')| \leq K|x - x'| \quad \forall x, x' \in U.
\]

Recall that Rademacher’s theorem ([15, Theorem 4.1.1]), states that a function which is Lipschitz on an open subset of \( \mathbb{R}^n \) is differentiable almost everywhere (in the sense of Lebesgue measure) on that set. This allows us to introduce the following definition.

**Definition 6.1 (Generalized Jacobian).** The generalized Jacobian \( \partial f(0) \) of \( f \) at 0 is the convex hull of all matrices obtained as limits of sequences of the Jacobi matrices of \( f \) at \( x_i \) where \( x_i \to 0, x_i \notin Z \). Here \( Z \) denotes the set of points at which \( f \) fails to be differentiable.

**Theorem 6.2 ([5]).** Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be a Lipschitz map-germ. If \( \partial f(0) \) does not contain singular matrices, then \( f \) has a Lipschitz inverse.

In this section, we are interested in blow-analytic maps satisfying the Lipschitz condition.

Let \( U \) be a convex open subset of \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R} \) be a continuous function with subanalytic graph. Then there is an nowhere dense closed subanalytic subset \( Z \) so that \( f \) is analytic on \( U - Z \).

**Lemma 6.3.** The function \( f \) is Lipschitz if and only if all partial derivatives of \( f \) are bounded on \( U - Z \).

**Theorem 6.4 ([13]).** Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be an arc-analytic map with subanalytic graph. If \( f \) is bilipschitz, i.e., there are positive constants \( c_1, c_2 \) so that
\[
c_1|y - y'| \leq |f(y) - f(y')| \leq c_2|y - y'|,
\]
then \( f^{-1} \) is arc-analytic.

Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be a homeomorphism which is blow-analytic and Lipschitz. The theorem asserts that the inverse \( f^{-1} \) is blow-analytic, if \( f^{-1} \) is Lipschitz.

**Corollary 6.5.** Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be an arc-analytic map with semi-algebraic graph. If \( f \) is bilipschitz, then \( f^{-1} \) is blow-analytic.
Theorem 6.6 ([13]). Let $F : \mathbb{R}^m \times \mathbb{R}^n, 0 \to \mathbb{R}^n, (x, y) \mapsto F(x, y)$, be an arc-analytic map with subanalytic graph. If there are positive constants $c_1, c_2$ so that

$$c_1 |y - y'| \leq |F(x, y) - F(x, y')| \leq c_2 |y - y'|,$$

then there is an arc-analytic and subanalytic map $\tau : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ such that

$$\{F(x, y) = 0\} = \{y = \tau(x)\}.$$

Remark 6.7. Let $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbb{R}, 0 \to \mathbb{R}^n, 0$ be an analytic map. Let $\text{ord}(\alpha)$ denote $\min\{\text{ord}(\alpha_1), \ldots, \text{ord}(\alpha_n)\}$. If an arc-analytic map $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is Lipschitz, then $\text{ord}(f \circ \alpha) \geq \text{ord}(\alpha)$. If the map $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is bilipschitz, then $\text{ord}(f \circ \alpha) = \text{ord}(\alpha)$. In particular, the image of a nonsingular curve by an arc-analytic bilipschitz map $\mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is a nonsingular curve.

Question 6.8. Does there exist a blow-analytic map (or an arc-analytic map with subanalytic graph) $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ with the following properties?

- there is a positive constant $c$ so that $c|y - y'| \leq |f(y) - f(y')| \quad \forall y, y' \in \mathbb{R}^n, 0$;
- $f$ is not Lipschitz.

7. Blow-analytic isomorphism and analytic arcs

A blow-analytic homeomorphism can be quite far from a bilipschitz homeomorphism.

Theorem 7.1 ([26]). For any unibranched curve $C \subset \mathbb{R}^2, 0$, there is a blow-analytic homeomorphism $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ such that $h(C)$ is nonsingular.

Theorem 7.1 motivates us to strengthen the conditions imposed to the definition of blow-analytic homeomorphisms.

Definition 7.2. We say that a map $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is a blow-analytic isomorphism (bai for short) if there are two neighbourhoods $U, U'$ of 0 in $\mathbb{R}^n$ so that the following conditions are satisfied.

- there are complex modifications $\pi : M \to U, \pi' : M' \to U'$, and an analytic isomorphism $F : (M, E) \to (M', E')$ of analytic spaces, where $E$ and $E'$ denote the critical loci of $\pi$ and $\pi'$ respectively.
- $f$ is a homeomorphism and $\pi' \circ F = f \circ \pi$.

A blow-analytic isomorphism is clearly a blow-analytic homeomorphism. But the converse is not true. For example, the blow-analytic homeomorphism in Example 7.1 is not a bai. In fact, the critical locus of the composites of horizontal arrows are normal crossing, and we have a correspondence between their irreducible components, but they have different multiplicities.

Let $\pi : M \to \mathbb{R}^n$ be a complex modification whose critical locus is a normal crossing divisor. We consider an analytic vector $\xi$ on $M$ which is tangent to each irreducible component of the critical locus. By integrating $\xi$, we obtain an analytic isomorphism of $M$. If it induces a homeomorphism of $\mathbb{R}^n$ near 0, this is a blow-analytic isomorphism. Thus, in all triviality theorems stated before, we can replace bah by bai.
Definition 7.3. Let $\pi : M \to U$ be a composition of blowing-ups with nonsingular centers. A blow-analytic function $P : U \to \mathbb{R}$ is said to be a blow-analytic unit (bau for short) via $\pi$ if $P\pi$ extends to an analytic unit (i.e. an analytic function which is nowhere vanishing). $P$ is said to be a blow-analytic unit (bau for short) if there is $\pi : M \to U$ such that $P$ is a bau via $\pi$.

Theorem 7.4. If $f : \mathbb{R}^n, 0 \to \mathbb{R}^m, 0$ is a blow-analytic isomorphism, then the Jacobian determinant $\det(df)$ is a blow-analytic unit.

Let $w_1, \ldots, w_n$ be real numbers. We consider the map

$$f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0, \quad x = (x_1, \ldots, x_n) \mapsto (x_1 P(x)^{w_1}, \ldots, x_n P(x)^{w_n}),$$

where $P : \mathbb{R}^n, 0 \to \mathbb{R}$ is a bounded blow-analytic function.

Theorem 7.5. Let $P$ be a non-negative blow-analytic function via some toric modification $\pi : M \to \mathbb{R}^n$. If $P + \sum_{i=1}^n w_i x_i \frac{\partial P}{\partial x_i}$ is a blow-analytic unit via the modification $\pi$, and if $P$ and $\sum_{i=1}^n w_i x_i \frac{\partial P}{\partial x_i}$ are continuously extendable on $\mathbb{R}^n - 0, 0$, then the map $f$ defined by (3) is a blow-analytic isomorphism.

Example 7.6. The map

$$f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0), \quad (x, y) \mapsto (xP^3, yP^2), \quad P = \frac{x^4 + 2y^6}{x^4 + y^6}$$

is a blow-analytic isomorphism.

Consider the map

$$f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0, \quad x = (x_1, \ldots, x_n) \mapsto (x_1 + Q(x_2, \ldots, x_n), x_2, \ldots, x_n),$$

where $Q : \mathbb{R}^{n-1}, 0 \to \mathbb{R}$ is a blow-analytic function. Since the map $(x_1, \ldots, x_n) \mapsto (x_1 - Q(x_2, \ldots, x_n), x_2, \ldots, x_n)$ is the inverse of $f$, $f$ is a homeomorphism.

Theorem 7.7. If $Q$ is blow-analytic, then $f$ is a blow-analytic isomorphism.

Example 7.8 ([38]). Consider a blow-analytic map $f : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ defined by

$$(x, y, z) \mapsto (x, y, z + \frac{2x^5 y}{x^6 + y^4}).$$

This is a blow-analytic isomorphism by Theorem 7.7. Let $\alpha : \mathbb{R}, 0 \to \mathbb{R}^3, 0$ be the map defined by $t \mapsto (t^2, t^3, 0)$. Observe that $f \circ \alpha(t) = (t^2, t^3, t)$. This means that the blow-analytic isomorphism $f$ sends a singular curve, the image of $\alpha$, to a regular curve.

We say that an analytic map $\alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0$ is irreducible if $\alpha$ cannot be written as $\alpha = \beta \circ \psi$, where $\beta : \mathbb{R}, 0 \to \mathbb{R}^n, 0$ and $\psi : \mathbb{R}, 0 \to \mathbb{R}, 0$, are analytic and $\psi'(0) = 0$.

Theorem 7.9. Let $\alpha : \mathbb{R}, 0 \to \mathbb{R}^n, 0, n \geq 3$, be an irreducible analytic map. Then there is a blow-analytic isomorphism $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ such that $f \circ \alpha$ is a regular map.
8. Jacobian of blow-analytic map

Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be a blow-analytic map. It is interesting to investigate what we can conclude when we assume that \( \det(df) \) is a blow-analytic unit. For example, is such a \( f \) a blow-analytic isomorphism?

**Example 8.1.** We identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) by the map \( (x, y) \mapsto z = x + \sqrt{-1}y \). Let \( k \) be a positive integer. Consider the continuous blow-analytic map

\[
  f : \mathbb{C}, 0 \rightarrow \mathbb{C}, 0, \quad z \mapsto z^{k+1}/z^k = z^{2k+1}/|z|^{2k}.
\]

Looking at the restriction to a small circle \( |z| = \varepsilon \), the mapping degree of \( f \) is \( 2k+1 \). In particular, \( f \) is not a homeomorphism. Since

\[
\det(df) = \begin{vmatrix} (k+1)z^k/z^k & -kz^{k+1}/z^{k+1} \\ -kz^{k+1}/z^{k+1} & (k+1)z^k/z^k \end{vmatrix} = (k+1)^2 - k^2 = 2k+1, \quad z \neq 0,
\]

\( \det(df) \) is a blow-analytic unit. We also have that \( f \) is Lipschitz, by Lemma 6.3. Let \( M \rightarrow \mathbb{C} \) denote the blowing-up at the origin. Since the map \( f \) is induced by an unbranched covering \( M \rightarrow M \) of degree \( 2k+1 \), \( f \) has the arc lifting property.

Example 8.1 shows that a blow-analytic map \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) may not be a homeomorphism, even though \( \det(df) \) is a blow-analytic unit. However, this kind of phenomenon is not possible in higher codimensional cases.

**Proposition 8.2.** Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be a blow-analytic map so that \( \det(df) \) is a blow-analytic unit. If there is a subset \( C \) of \( \mathbb{R}^n, 0 \), of codimension \( \geq 3 \), so that \( f|_{\mathbb{R}^n-C} \) is analytic, then \( f \) is a homeomorphism.

It is an open question whether \( f \) is a bai or not.

We have a version of the inverse mapping theorem via toric modification, which is the following

**Theorem 8.3.** Let \( h = (h_1, \ldots, h_n) : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be a continuous blow-analytic map via a toric modification. If \( \frac{\partial h_1}{\partial x_1}, \frac{\partial h_2}{\partial x_1}, \ldots, \frac{\partial h_n}{\partial x_1}, \ldots, \frac{\partial h_1}{\partial x_n}, \ldots, \frac{\partial h_n}{\partial x_n} \) are blow-analytic units and they are continuously extendable on \( \mathbb{R}^n - 0, 0 \), then \( h \) is a blow-analytic isomorphism.

If the map \( h = (h_1, \ldots, h_n) : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) satisfies the assumption of Theorem 8.3 after permutations of \( x_1, \ldots, x_n \) and \( h_1, \ldots, h_n \), then \( h \) is a blow-analytic isomorphism, by Theorem 8.3.

This is the corrected version of Theorem 6.1 in [12].

Lastly we have three more theorems.

**Theorem 8.4.** Let \( f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0 \) be a blow-analytic map so that \( \det(df) \) is a blow-analytic unit. If there are nonsingular subanalytic subsets \( C, C' \) so that \( f \) is blow-analytic via the blowing up with center \( C' \) and that \( f(C) = C' \), then \( \text{codim} \ C = \text{codim} \ C' \) and \( f \) has the arc lifting property. Moreover, there is an analytic map \( \tilde{f} : M \rightarrow M' \) such that \( \tilde{f} \) is locally an isomorphism and that \( \pi' \circ \tilde{f} = f \circ \pi \). where \( \pi : M \rightarrow \mathbb{R}^n \) is the blowing-up at \( C \) and \( \pi' : M' \rightarrow \mathbb{R}^n \) is the blowing-up at \( C' \).

**Theorem 8.5.** Let \( f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0) \) be a blow-analytic map. If \( \det(df) \) is a blow-analytic unit, then \( f \) is finite.
Theorem 8.6. Consider a blow-analytic map \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) defined by 
\[
(x_1, \ldots, x_n) \mapsto (x_1P_1(x), \ldots, x_nP_n(x)), \quad \text{where } P_i \text{ are blow-analytic units.}
\]
If \( f \) is blow-analytic via a toric modification and \( \det(df) \) is a blow-analytic unit, then \( f \) is a blow-analytic isomorphism.

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