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THE EXISTENCE OF MULTIPLE SOLUTIONS FOR NONLINEARLY PERTURBED PARABOLIC-ELLIPTIC SYSTEMS OF KELLER-SEGEL TYPE IN $\mathbb{R}^2$

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1. Introduction

This is a joint work with Professor F. Takahashi (Osaka city university) and Professor T. Ogawa (Tohoku university).

The motion of slime molds controlled by some chemical substance is referred as chemotaxis. This phenomena is described by the system of parabolic equations called the Keller-Segel system (see Keller-Segel [10], Herrero-Velázquez [8],[9], Nagai [14], [15], Biler [1], Nagai-Senba-Yoshida [17], Nagai-Senba-Suzuki [16] and Senba-Suzuki [18]). Among them, a simplified model

$$(1.1)\qquad \begin{cases} \partial_t u - \Delta u + \nabla(u\nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ \quad u(t,x),\ v(t,x) \geq 0, \\ \quad u(0,x) = u_0(x) \geq 0 \end{cases}$$

has been considered by several authors. The system (1.1) describes the case where the diffusion of the chemical substances is much slower than that of chemotaxis ameba.

For (1.1), the existence of a blowing up solution corresponding to the concentration of ameba and that of chemical substances is well known (Herrero-Velázquez [8], [9], Nagai [14]).

Later Chen-Zhong [5] introduced the perturbed system

$$(1.2)\qquad \begin{cases} \partial_t u - \Delta u + \nabla(u\nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + F(v) = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ \quad u(t,x),\ v(t,x) \geq 0, \\ \quad u(0,x) = u_0(x) \geq 0 \end{cases}$$

with $F(v) = v + v^p$ ($p > 1$). This model can be interpreted as describing the chemotaxis with nonlinear diffusion for the chemical substance. It is known that the system (1.2) with the particular choice of $F(v) = v + v^p$ has a close nature of the original system (1.1). Indeed, one can show the local existence theory and finite time blow up with mass concentration phenomena in the similar way to (1.1), see for Chen-Zhong [4] and Kurokiba-Senba-Suzuki [13].

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In this note, we are concerned with the case where $F(v) = v - v^p$ in (1.2):

$$\begin{cases} \partial_t u - \Delta u + \nabla(u\nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v - v^p = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(t,x),\ v(t,x) \geq 0, \\ u(0,x) = u_0(x) \geq 0. \end{cases} \tag{1.3}$$

Different from the case where $F(v) = v + v^p$, it is not the case to consider that our system is a simple perturbation of the original system (1.1) because of the nonmonotonicity of $F(v) = v - v^p$. Actually, the elliptic equation

$$-\Delta v + v - v^p = f, \quad x \in \mathbb{R}^2 \tag{1.4}$$

has at least two positive solutions when f is a sufficiently small nonnegative nontrivial function, while

$$-\Delta v + v + v^p = f, \quad x \in \mathbb{R}^2$$

admits only one solution. Moreover, for the equation (1.4), it is also known that if the external force f is large in an appropriate sense, then there is no positive solution. Hence one may even wonder whether the finite time blow up in the usual sense occur in (1.3) or not. In this sense, the structure of the time dependent positive solution of (1.3) seems to be very much different from that of the original system (1.1) or perturbed system (1.2) with $F(v) = v + v^p$.

In this note, we shall consider solutions of (1.3) in the following sense:

$$u \in C([0,\infty); L^2(\mathbb{R}^2)) \cap C^1((0,\infty); L^2(\mathbb{R}^2)) \cap C((0,\infty); \dot{H}^2(\mathbb{R}^2)),$$
$$v \in C((0,\infty); H^1(\mathbb{R}^2)) \cap C((0,\infty); W^{2,2}(\mathbb{R}^2)).$$

For a small nonnegative initial data, there exists a global-in-time solution for (1.3) which is, in a sense, "small" one, see [12]. On the other hand, as is mentioned above, the perturbed nonlinear elliptic equation

$$-\Delta v + v - v^p = f, \quad x \in \mathbb{R}^2$$

admits at least two positive solutions for small and nonnegative f. So it is natural to ask whether the time dependent equation (1.3) also has the second positive solution or not. The main issue of this note is to obtain for example a radially symmetric positive solution of the nonlinearly perturbed system (1.3) which is different from the solution obtained in [12]. Namely, we show that *the existence of two time-dependent solutions for the system* (1.3):

Theorem 1.1. (Multiple existence) *Let $1 < p < \infty$. Then there exists a constant $C_{**} > 0$ such that, if the radially symmetric nonnegative initial data $u_0 \in L^2$ satisfies*

$$\|u_0\|_2 \leq C_{**},$$

then there exist two positive radial pair of solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ for (1.3). One of them is different from the solution obtained in [12].

The main idea for the construction of the second time dependent solution relies on the variational structure of the elliptic part of the system. The v-component of the solution obtained in [12] is corresponding to the solution of (1.4) which is bifurcated from the trivial solution of the elliptic problem (1.4) with $f = 0$. On the other hand, it has been known that the problem (1.4) with $f = 0$ has a unique positive solution w (cf. Berestycki-Lions [2], Gidas-Ni-Nirenberg [7] and Kwong [11]). This solution is obtained as a mountain pass critical point of

$$I_0(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int |v|^2 - \frac{1}{p+1}\int |v|^{p+1}.$$

Then, if the second variation of the functional I_0 at w, namely, the Hessian operator of I_0 at w, is not degenerate and if f is small (in an appropriate sense), we may construct the solution v of (1.4) bifurcated from the mountain pass solution w. Of course, this is not possible in general since the kernel of the Hessian $(\nabla^2 I_0)_w$ of I_0 at w may be nontrivial. However, since the structure of the kernel of $(\nabla^2 I_0)_w$ is well-understood, we may construct the bifurcation branch from the nontrivial solution of (1.4) under the restriction of the radial symmetry. Therefore, there is a possibility that we can construct the second local-in-time solution of (1.3) if we restrict the class of initial data appropriately. In this note, we shall show that this is indeed the case.

2. Variational Structure of the Lagrangian Functional

The existence of multiple positive solutions for the semilinear elliptic equation

$$-\Delta v + v = v^p + f, \quad x \in \mathbb{R}^2 \tag{2.1}$$

is known for small nonnegative nontrivial external force f in H^{-1}, see e.g. Zhu [19] and Cao-Zhou [4]. According to their results, there exists a solution of (2.1) for small f (in the H^{-1} sense), which is not the local minimizer of the functional $I_f(u)$ where I_f is given by

$$I_f(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} - \int_{\mathbb{R}^2} fv\,dx, \quad v \in H^1(\mathbb{R}^2).$$

In this section, we give some analysis for the dependence of this nonminimal solution to f. In order to analyze the continuous dependence of the nonminimal solution branch with respect to f, we need to refine results of Zhu [19] and Cao-Zhou [4] from a bifurcation theoretical point of view. As is mentioned in the introduction, it is well known that the nonlinear elliptic problem (2.1) with $f \equiv 0$

$$-\Delta v + v = v^p, \quad x \in \mathbb{R}^2 \tag{2.2}$$

has a radially symmetric positive solution w which is unique up to translation [2, 7, 11]. This solution is obtained as a critical point of the variational functional

$$I_0(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}$$

by the well known mountain pass lemma in H^1. Note that the Hessian operator $(\nabla^2 I_0)_u$ of this variational functional I_0 in H^1 at $u \in H^1$ is $L_u := -\Delta + 1 - p|u|^{p-1}$ (we regard L_u as an operator from H^1 to H^{-1}).

Since the problem (2.2) is invariant under the translation with respect to space variables, $w(\cdot - y)$ is also a solution of (2.2) for any $y \in \mathbb{R}^2$. Hence, it is reasonable to think that L_w has some degeneracy. Indeed, the following is well-known.

Proposition 2.1 (Kernel of the linearized operator). *For $u \in H^1$, let $L_u := -\Delta + 1 - p|u|^{p-1}$ with $1 < p < \infty$. Then for the solution w of* (2.2)*, the kernel of the operator L_w is spanned by $\partial_{x_1} w$ and $\partial_{x_2} w$.*

By making use of these facts, we can construct a solution branch of the nonminimal solution of (2.1) with the aid of the implicit function theorem. Hereafter H^1_r denotes the subspace of H^1 which consists of radially symmetric functions and $(H^1_r)^*$ its dual.

Proposition 2.2. *There exists $\delta > 0$ and $h \in C(B_{\delta,(H^1_r)^*}; H^1_r)$ such that $h(f)$ is a critical point of I_f and $h(0) = w$ where*

$$B_{\delta,(H^1_r)^*} := \{f \in (H^1_r)^*;\ \|f\|_{(H^1_r)^*} < \delta\}.$$

Moreover, h is a Lipschitz continuous mapping in $B_{\delta,(H^1_r)^*}$, namely, there exists $C > 0$ such that

$$\|h(f_1) - h(f_2)\|_{H^1} < C\|f_1 - f_2\|_{(H^1_r)^*}, \quad \forall f_1,\ f_2 \in B_{\delta,(H^1_r)^*}. \tag{2.3}$$

If $f \geq 0$, then $h(f) \geq 0$ holds.

The following corollary immediately follows from Proposition 2.2.

Corollary 2.3. *There exists $\rho > 0$ such that the conclusion of Proposition 2.2 holds $B_{\delta;(H^1_r)^*}$ and $(H^1_r)^*$ replaced by*

$$B_{\rho,L^2_r} := \{f \in L^2_r;\ \|f\|_2 < \rho\}$$

and L^2_r respectively, where L^2_r denotes the subspace of L^2 which consists of radially symmetric functions.

3. Proof of Theorem

In this section, we give the proof of Theorem 1.1.

Let $1 < p < \infty$. We choose M with $M < \rho$ where ρ is the number which appears in Corollary 2.3. The solution of (1.3) will be constructed in the complete metric space

$$X_{T,M} = \Big\{\phi \in C([0,T); L^2_r) \cap L^2(0,T; \dot{H}^1);\ \phi \geq 0,\ \|\!|\phi|\!\|_X \leq M\Big\}$$

with the metric $d(\phi,\psi) \equiv \sup_{t\in[0,T]} \|\!|\phi - \psi|\!\|_X$, where

$$\|\!|\phi|\!\|_X \equiv \Big(\sup_{\tau\in[0,T)} \|\phi(\tau)\|_2^2 + \int_0^T \|\nabla\phi(\tau)\|_2^2 d\tau\Big)^{\frac{1}{2}}$$

and $T > 0$ is chosen to be small later.

For $f \in X_{T,M}$ and radially symmetric nonnegative function $a \in L^2$, we define a map $\Phi_a : X_{T,M} \ni f \to u \in X_{T,M}$ such that u solve the following linear system (with respect to u):

$$\begin{cases} \partial_t u - \Delta u + \nabla(u\nabla v) = 0, & t > 0,\ x \in \mathbf{R}^2, \\ -\Delta v + v = v^p + f, & t > 0,\ x \in \mathbf{R}^2, \\ \quad u(0,x) = a. \end{cases} \tag{3.1}$$

Here we choose the solution $v(t)$ of the elliptic part of the above system as $h(f(t))$ where h appears in Corollary 2.3. Note that this is always possible from Corollary 2.3, since

$$\sup_{\tau\in[0,T)} \|f(\tau)\|_2 \leq \|\!|f|\!\|_X \leq M < \rho.$$

Again by virtue of Corollary 2.3, it is easy to see that

$$\begin{aligned}(3.2)\qquad &\sup_{\tau\in[0,T)} \|h(f(\tau))\|_{H^1} \le \sup_{\tau\in[0,T)} \|h(f(\tau)) - h(0)\|_{H^1} + \|h(0)\|_{H^1} \\ &\le C \sup_{\tau\in[0,T)} \|f(\tau) - 0\|_2 + \|w\|_{H^1} \le CM + \|w\|_{H^1} =: \sigma\end{aligned}$$

where w is a unique positive solution of $-\Delta w + w = w^p$ in $\mathbb{R}^2$. Hereafter for f and $\bar{f} \in X_{T,M}$, we denote $h(f(\tau))$ and $h(\bar{f}(\tau))$ by $v(\tau)$ and $\bar{v}(\tau)$ (or simply v and $\bar{v}$), respectively.

Our first lemma is:

Lemma 3.1. *There exists $C > 0$ such that*

$$\|\nabla v(\tau)\|_\infty^2 \le C(1 + \|\nabla f(\tau)\|_2) \tag{3.3}$$

for $\tau \in [0, T)$.

Proof of Lemma 3.1. The second equation of (3.1), the Sobolev embedding $H^1 \hookrightarrow L^{2p}$ and (3.2) yield

$$\|\Delta v\|_2^2 \le \|v\|_2 + \|v^p\|_2 + \|f\|_2 < C_1(M, \sigma)$$

for some $C_1 > 0$. Hence by using a version of the Brezis-Gallouet inequality [3],

$$\|f\|_\infty^2 \le C\left(\|f\|_{H^1}^2(1 + \|\Delta f\|_2^{1/2}) + \|\Delta f\|_2\right), \quad \forall f \in H^2(\mathbb{R}^2),$$

we have

$$\|\nabla v\|_\infty^2 \le C_2(\|\Delta v\|_2^2(1 + \|\nabla\Delta v\|_2^{1/2}) + \|\nabla\Delta v\|_2) \le C_3(1 + \|\nabla\Delta v\|_2). \tag{3.4}$$

Note that by the boundedness of $(-\Delta + 1)^{-1}$ from L^2 to $W^{1,2p}$, the Sobolev embedding $H^1 \hookrightarrow L^{2p}$ and (3.2),

$$\|\nabla v^p\|_2^2 = p^2 \int_{\mathbb{R}^2} |v|^{2(p-1)}|\nabla v|^2 \le p^2\|v\|_{2p}^{2(p-1)}\|\nabla v\|_{2p}^2 < C_4 \tag{3.5}$$

holds. Then the second equation of (3.1) together with (3.2) and (3.5) yields

$$\|\nabla\Delta v\|_2 \le \|\nabla v\|_2 + \|\nabla v^p\|_2 + \|\nabla f\|_2 \le C_5(1 + \|\nabla f\|_2).$$

Hence combining this relation with (3.4), we have

$$\|\nabla v\|_\infty^2 \le C_6(1 + \|\nabla f\|_2),$$

thus the conclusion.

□

The following proposition is a key estimate for the verification of Theorem 1.1.

Proposition 3.2. *Let a, $\bar{a} \in L^2$ be a smooth nonnegative radial functions. Then for some $C > 0$,*

(3.6) $(1 - CT^{1/2}(T^{1/2} + M))\|\Phi_a(f)\|_X^2 \le \|a\|_2^2$,

(3.7) $(1 - CT^{1/2}(T^{1/2} + M))\|\Phi_a(f) - \Phi_{\bar{a}}(\bar{f})\|_X^2 \le \|a - \bar{a}\|_2^2 + C\|\Phi_a(f)\|_X^2 T^{1/2}\|f - \bar{f}\|_X^2$

hold for f, $\bar{f} \in X_{T,M}$.

Proof of Proposition 3.2. The existence of a smooth solution for the system (3.1) with smooth initial data follows from the standard theory of evolution equations. Under the assumption of the proposition, we denote solutions $\Phi_a(f(\tau))$ and $\Phi_{\bar{a}}(\bar{f}(\tau))$ of (3.1) by $u(\tau)$ and $\bar{u}(\tau)$ (or simply u and $\bar{u}$), respectively. We also denote $h(f(\tau))$ and $h(\bar{f}(\tau))$ by $v(\tau)$ and $\bar{v}(\tau)$ (or simply v and $\bar{v}$), respectively. Now by multiplying the first equation of (3.1) by $u = u(\tau)$ and integrating it by parts in x, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\|u(\tau)\|_2^2 + \|\nabla u(\tau)\|_2^2 = \int_{\mathbb{R}^2} u(\tau)\nabla v(\tau)\cdot\nabla u(\tau)dx \\ &\leq \left|\int_{\mathbb{R}^2} u\nabla v\cdot\nabla u dx\right| \leq \|u\|_2\|\nabla v\|_\infty\|\nabla u\|_2 \\ &\leq \frac{1}{2}\|u\|_2^2\|\nabla v\|_\infty^2 + \frac{1}{2}\|\nabla u\|_2^2. \end{aligned} \tag{3.8}$$

Then, the integration of (3.8) from 0 to t in τ leads

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|a\|_2^2 + \int_0^t \|u(\tau)\|_2^2\|\nabla v(\tau)\|_\infty^2 d\tau. \tag{3.9}$$

As for the right hand side of (3.9), by Lemma 3.1, we have

$$\begin{aligned} \int_0^t \|u(\tau)\|_2^2\|\nabla v(\tau)\|_\infty^2 d\tau &\leq \sup_{\tau\in[0,T)} \|u(\tau)\|_2^2 C\int_0^t (1+\|\nabla f(\tau)\|_2)d\tau \\ &\leq \sup_{\tau\in[0,T)} \|u(\tau)\|_2^2 CT^{1/2}(T^{1/2}+M) \end{aligned}$$

for $t \in [0,T)$, here we use that $\sqrt{\int_0^t \|\nabla f\|_2^2 d\tau} \leq M$. Then this relation together with (3.9) yields

$$(1 - CT^{1/2}(T^{1/2}+M)) \sup_{\tau\in[0,T)} \|u(\tau)\|_2^2 + \int_0^T \|\nabla u(\tau)\|_2^2 d\tau \leq \|a\|_2^2,$$

hence we have (3.6).

Next we show (3.7). For f and $\bar{f} \in X_{T,M}$, set $v = v(\tau) = h(f(\tau))$ and $\bar{v} = \bar{v}(\tau) = h(\bar{f}(\tau))$. We consider linear equations

$$\begin{aligned} \partial_t u - \Delta u + \nabla(u\nabla v) = 0, &\qquad t>0, \quad x\in\mathbb{R}^2, \\ \partial_t \bar{u} - \Delta \bar{u} + \nabla(\bar{u}\nabla \bar{v}) = 0, &\qquad t>0, \quad x\in\mathbb{R}^2 \end{aligned}$$

with $u(0) = a$ and $\bar{u}(0) = \bar{a}$. We denote $u = u(\tau) = \Phi_a(f(\tau))$ and $\bar{u} = \bar{u}(\tau) = \Phi_{\bar{a}}(\bar{f}(\tau))$. Multiplying $u - \bar{u}$ to the difference of these equations and integrating it by parts in $\mathbb{R}^2$, we see

(3.10)

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\|u(\tau)-\bar{u}(\tau)\|_2^2 + \|\nabla(u(\tau)-\bar{u}(\tau))\|_2^2 = \int_{\mathbb{R}^2}(u(\tau)\nabla v(\tau) - \bar{u}(\tau)\nabla\bar{v}(\tau))\cdot\nabla(u(\tau)-\bar{u}(\tau))dx \\ &= \int_{\mathbb{R}^2} u\nabla(v-\bar{v})\cdot\nabla(u-\bar{u})dx + \int_{\mathbb{R}^2}(u-\bar{u})\nabla\bar{v}\cdot\nabla(u-\bar{u})dx. \end{aligned}$$

Then the similar argument as for the verification of Lemma 3.1 gives

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2} u\nabla(v-\bar{v})\nabla(u-\bar{u})\right| \\
(3.11)\qquad &\le \|u\|_4\|\nabla(v-\bar{v})\|_4\|\nabla(u-\bar{u})\|_2 \le \|u\|_4^2\|\nabla(v-\bar{v})\|_4^2 + \frac{1}{4}\|\nabla(u-\bar{u})\|_2^2 \\
&\le \|u\|_4^2 C_1 \sup_{\tau\in[0,T)} \|f(\tau)-\bar{f}(\tau)\|_2^2 + \frac{1}{4}\|\nabla(u-\bar{u})\|_2^2.
\end{aligned}
$$

Moreover, as for the second term in the right hand side of (3.10), we have

(3.12)

$$
\left|\int_{\mathbb{R}^2}(u-\bar{u})\nabla\bar{v}\cdot\nabla(u-\bar{u})dx\right| \le \|u-\bar{u}\|_2\|\nabla\bar{v}\|_\infty\|\nabla(u-\bar{u})\|_2 \le \|u-\bar{u}\|_2^2\|\nabla\bar{v}\|_\infty^2 + \frac{1}{4}\|\nabla(v-\bar{v})\|_2^2
$$

$$
\le C_2(1+\|\nabla f\|_2) \sup_{\tau\in[0,T)} \|u(\tau)-\bar{u}(\tau)\|_2^2 + \frac{1}{4}\|\nabla(v-\bar{v})\|_2^2.
$$

Then plugging (3.11) and (3.12) into (3.10) and integrating it from 0 to t in τ, we have

$$
\begin{aligned}
&\|u(t)-\bar{u}(t)\|_2^2 + \int_0^t \|\nabla(u(\tau)-\bar{u}(\tau))\|_2^2 d\tau \le \\
(3.13)\qquad &\|a-\bar{a}\|_2^2 + 2C_1\int_0^T \|u(\tau)\|_4^2 d\tau \sup_{\tau\in[0,T)} \|f(\tau)-\bar{f}(\tau)\|_2^2 \\
&+2C_2 \sup_{\tau\in[0,T)} \|u(\tau)-\bar{u}(\tau)\|_2^2 \int_0^T (1+\|\nabla f(\tau)\|_2)d\tau.
\end{aligned}
$$

We here recall the Ladyzhenskaya inequality (see e.g., [6]):

$$
\left(\int_0^T \|\varphi(\tau)\|_4^4 d\tau\right)^{1/2} \le \sup_{\tau\in[0,T)} \|\varphi(\tau)\|_2^2 + \int_0^T \|\nabla\varphi(\tau)\|_2^2 d\tau
$$

for $\varphi\in C([0,T);L^2)\cap L^2(0,T;\dot{H}^1)$. Then we obtain the following for the second term in the right hand side of (3.13):

$$
(3.14)\qquad \int_0^T \|u(\tau)\|_4^2 d\tau \le \left(\int_0^T \|u(\tau)\|_4^4 d\tau\right)^{1/2} T^{1/2} \le \|u\|_X^2 T^{1/2}.
$$

Moreover, since $\sqrt{\int_0^T \|\nabla f(\tau)\|_2^2 d\tau} \le M$,

$$
(3.15)\qquad \int_0^T (1+\|\nabla f(\tau)\|_2)d\tau \le T^{1/2}(T^{1/2}+M).
$$

Hence by (3.13)-(3.15),

$$
\begin{aligned}
&\left(1-C_3T^{1/2}(T^{1/2}+M)\right) \sup_{\tau\in[0,T)} \|u(\tau)-\bar{u}(\tau)\|_2^2 + \int_0^t \|\nabla(u(\tau)-\bar{u}(\tau))\|_2^2 d\tau \\
&\le \|a-\bar{a}\|_2^2 + C_3\|u\|_X^2 T^{1/2} \sup_{\tau\in[0,T)} \|f(\tau)-\bar{f}(\tau)\|_2^2,
\end{aligned}
$$

whence (3.7) holds.

□

Now we are in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Take any $M < \rho$ where ρ is the number which appears in Corollary 2.3. Then choose $T > 0$ so small that

$$\tfrac{1}{2} \leq 1 - CT^{1/2}(T^{1/2} + M), \tag{3.16}$$

$$CM^2T^{1/2} \leq \tfrac{1}{4} \tag{3.17}$$

where C is the constant in Proposition 3.2. Let $u_0 \in L^2_r(\mathbb{R}^2)$ be a nonnegative initial data with $\|u_0\|_2^2 < M^2/2 =: C_{**}^2$ and let $(a_n) \subset C_0^\infty(\mathbb{R}^2)$ be a sequence such that

$$a_n \to u_0 \quad \text{in} \quad L^2. \tag{3.18}$$

Take any $f \in X_{T,M}$ and let $u_n := \Phi_{a_n}(f)$. Then by (3.6), (3.16) and (3.18),

$$\frac{1}{2}\|u_n\|_X^2 \leq \|a_n\|_2^2 \leq \frac{M^2}{2}, \tag{3.19}$$

hence $u_n \in X_{T,M}$. Moreover, by (3.7) and (3.17),

$$\frac{1}{2}\|u_n - u_m\|_X^2 \leq \|a_n - a_m\|_2^2 \to 0,$$

so (u_n) is a Cauchy sequence in $X_{T,M}$. Hence there exists a limit of (u_n) in $X_{T,M}$. Now we define $\Phi_{u_0} : X_{T,M} \to X_{T,M}$ by $\Phi_{u_0}(f) := \lim_{n\to\infty} \Phi_{a_n}(f)$. Note that by (3.19) and (3.17),

$$C\|\Phi_{a_n}\|_X^2 T^{1/2} \leq CM^2T^{1/2} \leq \frac{1}{4}.$$

Hence this relation together with (3.7), (3.16) and (3.17) gives

$$\frac{1}{2}\|\Phi_{a_n}(f) - \Phi_{a_n}(\bar{f})\|_X^2 \leq \frac{1}{4}\|f - \bar{f}\|_X^2$$

for $f, \bar{f} \in X_{T,M}$. Therefore

$$\begin{aligned}\|\Phi_{u_0}(f) - \Phi_{u_0}(\bar{f})\|_X &\leq \|\Phi_{u_0}(f) - \Phi_{a_n}(f)\|_X + \|\Phi_{a_n}(f) - \Phi_{a_n}(\bar{f})\|_X + \|\Phi_{a_n}(\bar{f}) - \Phi_{u_0}(\bar{f})\|_X \\ &\leq o(1) + \frac{1}{\sqrt{2}}\|f - \bar{f}\|_X,\end{aligned}$$

which says that Φ_{u_0} is a contraction mapping from $X_{T,M}$ to $X_{T,M}$. Therefore, the Banach fixed point theorem yields that there exists a unique solution of $u = \Phi_{u_0}(u)$. It is obvious that $(u, v) = (u, h(u))$ gives a solution of (1.3). The standard parabolic regularity argument gives that the solution becomes regular immediately after $t > 0$. □

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