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The uniqueness and existence of level sets for motion of spirals

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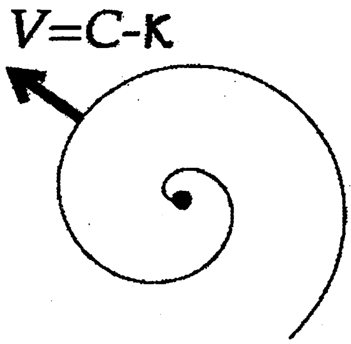
1. Introduction

The aim of this paper is to give a brief summary of results in [GNO]. We here discuss about the existence and uniqueness of generalized solutions of spiral curves which move by

$$V = C - \kappa, \tag{1.1}$$

with the Neumann boundary condition, where V and κ is the normal velocity and the curvature of the curve, and C is a constant.



In 1949, F. C. Frank pointed out important roles of screw dislocations to growth of crystals in [F]. The theory of a crystal growth by screw dislocations is proposed by [BCF]. This is what is called 'spiral crystal growth'. On a surface of crystals which grow by spiral crystal growth, various spiral curves are observed on its surface. According to the theory in [BCF], the spiral curves are described by 'steps' on a surface of a crystal, and they move by (1.1). This is an interface model of spiral crystal growth.

Two types of mathematical models for spiral crystal growth are proposed. A. Karma and M. Plapp ([KP]), and R. Kobayashi ([K]) proposed phase-field models for spiral crystal growth. P. Smereka ([S]), and the author ([O]) proposed level set formulations of the interface model.

In this paper we consider a model by [O] with the case that there is only one spiral on a surface of a crystal. We define generalized solution of (1.1) with the Neumann boundary condition by the level set formulation. The aim of this paper is to discuss

(i) the uniqueness of level sets with respect to the initial spiral,

(ii) how to construct an initial datum of the level set equation for a given initial spiral.

We shall recall a level set formulation by [O] in section 2. We note that a spiral is not described by the usual level set formulation since a spiral does not divide a domain into two domains. To overcome this difficulty, the level set formulation in [O] describes a spiral by a level set on a helical covering space of a surface, whose idea is based on an idea of a 'sheet structure function' by [K] and a similar idea appears in [KP]. However, we need to show the uniqueness of level sets and existence of initial data for spirals because of this revised level set method.

We shall discuss about the uniqueness in section 3. We use the strategy of [CGG, §5] or [ES, §5]. As in them, we prove the comparison of 'super-' or 'sub-level' sets so that we obtain the uniqueness. However, there is no classification of 'super-' or 'sub-level' sets on the domain for a spiral because of its shape. The crucial idea to overcome this difficulty is to introduce a helical covering space as in [O]. This idea is natural and useful to consider a motion of spirals. In fact, a level set method in [O] looks like a cross section of an auxiliary surface by a covering space. Therefore we can make sense of 'super-' or 'sub-level' sets if we see spirals on the covering space.

We shall discuss about the construction of an initial datum in section 4. The crucial difficulty for the construction lies in how to determine an initial datum on a domain except a spiral curve, because there is only one constraint for an initial datum; it is continuous. To overcome this difficulty we revise the definition of a sheet structure function for the construction of an initial datum. According to the idea of [K], he illustrates a structure of the lattice of a crystal by using an argument of a vector whose origin is a screw dislocation. The similar idea appears in [KP], who says that the 'sheet structure function' denotes a shape of an initial surface. Hence we here combine their ideas to construct an initial datum; define a sheet structure function as a single-valued function with a discontinuity on a given spiral curve. Consequently we obtain the desired function by adding some small constant and mollifying the function.

There are interesting results which mention a motion of spirals. N. Ishimulra ([I]) considered a motion of spirals by the curve shortening flow, and discussed about the existence and nonexistence of self-similar spiral-like solution. T. Imai, N. Ishimura and T. Ushijima ([IIU]) showed the existence of solutions for an ordinary differential equation which describes the motion of spirals by crystalline curvature, and showed numerical examples. T. Ogiwara and K.-I. Nakamura ([ON1, ON2]) showed the existence and the stability in the sense of Liapunov of 'spiral traveling wave solutions' for a phase-field model by [K]. Y. Giga, N. Ishimura and Y. Kohsaka ([GIK]) showed the existence and the stability in the sense of Liapunov of 'spiral solutions' for the interface model of spirals.

2. Main result

2.1. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^2$. Assume that $0 \in \Omega$. Take $\rho > 0$ small enough so that $B_\rho = \{x \in \mathbb{R}^2;\ |x| < \rho\} \subset \Omega$. Set $W = \Omega \setminus \overline{B_\rho}$. We first define a 'spiral curve' on $\overline{W}$.

Definition 2.1. *We say Γ is a principal spiral on $\overline{W}$ if $\Gamma = \{P(s) \in \overline{W};\ s \in [0, l]\}$ satisfies the following properties;*

(A1) Γ *is a* C^1 *curve,*

(A2) $P(0) \in \partial B_\rho$, $P(l) \in \partial\Omega$, *and* $P(s) \notin \partial W$ *for* $s \in (0, l)$,

(A3) $P(s_1) \neq P(s_2)$ *if* $s_1 \neq s_2$,

(A4) there exists $\delta > 0$ *such that, for any* $x \in \{y \in \overline{W};\ \mathrm{dist}(x, \Gamma) < \delta\} =: \Gamma^\delta$, *there exists* $z \in \Gamma$ *uniquely such that* $\mathrm{dist}(x, \Gamma) = |x - z|$.

We here consider a principal spiral for an initial curve. The aim of this paper is to show the existence and uniqueness of 'generalized' solution Γ_t moving by

$$V = C - \kappa \quad \text{on } \Gamma_t, \tag{2.1}$$

$$\Gamma_t \perp \partial W, \tag{2.2}$$

where V and κ is the normal velocity and the curvature of Γ_t, respectively, and C is a constant.

For this purpose, we recall a level set formulation by [O]. Let us introduce a formulation of Γ_t of the form

$$\Gamma_t := \{x \in \overline{W}; u(t, x) - \theta(x) \equiv 0 \bmod 2\pi\mathbb{Z}\}, \tag{2.3}$$

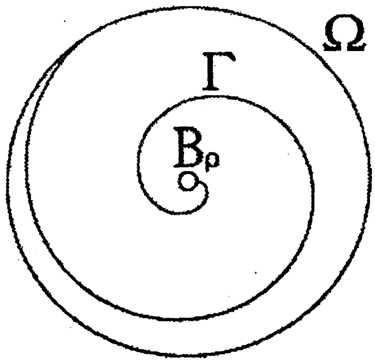


Figure 1: A example of a principal spiral.

where u is an auxiliary function, and $\theta(x) = \arg x$. We here treat $\arg x$ as a multi-valued function. This formulation means that we consider a spiral curve to be a cross section of an auxiliary surface by the Riemannian surface, and see its projection on $\mathbb{R}^2$. Indeed, we observe that Γ_t is a rotating Archmedean spiral if $u(t,x) = t + |x|$. By this formulation we derive a level set equation of the form

$$u_t - |\nabla(u-\theta)|\left\{\operatorname{div}\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C\right\} = 0 \quad \text{in} \quad (0,T)\times W, \qquad (2.4)$$

$$\langle \vec{\nu}, \nabla(u-\theta)\rangle = 0 \quad \text{on} \quad (0,T)\times \partial W, \qquad (2.5)$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^2$, and $\vec{\nu}$ is the outer unit normal vector field of ∂W. We note that (2.4) and (2.5) are well-defined since $\nabla\theta$ is determined as a single-valued function. The mathematical results, especially the solvability for continuous initial data and the comparison principle of (2.4)–(2.5) in the viscosity solution sense, have been established by [O]. We thereby define a generalized solution of the initial and boundary value problem of (2.1)–(2.2) with an initial principal spiral.

Definition 2.2. *Let Γ_0 be a principal spiral on $\overline{W}$. We say a family of $\{\Gamma_t\}_{t\geq 0}$ is a generalized solution of a spiral of (2.1)–(2.2) if Γ_t is given by (2.3) with a viscosity solution u of (2.4)–(2.5) such that $u(0,x) = u_0(x) \in C(\overline{W})$, where u_0 satisfies*

$$\Gamma_0 = \{x \in \overline{W};\ u_0(x) - \theta(x) \equiv 0 \ mod\ 2\pi\mathbb{Z}\}. \qquad (2.6)$$

2.2. Main results

We here mention about the main results of this paper. The aim is to show the uniqueness and existence of a generalized solution of a spiral for (2.1)–(2.2). The uniqueness of generalized solutions is established by showing the uniqueness of level sets as follows;

Theorem 2.3. *Let u_0 and v_0 be continuous functions satisfying*

$$\{x \in \overline{W};\ u_0(x) - \theta(x) \equiv 0 \ mod\, 2\pi\mathbb{Z}\} = \{x \in \overline{W};\ v_0(x) - \theta(x) \equiv 0 \ mod\, 2\pi\mathbb{Z}\}.$$

Let u and v be a viscosity solution of (2.4)–(2.5) *satisfying $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$, respectively. For $T > 0$, the property $\Gamma_t^u = \Gamma_t^v$ holds for $t \in (0,T)$, where*

$$\Gamma_t^u := \{x \in \overline{W};\ u(t,x) - \theta(x) \equiv 0 \ mod\, 2\pi\mathbb{Z}\}.$$

We remark that the usual method as in [CGG, §5] or [ES, §5] does not work well since there is no classification of 'sub-' or 'super-level' sets for the level set formulation of spirals. To overcome this difficulty, we introduce a helical covering space as in [O].

The existence of a generalized solution is established by the construction of an initial datum for a given initial principal spiral, since the existence and uniqueness of (2.4)–(2.5) are shown by [O].

Theorem 2.4. *Let Γ_0 be a principal spiral. Then there exists $u_0 \in C(\overline{W})$ such that u_0 satisfies* (2.6).

We remark that Theorem 2.4 is not obvious. In the case of the usual level set method, we observe that the signed distance functions of the initial closed curves give initial data of a level set equation. However, this way does not work well, since a signed distance function of a principal spiral is nonnegative or nonpositive generally. The basic strategy of the construction is to make a step-like function along to a initial curve. For this purpose we introduce the other branch of $\arg x$ whose discontinuity lies only on Γ_0. Once we have such a function, we then obtain a desired initial datum by adding some small constant and mollifying it.

3. Uniqueness

In this section we shall prove Theorem 2.3. We use a strategy as in [ES] or [CGG], which is for the usual level set method. The simple interpretation of them to our problem is organized as follows.

Step1. We make a rescaling function G satisfying '$u_0 - \theta \le G(v_0 - \theta)$', and '$G(v - \theta) \le 0$' if '$v - \theta \le 0$', where v is a viscosity supersolution of the mean curvature flow with a driving force,

Step2. (Rescaling invariance.) Show that '$G(v - \theta)$' is still a viscosity supersolution,

Step3. Apply the comparison principle and obtain '$u - \theta \le G(v - \theta)$' for a viscosity sub- and super-solution u and v, respectively. This yields that a set where '$u - \theta$ is positive' is contained by a set where '$v - \theta$ is positive'.

We now clarify the sense of '$u - \theta$ is positive', which is actually not clear because θ is multi-valued. We recall a covering space as in [O] of the form

$$\mathfrak{X} := \{(x, \xi) \in \overline{W} \times \mathbb{R};\ (\cos\xi, \sin\xi) = x/|x|\}. \tag{3.1}$$

By using this set we make sense of sets where '$u - \theta$ is positive' or '$u - \theta$ is negative' by

$$\begin{aligned} O_u(t;k) &= \{(x,\xi) \in \mathfrak{X};\ u(t,x) - \xi > 2\pi k\}, &(3.2)\\ D_u(t;k) &= \{(x,\xi) \in \mathfrak{X};\ u(t,x) - \xi < 2\pi k\}. &(3.3)\end{aligned}$$

The conclusion in Step 3 says that $O_u(t;0) \subset O_v(t;0)$. By similar arguments we also obtain $O_u(t;k) \subset O_v(t;k)$, or $D_u(t;k) \supset D_v(t,k)$ for $k \in \mathbb{Z}$.

3.1. Rescaling invariance

We now recall a rescaling invariance of dependent variables. See [CGG, §5], [ES, §5], or [GS, Lemma 4.1] for a rescaling invariance. We also find the rescaling invariance for spirals in [O, Lemma 4.8]. However, it is not clear since the statement includes the notation of '$G(v-\theta)$' without its explanation. Therefore we here clarify the sense of it. We now consider a line

$$\mathcal{L} = \{x \in \overline{W};\ x/|x| = (-1, 0)\},$$

and let $\Theta\colon \overline{W} \setminus \mathcal{L} \to \mathbb{R}$ be a function defined by

$$\Theta(x) = \Xi\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\right) \quad \text{for } x = (x_1, x_2) \in \overline{W} \setminus \mathcal{L},$$

where $\Xi\colon (-1,1] \to (-\pi,\pi)$ be a function satisfying $\cos \Xi(r) = r$.

Definition 3.1. *For a lower semicontinuous function $f\colon \overline{W} \to \mathbb{R}$ and a continuous, nondecreasing function $G\colon \mathbb{R} \to \mathbb{R}$ satisfying*

$$G(s+2\pi) = G(s) + 2\pi \quad \text{for } s \geq s_0 \tag{3.4}$$

for some $s_0 \in \mathbb{R}$, we define a function $g\colon \overline{W} \to \mathbb{R}$ defined by

$$g(x) := \begin{cases} G(f(x) - \bar{\Theta}(x)) + \bar{\Theta}(x) & \text{if } x \in \overline{W} \setminus \mathcal{L}, \\ \lim_{y\to x}[G(f(x) - \bar{\Theta}_*(y)) + \bar{\Theta}_*(y)] & \text{if } x \in \mathcal{L}, \end{cases}$$

where $\bar{\Theta}(x) = \Theta(x) + 2\pi\lambda_0$, $\lambda_0 \in \mathbb{Z}$ such that $\inf_{\overline{W}}(f - \bar{\Theta}_*) \geq s_0$, and $\bar{\Theta}_*\colon \overline{W} \to \mathbb{R}$ be a lower semicontinuous envelope of $\bar{\Theta}$. We denote g by $G(f-\theta)+\theta$.

We remark that $G(f-\theta)+\theta$ is well-defined. Indeed, it suffices to see the existence of the limit $\lim_{y\to x}[G(f(x) - \bar{\Theta}(y)) + \bar{\Theta}(y)]$ for $x \in \mathcal{L}$. Let $\{y_n\}$ and $\{z_n\}$ be sequences which converge to $x \in \mathcal{L}$. We may assume that $\Theta(y_n) < 0$ and $\Theta(z_n) > 0$. We observe that

$$\lim_{n\to\infty}(\Theta_*(y_n)) + 2\pi = \lim_{n\to\infty}(\Theta_*(z_n)).$$

Therefore we obtain

$$\begin{aligned} \lim_{n\to\infty}[G(f(x) - \bar{\Theta}_*(z_n)) + \bar{\Theta}_*(z_n)] &= \lim_{n\to\infty}[G(f(x) - \bar{\Theta}_*(z_n) + 2\pi) + \bar{\Theta}_*(z_n) - 2\pi] \\ &= \lim_{n\to\infty}[G(f(x) - \bar{\Theta}_*(y_n)) + \bar{\Theta}(y_n)]. \end{aligned}$$

Therefore $G(f-\theta)+\theta$ is well-defined.

We verify the regularity of $G(f-\theta)+\theta$ for smooth f.

Lemma 3.2. *Let $G\colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying* (3.4). *Let $f \in C^{1,2}((0,T)\times\overline{W})$. We have that $g = G(f-\theta)+\theta \in C^{1,2}((0,T)\times\overline{W})$. Moreover we obtain*

$$\begin{aligned} g_t(t,x) &= G'(f(t,x) - \bar{\Theta}_*(x))f_t(t,x), \\ \nabla g(t,x) &= G'(f(t,x) - \bar{\Theta}_*(x))\nabla(f-\theta) + \nabla\theta, \\ \nabla^2 g(t,x) &= G''(f(t,x) - \bar{\Theta}_*(x))\nabla(f-\theta)\otimes\nabla(f-\theta) \\ &\quad + G'(f(t,x) - \bar{\Theta}_*(x))\nabla^2(f-\theta) + \nabla^2\theta. \end{aligned}$$

It suffices to see the regularity of $G(f-\theta)+\theta$ on $\mathcal{L}$, and this is obtained by considering the another branch of the argument which is smooth on $\mathbb{R}^2 \setminus \{(r,0);\ r>0\}$.

We are now in position to state an adapted rescaling invariance for spiral crystal growth.

Lemma 3.3. *Let G be a uniform continuous, nondecreasing function satisfying* (3.4). *Let v be a viscosity supersolution of* (2.4)–(2.5). *For $T>0$ let $w\colon [0,T)\times\overline{W}\to\mathbb{R}$ be a function defined by $w(t,x)=G(v(t,x)-\theta(x))+\theta(x)$ in the sense of Definition 3.1. Then w is a viscosity supersolution of* (2.4)–(2.5) *with a initial datum $w(0,\cdot)=G(v_0-\theta)+\theta$.*

Sketch of the proof. Since we consider the approximation of a uniform continuous G by smooth G_k such that $G_k'>0$ and (3.4) holds, it suffices to consider the case that G is smooth and satisfies $G'>0$. For the simplicity we only demonstrate that Lemma 3.3 is true in $(0,T)\times W$ if v is smooth.

Let $(\hat{t},\hat{x})\in(0,T)\times W$, and $\varphi\in C^{1,2}((0,T)\times W)$ satisfy

$$w(t,x)-\varphi(t,x)\geq w(\hat{t},\hat{x})-\varphi(\hat{t},\hat{x})=0 \quad \text{for } (t,x)\in[0,T)\times\overline{W}.$$

We may assume that $\varphi(t,x)-\bar{\Theta}_*(x)\geq s_0$ without loss of generality. Since $G'>0$, there exists $H=G^{-1}$ and satisfies

$$H'>0,\quad H(s+2\pi)=H(s)+2\pi \quad \text{for} s\ \geq G(s_0).$$

Set

$$\psi(t,x)=H(\varphi(t,x)-\theta(x))+\theta(x),$$

then we have $\psi\in C^{1,2}((0,T)\times W)$ and

$$v(t,x)-\psi(t,x)\geq v(\hat{t},\hat{x})-\psi(\hat{t},\hat{x}) \quad \text{for } [0,T)\times\overline{W}.$$

Since v is a viscosity supersolution of (2.4)–(2.5) we have

$$\psi_t+F^*(\nabla(\psi-\theta),\nabla^2(\psi-\theta))\geq 0 \quad \text{at } (\hat{t},\hat{x}), \tag{3.5}$$

where $F\colon\mathbb{R}^2\setminus\{0\}\times\mathbb{S}^2\to\mathbb{R}$ is a function satisfying

$$F(\nabla(\varphi-\theta),\nabla^2(\varphi-\theta))=-|\nabla(\varphi-\theta)|\left\{\operatorname{div}\frac{\nabla(\varphi-\theta)}{|\nabla(\varphi-\theta)|}+C\right\}.$$

By Lemma 3.2 we have

$$\begin{aligned}
\psi_t &= G'(\varphi-\theta)\varphi_t,\\
\nabla\psi &= G'(\varphi-\theta)\nabla(\varphi-\theta)+\nabla\theta,\\
\nabla^2\psi &= G''(\varphi-\theta)\nabla(\varphi-\theta)\otimes\nabla(\varphi-\theta)+G'(\varphi-\theta)\nabla^2(\varphi-\theta)+\nabla^2\theta.
\end{aligned}$$

Here we have denoted $G'(\varphi - \bar{\Theta}_*)$ by $G'(\varphi - \theta)$ for the simplicity. Since we have

$$F^*(\lambda p, \lambda X + \mu p \otimes p) = \lambda F^*(p, X) \quad \text{for } (p, X) \in \mathbb{R}^2 \times \mathbb{S}^2,\ \lambda > 0, \text{ and } \mu \in \mathbb{R}$$

and (3.5), we thereby obtain

$$\varphi_t + F^*(\nabla(\varphi - \theta), \nabla^2(\varphi - \theta)) \geq 0 \quad \text{at } (\hat{t}, \hat{x}).$$

For an approximation of a uniform continuous G, we consider

$$G_k(s) = (1 - k^{-1})(G * \mu_k)(s) + k^{-1}s,$$

where $\mu_k \in C_0^\infty(\mathbb{R})$ is a mollifier. We observe that $\lim_{k\to\infty} G_k = G$, this convergence is locally uniform, $G_k' > 0$, and G_k satisfies (3.4). □

3.2. Rescaling function

In this section we construct a rescaling function for spirals. As in Lemma 3.3 we need to construct a uniformly continuous function. However, we here mention only about a construction of upper semicontinuous rescaling function, since the approximation is given by a standard way as in [G]. For classification of a branch of θ we use an idea of a covering space in (3.1).

Lemma 3.4. *Let u_0 and v_0 be continuous functions on $\overline{W}$, and assume that*

$$\{(x, \xi) \in \mathfrak{X};\ u_0(x) - \xi > 0\} \subset \{(x, \xi) \in \mathfrak{X};\ v_0(x) - \xi > 0\}. \tag{3.6}$$

There exists $G_1 \colon \mathbb{R} \to \mathbb{R}$ satisfying

(i) G_1 is nondecreasing,

(ii) $G_1(s) = 0$ if $s \leq 0$,

(iii) $u_0(x) - \xi \leq G(v_0(x) - \xi)$ for $(x, \xi) \in \mathfrak{X}$,

(iv) $G_1(s + 2\pi) = G_1(s) + 2\pi$ for $s \in G_1^{-1}((0, +\infty))$,

(v) G_1 is upper semicontinuous and continuous on $G^{-1}(0)$.

We propose the candidate of G_1 as in Lemma 3.4. We define

$$G_1(s) := \sup\{(u_0(x) - \xi)_+;\ (x, \xi) \in \mathfrak{X},\ v_0(x) - \xi \leq s\},$$

where $(r)_+ = \max(r, 0)$. We here mention only the proof of (iv), since the other properties are obtained by standard arguments. Let $s \in G_1^{-1}(\{s;\ s > 0\})$. For $k \in \mathbb{N}$ we have $(x_k, \xi_k) \in \mathfrak{X}$ satisfying

$$G_1(s) - k^{-1} \leq u_0(x_k) - \xi_k, \text{ and } v_0(x_k) - \xi_k \leq s.$$

We now consider $(x, \xi_k - 2\pi)$. By the definition of $\mathfrak{X}$ we have $(x, \xi_k - 2\pi) \in \mathfrak{X}$. Moreover we obtain $v_0(x_k) - (\xi_k - 2\pi) \leq s + 2\pi$. Therefore we obtain

$$u_0(x_k) - (\xi_k - 2\pi) \leq G_1(s + 2\pi).$$

By letting $k \longrightarrow \infty$ we obtain

$$G_1(s) + 2\pi \leq G_1(s + 2\pi).$$

We next consider $(x, \xi) \in \mathfrak{X}$ satisfying $v_0(x) - \xi \leq s + 2\pi$. We now have $(x, \xi + 2\pi) \in \mathfrak{X}$, $v_0(x) - (\xi + 2\pi) \leq s$, and

$$(u_0(x) - \xi)_+ \leq (u_0(x) - (\xi + 2\pi))_+ + 2\pi \leq G_1(s) + 2\pi.$$

By taking supremum on above with respect to $(x, \xi) \in \mathfrak{X}$ such that $v_0(x) - \xi \leq s + 2\pi$ we obtain

$$G_1(s + 2\pi) \leq G_1(s) + 2\pi. \ \Box$$

4. Existence

In this section we shall prove Theorem 2.4. For this purpose we construct a step-like function along to an initial curve.

Lemma 4.1. *Let Ω be a bounded domain with C^2 boundary. Assume that $0 \in W$. Take $\rho > 0$ such that $B_\rho \subset \Omega$. Set $W = \Omega \setminus \overline{B_\rho}$. Let Γ be a principal spiral on $\overline{W}$. There exists $\varphi \in C^\infty(W \setminus \Gamma) \cap C(\overline{W} \setminus \Gamma)$ satisfying*

$$\varphi(x) - \arg x \equiv 0 \ \mathit{mod}\ 2\pi\mathbb{Z}$$

for $x \in \overline{W} \setminus \Gamma$.

The function φ in Lemma 4.1 gives other branch of $\arg x$ on $\overline{W} \setminus \Gamma$. Usually, we consider the domain $\mathbb{R}^2 \setminus (\{0\} \times \bar{\mathbb{R}}_-)$ to define $\arg x$, where $\bar{\mathbb{R}}_- = \{x \in \mathbb{R};\ x \leq 0\}$. Lemma 4.1 says that one can define a branch of $\arg x$ if we remove a principal spiral from a domain.

Idea of the proof. We now introduce a polar coordinate. Let $\Psi\colon (0,\infty)\times\mathbb{R}\to\mathbb{R}^2$ be a map defined by $\Psi(r,\tau)=(r\cos\tau, r\sin\tau)$. Set

$$\mathcal{D} := \{(r,\tau)\in(0,\infty)\times\mathbb{R};\ \Psi(r,\tau)\in W\},$$
$$\tilde{\Gamma} := \{(r,\tau)\in\overline{\mathcal{D}};\ \Psi(r,\tau)\in\Gamma\}.$$

We find curves $\mathcal{C}_1$ of $\partial\Omega$, $\mathcal{C}_2$ of ∂B_ρ, and infinite curves $\tilde{\Gamma}_n$ of Γ, i.e.,

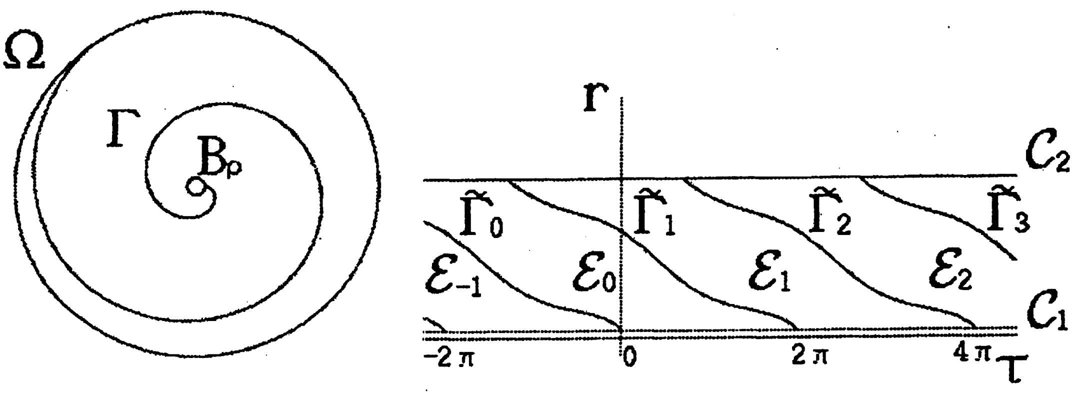


Figure 2: Domain and spiral in polar coordinate.

$$\mathcal{C}_1 := \{(\rho,\tau);\ \tau\in\mathbb{R}\},$$
$$\mathcal{C}_2 := \{(p(s),q(s));\ \Psi(p(s),q(s))\in\partial\Omega\},$$
$$\tilde{\Gamma}_n := \{\tilde{P}_n(s)=(r(s),\tau_n(s));\ s\in[0,l],\ \Psi(\tilde{P}_n(s))=P(s),\ \tau_n(0)=2\pi n\}.$$

We observe that curves $\mathcal{C}_1$, $\tilde{\Gamma}_{n+1}$, $\mathcal{C}_2$ and $\tilde{\Gamma}_n$ yield a closed curve so that there exists a bounded domain $\mathcal{E}_n$ which is enclosed by them. By using these notation we define $\tilde{\varphi}\colon \overline{\mathcal{D}}\setminus\tilde{\Gamma}\to\mathbb{R}$ by

$$\tilde{\varphi}(r,\tau)=\tau-2\pi n \quad \text{if } (r,\tau)\in\mathcal{E}_n,\ n\in\mathbb{Z},$$

and define

$$\varphi(x)=\tilde{\varphi}(|x|,\mathrm{Arg}x),$$

where $\mathrm{Arg}x\in[0,2\pi)$ is the principal value of $\arg x$ for $x\in\mathbb{R}^2\setminus\{0\}$. This φ is a desired function. □

Proof of Theorem 2.4. We may assume that $P(0)=(\rho,0)$ without loss of generality. Let φ be a function obtained by Lemma 4.1. We consider a tubular neighborhood of Γ_0. Set

$$\Gamma_0^\delta=\{x\in\overline{W};\ \mathrm{dist}(x,\Gamma_0)<\delta\}.$$

We here take δ small enough so that $\delta < \rho$ in addition to the condition (A4). Then we observe that Γ_0 divide Γ_0^δ into two domains $\Gamma_{0,\pm}^\delta$, which satisfy

$$\Gamma_{0,-}^\delta = \Gamma_0^\delta \setminus \overline{\Gamma_{0,+}^\delta},$$
$$x + hn(x) \in \Gamma_{0,-}^\delta, \ x - hn(x) \in \Gamma_{0,+}^\delta \text{ for } x \in \Gamma_0,$$

where $n(x)$ is the unit normal vector field of Γ_0 in the direction of the normal velocity. We define $\varphi_\pm \colon \Gamma_{0,\pm}^\delta \to \mathbb{R}$ by the restriction of φ on $\overline{\Gamma_{0,\pm}^\delta} \setminus \Gamma_0$, respectively. We next extend $\varphi_\pm$ onto $\overline{\Gamma_0^\delta}$ satisfying $\varphi_\pm$ is continuous, $\varphi_\pm \equiv \arg x \mod 2\pi\mathbb{Z}$ on $\overline{\Gamma_0^\delta}$. We still the extension of $\varphi_\pm$ by $\varphi_\pm$. We remark that $|\varphi_+(x) - \varphi_-(x)| = 2\pi$ for $x \in \Gamma_0^\delta$.

Let d_0 be a signed distance function of Γ_0 on Γ_0^δ, whose sign is same as the sign of $\Gamma_{0,\pm}^\delta$, in other words, $d_0 > 0$ in $\Gamma_{0,+}^\delta$ and $d_0 < 0$ in $\Gamma_{0,-}^\delta$. By using this function we define

$$\psi(x) := \frac{\delta + d_0(x)}{2\delta}\varphi_+(x) + \frac{\delta - d_0(x)}{2\delta}\varphi_-(x).$$

We observe that $\psi = \varphi$ on $\partial\Gamma_0^\delta \cap W$, $\psi \in C(\overline{\Gamma_0^\delta})$ and $\psi - \arg x \equiv \pi$ only on Γ_0.

By using ψ and φ we define

$$u_0(x) = \begin{cases} \varphi(x) + \pi & \text{if } \ x \in \overline{W} \setminus \overline{\Gamma_0^\delta}, \\ \psi(x) + \pi & \text{if } \ x \in \overline{\Gamma_0^\delta}. \end{cases}$$

This is the desired function. □

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