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Asymptotic solutions of Hamilton-Jacobi equations with state constraints

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This article reviews recent results on the asymptotic behavior of solutions of the Cauchy problem for Hamilton-Jacobi equations and describe briefly some results obtained in [26].

1 Introduction.

We are concerned with the Cauchy problem for Hamilton-Jacobi equations:

$$\begin{cases} u_t + H(x, Du(x,t)) = 0 & \text{in } \Omega\times(0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^n$, $H = H(x,p)$ is a function: $\Omega\times\mathbb{R}^n \to \mathbb{R}$, which is assumed to be coercive and convex in the variable p, $u : \Omega\times[0,\infty) \to \mathbb{R}$ is the unknown function, $u_t = \partial u/\partial t, Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$, and $u_0 : \Omega \to \mathbb{R}$ is a given initial data. The function H will be called the Hamiltonian.

In recent years, many researchers have investigated in the large-time behavior of the solution $u(x,t)$ of (1.1) as $t\to\infty$ and established convergence results which claim under appropriate hypotheses that there exist a constant c and a solution $v\in C(\Omega)$ of $H(x, Du) = c$ in Ω such that

$$u(x,t) + ct - v(x) \to 0 \quad \text{locally uniformly for } x\in\Omega \text{ as } t\to\infty. \tag{1.2}$$

In this paper, we consider (1.1) with state constraints and establish a convergence result.

Associated with the Cauchy problem (C) is the *additive eigenvalue* problem for H:

$$H(x, Du(x)) = a \quad \text{in } \Omega. \tag{1.3}$$

Here one seeks for a pair (u,a) of $u\in C(\overline{\Omega})$ and $a\in\mathbb{R}$ such that u is a solution of (1.3). If (u,a) is such a pair, we call u an *additive eigenfunction* and a an *additive eigenvalue.*

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A simple observation related to this is that, for any $(v,c) \in C(\Omega \times \mathbb{R})$, the function $v(x) - ct$ is a solution of (1.1) if and only if (v,c) is a solution of the additive eigenvalue problem for H. We call such a function $v(x) - ct$ an *asymptotic solution* of (1.1) provided (v,c) is a solution of the additive eigenvalue problem for H.

The following classical theorem solves the additive eigenvalue problem for H which is periodic in x.

Theorem 1.1 (Lions-Papanicolaou-Varadhan [24]). *Let $\Omega = \mathbb{R}^n$ and $H \in C(\Omega \times \mathbb{R}^n)$ be coercive and periodic in x. Then for any $p \in \mathbb{R}^n$,*

$$H(x, p + Du) = a \quad \text{in } \Omega$$

has a solution $(v,c) \in C(\Omega) \times \mathbb{R}$ and the constant c is unique.

Next we look back on a short history of asymptotic problems. This study goes back to the works of Kružkov [22], Lions [23] and Barles [1], who studied the case where $\Omega = \mathbb{R}^n$ and $H = H(p)$ does not depend on x variable.

In the case where $H = H(x,p)$ depends both on x and p, the first general results were obtained by Namah-Roquejoffre [28] and Fathi [11, 12]:

Theorem 1.2. *Let M be a compact manifold without boundary. Let $H : M \times \mathbb{R}^n \to \mathbb{R}$ be smooth, superlinear and strictly convex. Then for any $u_0 \in C(M)$ and a solution u of* (1.1)*, there exists a solution (v,c) of* (1.3) *such that the convergence* (1.2) *holds.*

Afterwards Roquejoffre [30] and Davini-Siconolfi [9] improved the above approach.

By another approach based on the theory of partial differential equations and viscosity solutions, this type of results have been obtained by Namah-Roquejoffre [28] and Barles-Souganidis [4].

More recently the large-time asymptotic problem of the same kind has been studied in the case where $\Omega = \mathbb{R}^n$ by Fujita-Ishii-Loreti [14], Barles-Roquejoffre [3], Ishii [18], and Ichihara-Ishii [16].

On the other hand, there are not many results on the large-time asymptotic problem which treat Hamilton-Jacobi equations with boundary conditions. Hamilton-Jacobi equations on n-dimensional torus can be also considered to be set on $\mathbb{R}^n$ with the periodic boundary. The periodic boundary condition is thus covered by the results quoted above. As far as the author knows, only the periodic boundary condition and the Dirichlet boundary condition are treated for the large-time asymptotic problem.

We here study the asymptotic problem for Hamilton-Jacobi equations with state constraints or, in other words, with the state constraint boundary condition:

$$\text{(C)} \qquad \begin{cases} u_t + H(x, Du(x,t)) \le 0 & \text{in } \Omega \times (0,\infty), & (1.4) \\ u_t + H(x, Du(x,t)) \ge 0 & \text{in } \overline{\Omega} \times (0,\infty), & (1.5) \\ u(\cdot,0) = u_0 & \text{on } \overline{\Omega}. & (1.6) \end{cases}$$

State constraint problems arise naturally in optimal control, and their dynamic programming equations have the form (1.4)–(1.5), where the boundary condition is implicit in the fact that inequality (1.5) is required on the closure $\overline{\Omega}$. This formulation, in terms of PDE, of state constraint problems has been introduced by Soner [31]. Pairs of inequalities such as (1.4)–(1.5) are referred as Hamilton-Jacobi equations with state constraints or the state constraint problem for Hamilton-Jacobi equations.

The additive eigenvalue problem with state constraints is formulated as follows:

$$(\mathrm{E})_a \qquad \begin{cases} H(x, Du(x)) \leq a & \text{in } \Omega, \qquad (1.7) \\ H(x, Du(x)) \geq a & \text{on } \overline{\Omega}. \qquad (1.8) \end{cases}$$

The additive eigenvalue problem for H gives the "stationary states" for solutions of (C) as our main result shows. We call a function $v(x) - ct$ an *asymptotic solution* of (C) provided (v, c) is a solution of (1.7) and (1.8).

Our main purpose of this paper is to show that under appropriate hypotheses on H and Ω any solution $u(x,t)$ of (C) converges to an asymptotic solution $v(x) - ct$ in $C(\overline{\Omega})$ as $t \to \infty$. That is, as $t \to \infty$,

$$u(x,t) + ct - v(x) \to 0 \quad \text{uniformly for } x \in \overline{\Omega}.$$

2 Assumptions.

Let $A \subset \mathbb{R}^k$, $B \subset \mathbb{R}^l$ for some $k, l \in \mathbb{N}$ and $r > 0$. Write $U(x,r) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$. We denote by $C(A,B), C^{0,1}(A,B)$ and $\mathrm{LSC}(A,B)$ the sets of continuous, Lipschitz continuous and lower semicontinuous functions on A with values in B, respectively. We denote by $W^{1,\infty}(A,B)$ the set of functions on A with values in B which is differentiable and the distributional first derivatives are bounded almost everywhere on A. When the set B is clear by the context, we may omit writing B in the above notation: for instance, we may write $C(A)$ for $C(A,B)$. We also use the symbol $\mathrm{AC}([a,b],B)$ to denote the set of absolutely continuous functions on $[a,b]$ with values in B.

We call a function $m : [0,\infty) \to [0,\infty)$ a *modulus* if it is continuous and nondecreasing on $[0,\infty)$ and if $m(0) = 0$.

We make the following assumptions on the Hamiltonian H, the initial data u_0 and the domain Ω:

(H1) $H \in C(\overline{\Omega} \times \mathbb{R}^n)$.

(H2) The function $p \mapsto H(x,p)$ is strict convex for each $x \in \overline{\Omega}$.

(H3) The function H is coercive, i.e.

$$\lim_{r \to \infty} \inf\{H(x,p) \mid x \in \overline{\Omega}, p \in \mathbb{R}^n \setminus U(0,r)\} = \infty.$$

(u1) $u_0 \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.

(B1) The domain Ω is bounded and a Hölder domain with exponent $\alpha > 2/3$.

Remark 2.1. In fact, we can remove the restriction that $u_0 \in W^{1,\infty}(\Omega)$. See Section 7 in [26].

Remark 2.2. An equivalent formulation of (B1) is that there exists a constant $2/3 < \alpha \leq 1$ such that for any $z \in \partial\Omega$ and for some $\eta_z \in \mathbb{R}^n$ and $b_z > 0$,

$$\bigcup_{x \in \overline{\Omega} \cap U(z,b_z)} \bigcup_{0<s<b_z} U(x + s^\alpha \eta_z, sb_z) \subset \Omega.$$

3 Solutions of (C).

Now we give a comparison result for (C).

Theorem 3.1 (Theorem 2.1 in [26]). *Let $T > 0$, and let $u \in C(\overline{\Omega} \times [0,T])$ and $v \in$ LSC$(\overline{\Omega} \times [0,T])$ satisfy $u_t + H(x, Du) \leq 0$ in $\Omega \times (0,T)$ and $v_t + H(x, Dv) \geq 0$ on $\overline{\Omega} \times (0,T)$ in the viscosity sense, respectively. Then, if $u \leq v$ on $\overline{\Omega} \times \{0\}$, $u \leq v$ on $\overline{\Omega} \times [0,T)$.*

For a proof, we refer to the reader [26, Section 4]. Uniqueness of solutions of (C) follows from the above theorem. It is worth pointing out that in the literature on (C) or its stationary version, it is usually assumed for uniqueness of solutions that Ω is a Lipschitz domain. Here we take advantage of assumption (H3) to reduce the standard Lipschitz regularity of Ω to the Hölder regularity (B1), which seems to be a new observation. This new generality of domains Ω is obtained with help of the coercivity assumption (H3) on H.

We consider the function $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ defined by

$$u(x,t) := \inf\{ \int_0^t L(\gamma(s), \dot\gamma(s))\, ds + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x;t)\}, \tag{3.1}$$

where L is the Lagrangian of H, i.e, $L(x,p) := \sup_{\xi \in \mathbb{R}^n}\{p \cdot \xi - H(x,\xi)\}$ and $\mathcal{C}(x;t)$ denotes the set of all trajectories $\gamma \in \mathrm{AC}([0,t], \overline{\Omega})$ such that $\gamma(t) = x$.

The regularity and continuity of u is obtained by our next theorem.

Theorem 3.2. *Let u be the function defined by* (3.1). *Then*

(a) $u \in C^{0,1}(\Omega \times [0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$;

(b) *There is a constant $C > 0$ such that $|Du(x,t)| + |u_t(x,t)| \leq C$ a.e. $(x,t) \in \Omega \times (0,\infty)$;*

(c) *u is a* (unique) *solution of* (C).

4 Additive eigenvalue problems.

We define the constant c_H by

$$c_H := \inf\{a \in \mathbb{R} \mid (1.7) \text{ has a solution}\}, \tag{4.1}$$

and consider the following inequalities:

$$H(x, Du(x)) \leq c_H \quad \text{in } \Omega, \tag{4.2}$$

$$H(x, Du(x)) \geq c_H \quad \text{on } \overline{\Omega}. \tag{4.3}$$

The following theorem ensures the existence of the additive eigenvalue problem and the uniqueness of the constant.

Theorem 4.1. *Problem* $(\mathrm{E})_a$ *has a solution* $v \in C(\overline{\Omega})$ *if and only if* $a = c_H$.

Using Theorem 3.1 and Theorem 4.1, we see that the function $u(x,t)+c_H t$ is bounded on $\overline{\Omega} \times [0, \infty)$, where u is the solution of (C).

Proposition 4.2. *There exists a constant* $C > 0$ *such that*

$$|u(x,t) + c_H t| \leq C \quad \text{on } \overline{\Omega} \times [0, \infty).$$

We assume that $c_H = 0$ by replacing H by $H - c_H$.

The following lemma is important for our proof of Theorem 5.2.

Lemma 4.3 (Theorem 8.1 in [26]). *Let* $x \in \overline{\Omega}$ *and* $\phi \in C(\overline{\Omega})$ *be a viscosity solution of* $(\mathrm{E})_0$. *Then there exists a curve* $\gamma \in C((-\infty, 0], \overline{\Omega})$ *such that* $\gamma(0) = x$ *and for any* $[a, b] \subset (-\infty, 0]$,

$$\gamma \in \mathrm{AC}([a,b], \overline{\Omega}) \quad \text{and} \quad \int_a^b L(\gamma(s), \dot{\gamma}(s)) + c_H \, ds = \phi(\gamma(b)) - \phi(\gamma(a)). \tag{4.4}$$

Following [30, 18], we call curves satisfying (4.4) *extremal curves* for ϕ and hereinafter we write $\mathcal{E}(\phi)$ to denote the set of all extremal curves for ϕ.

5 Convergence.

Let $u(x,t)$ be the unique viscosity solution of (C).

Lemma 5.1 (Proposition 8.2 in [26]). *There exist a constant* $\delta \in (0,1)$ *and a modulus* ω *for which if* $u_0 \in C(\overline{\Omega})$, ϕ *is a solution of* $(\mathrm{E})_0$, $\gamma \in \mathcal{E}(\phi)$, $T > \tau \geq 0$ *and* $\frac{\tau}{T-\tau} \leq \delta$, *then*

$$u(\gamma(0), T) - u(\gamma(-T), \tau) \leq \phi(\gamma(0)) - \phi(\gamma(-T)) + \frac{\tau T}{T-\tau}\omega\left(\frac{\tau}{T-\tau}\right).$$

Lemma 5.1 is a variant of [18, Proposition 7.1]. We remark that the "strict" convexity of H is only needed in Lemma 5.1 in our approach to Theorem 5.2. We here note that it is known that there are some examples of Hamilton-Jacobi equations to show that the Hamiltonian is not strict convex but convex and the convergence (1.2) is not true. We refer the reader to [1, 15, 4, 5, 19].

We state our main theorem:

Theorem 5.2 (Theorem 2.2 in [26]). *For any u_0 there exists a solution $(v, c) \in C(\overline{\Omega}) \times \mathbb{R}$ of the additive eigenvalue problem for H such that if $u \in C(\overline{\Omega} \times [0, \infty))$ is the viscosity solution of* (C)*, then, as $t \to \infty$,*

$$u(x, t) + ct - v(x) \to 0 \quad \textit{uniformly on } \overline{\Omega}.$$

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