

氷の内部融解・凍結による六角板状の真空泡形成のモデリングに向けて
Towards modelling the formation of negative ice crystals or vapor figures
produced by freezing of internal melt figures*¹

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Abstract

When so-called Tyndall figure in a single ice crystal is refrozen, a vapor bubble in the figure remains in ice, and it is transformed to a hexagonal disk which is called negative crystal or vapor figure. Under some assumptions, this transformation process will be simulated by application of an area-preserving crystalline curvature flow, i.e., an area-preserving gradient flow of total interfacial energy.

Key Words: single ice crystal, internal melting, Tyndall figure, negative crystal, vapor figure, interfacial energy, area-preserving crystalline curvature flow

1 Introduction

When a block of ice is exposed to solar beams of other radiation, internal melting of ice occurs. That is, internal melting starts from some interior points of ice without melting the exterior portions, and each water region forms a flower of six petals, which is called “Tyndall figure” (Figure 1 (left)). And Figure 1 (right) indicates that Tyndall figures are almost two dimensional figures.

The figure is filled with water except for a vapor bubble. This phenomenon was first observed by Tyndall (1858). When Tyndall figure is refrozen, the vapor bubble remains in the ice as a hexagonal disk (see Figure 2). This hexagonal disk is filled with water vapor saturated at that temperature and surrounded by ice. McConnell found these disks in the ice of Davos lake [9]. Nakaya called this hexagonal disk “vapor figure (空像)” and investigated its properties precisely [10]. Adams and Lewis (1934) called it “negative

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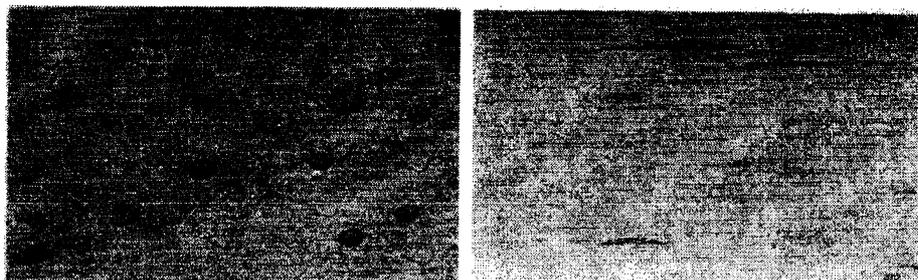


Figure 1: (Left) Tyndall figures seen from the direction of 45° to c-axis [10, No.17]; (right) Tyndall figures seen from the direction of 90° to c-axis [10, No.18].

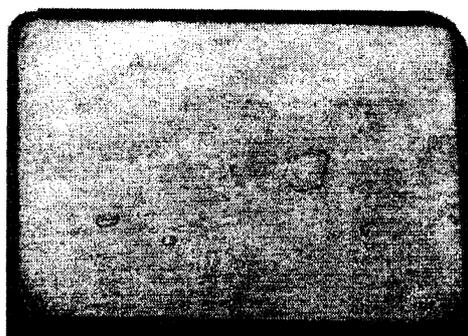


Figure 2: Natural vapor figures in an ice single crystal [10, No.1].

crystal (負結晶).” (Although Nakaya said “this term does not seem adequate” with a certain reason, hereafter we use the term negative crystal for avoiding confusion.)

Negative crystal is useful to determine the structure and orientation of ice or solids. Because, within a single ice crystal, all negative crystals are similarly oriented, that is, corresponding edges of hexagon are parallel each other (see Figure 3). Furukawa and



Figure 3: A cluster of minute vapor figures [10, No.53 (magnified)].

Kohata made hexagonal prisms experimentally in a single ice crystal, and investigated the habit change of negative crystals with respect to the temperatures and the evaporation mechanisms of ice surfaces [3].

To the best of author's knowledge, after the Furukawa and Kohata's experimental research, there have been no published results on negative crystals, and there are no

dynamical model equations describing the process of formation of negative crystals. In this paper, we will focus on the process of formation of negative crystals after Tyndall figures are refrozen, and propose a model equation of interfacial motion which tracks the deformation of negative crystals in time.

2 Formation of negative crystals

Figure 4 indicates aftereffect of freezing of Tyndall figures from the initial stage of refrozen process to the final stage of the formation of negative crystals. The aim of this talk is to

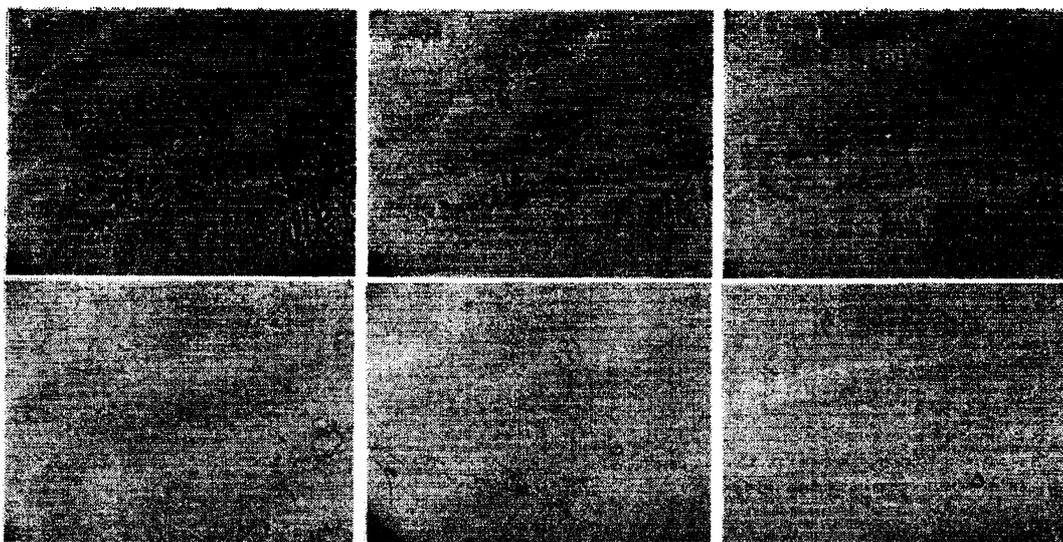


Figure 4: From left to right, upper to lower: (a) Start of freezing, $t = 0\text{min}$. (b) Freezing proceeds, $t = 3\text{min}$. (c) Freezing proceeds further, $t = 11\text{min}$. (d) The bubble is separated, $t = 17\text{min}$. (e) The separated liquid film migrates, $t = 28\text{min}$. (f) After freezing, cloudy layers and a vapor figure are left, $t = 1\text{hr } 21\text{min}$. [10, No. 52a–52f].

propose a model equation, revealed the process in Figure 4 from (d) to (f). This process may be described as the following:

Negative crystal changes the shape from an oval to the hexagon.

Thus, our model will be assumed the followings.

- (A1) water vapor region is simply connected and bounded region in the plane \mathbb{R}^2 (we denote it by Ω);
- (A2) Ω is surrounded by a single ice crystal (i.e., a single ice region is included in $\mathbb{R}^2 \setminus \Omega$);
- (A3) moving boundary $\partial\Omega$ is interface of the water vapor region and the single ice crystal;

- (A4) the crystallographic c-axis (main axis) of the single ice crystal is perpendicular to the plane;
- (A5) Ω is filled with water vapor saturated at that temperature, and the temperature is a constant (i.e., there is no thermal gradient);
- (A6) the initial figure of negative crystal (we denote it by Ω_0) is strictly convex set with a smooth boundary.

Remark: Under (A5), the thickness of negative crystal is regarded as constant in deformation process. In this sense, (A1) is a natural assumption. Meaning of (A2) and (A3) is that after the bubble in Tyndall figure is captured in ice phase, there is no water region between the bubble and the ice, and they are separated each other by a sharp boundary interface curve. (A4) means that we observe figures from the direction of main axis in structure of single ice crystal. To observe effect of interfacial energy, we assume (A5). Shape of the bubble in Tyndall figure is circle by virtue of surface tension. Hence immediately after the bubble is captured in ice phase, the shape can be regarded as a circle or a convex set, and this follows (A6). The primary idea of our assumptions can be found in [8].

The evolution law of moving interface is similar to the growth of snow crystal, since deformation of negative crystal is regarded as crystal growth in the air. As a model of snow crystal, we refer the Yokoyama-Kuroda model [15], which is based on the diffusion process and the surface kinetic process by Burton-Cabrera-Frank (BCF) theory [2]. Meanwhile, we assume the existence of interfacial energy (density) on the boundary $\partial\Omega$. The equilibrium shape of negative crystal is a regular hexagon, and if the region Ω is very close to a regular hexagon, then the evolution process may be described as a gradient flow of total interfacial energy subject to a fixed enclosed area. Under (A5), the diffusion process and the surface kinetic process will not give so much effect on deformation of negative crystal. Nakaya explained the transformation of an apparently circular form into a hexagon (Figure 4 (d)(e)(f) or Figure 5) by “the principle of minimum surface — to find the shape which has the minimum circumference for the given area” [10, section 24].

In this paper, we will explain the deformation process by using an area-preserving gradient flow of total interfacial energy. Figure 6 is a numerical example by an area-preserving crystalline curvature flow. We can observe that the initial shape converges to a regular hexagon.

In the next section, we will introduce an area-preserving crystalline curvature flow, and show known mathematical results. In the last section, we will summarize and discuss future works.



Figure 5: Transformation of a circular figure into a hexagon [10, No.93]; from left to right: a) $t = 0$, b) $t = 2.5$, c) $t = 5$, d) $t = 39\text{min}$. Although Nakaya did not mention on thermal gradient in this process, it seems that (A5) is not satisfied.

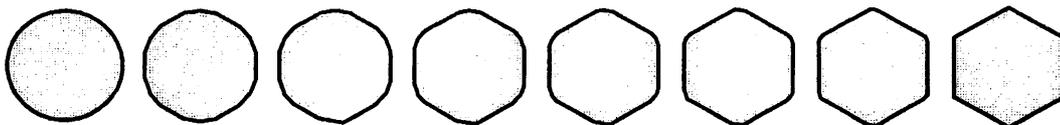


Figure 6: Transformation of a circular figure (the initial 36-sided regular polygon (far left)) into a hexagon (far right). Numerical computation is based on the scheme in Ushijima and Yazaki [14].

3 Area-preserving crystalline curvature flow

In this section, crystalline curvature flow and its area-preserving version will be introduced, and known mathematical results will be shown. Crystalline curvature flow is very singular weighted curvature flow, and it was proposed by J. E. Taylor, and S. Angenent and M. E. Gurtin at the end of 1980's. We refer the reader to the pioneer works Taylor [11, 12] and Angenent and Gurtin [1], and the surveys by Taylor, Cahn and Handwerker [13] and Giga [5], and the books by Gurtin [6] for a geometric and physical background. In what follows, we will introduce an area-preserving crystalline curvature flow for convex polygon in the plane.

Polygons. Let \mathcal{P} be an N -sided convex polygon in the plane \mathbb{R}^2 , and label the position vector of vertices \mathbf{p}_i ($i = 1, 2, \dots, N$) in an anticlockwise order: $\mathcal{P} = \bigcup_{i=1}^N \mathcal{S}_i$, where $\mathcal{S}_i = [\mathbf{p}_i, \mathbf{p}_{i+1}]$ is the i -th edge ($\mathbf{p}_{N+1} = \mathbf{p}_1$). The length of \mathcal{S}_i is $d_i = |\mathbf{p}_{i+1} - \mathbf{p}_i|$, and then the i -th unit tangent vector is $\mathbf{t}_i = (\mathbf{p}_{i+1} - \mathbf{p}_i)/d_i$ and the i -th unit outward normal vector is $\mathbf{n}_i = -\mathbf{t}_i^\perp$, where $(a, b)^\perp = (-b, a)$. We define a set of normal vectors of \mathcal{P} by $\mathcal{N} = \{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_N\}$. Let θ_i be the exterior normal angle of \mathcal{S}_i such as $\mathbf{n}_i = \mathbf{n}(\theta_i)$ and $\mathbf{t}_i = \mathbf{t}(\theta_i)$, where $\mathbf{n}(\theta) = (\cos \theta, \sin \theta)$ and $\mathbf{t}(\theta) = (-\sin \theta, \cos \theta)$. We define the i -th hight function $h_i = \mathbf{p}_i \cdot \mathbf{n}_i = \mathbf{p}_{i+1} \cdot \mathbf{n}_i$. By using N -tuple $h = (h_1, h_2, \dots, h_N)$, d_i is described as follows:

$$d_i[h] = -(\cot \vartheta_i + \cot \vartheta_{i+1})h_i + h_{i-1} \operatorname{cosec} \vartheta_i + h_{i+1} \operatorname{cosec} \vartheta_{i+1}, \quad (3.1)$$

where $\vartheta_i = \theta_i - \theta_{i-1}$ for $i = 1, 2, \dots, N$. Note that $0 < \vartheta_i < \pi$ holds for all i .

Interfacial energy. In the field of material sciences and crystallography, we need to explain the anisotropy: phenomenon of interface motion which depends on the normal direction \mathbf{n} . To explain the anisotropy, it is convenient to define an interfacial energy on the interface or the curve which has line density $\gamma(\mathbf{n}) > 0$. The function $\gamma(\mathbf{n})$ can be extended to the function $\mathbf{x} \in \mathbb{R}^2$ by putting $\gamma(\mathbf{x}) = |\mathbf{x}|\gamma(\mathbf{x}/|\mathbf{x}|)$ if $\mathbf{x} \neq \mathbf{0}$, otherwise $\gamma(\mathbf{0}) = 0$. This extension is called the extension of positively homogeneous of degree 1, since $\gamma(\lambda\mathbf{x}) = \lambda\gamma(\mathbf{x})$ holds for $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^2$. We will use the same notation γ for the extended function. To observe the characteristic of γ , the following *Frank diagram* is useful: $\mathcal{F}_\gamma = \{\mathbf{n}(\theta)/\gamma(\mathbf{n}(\theta)); \theta \in S^1\} = \{\mathbf{x} \in \mathbb{R}^2; \gamma(\mathbf{x}) = 1\}$. If the Frank diagram \mathcal{F}_γ is a convex polygon, γ is called *crystalline energy*. When \mathcal{F}_γ is a J -sided convex polygon, there exists a set of angles $\{\phi_i | \phi_1 < \phi_2 < \dots < \phi_J < \phi_1 + 2\pi\}$ such that the position vectors of vertices are labeled $\mathbf{n}(\phi_i)/\gamma(\mathbf{n}(\phi_i))$ in an anticlockwise order ($\phi_{J+1} = \phi_1$): $\mathcal{F}_\gamma = \bigcup_{i=1}^J [\xi_i, \xi_{i+1}]$, $\xi_i = \nu_i/\gamma(\nu_i)$. Here and hereafter, we denote $\nu_i = \mathbf{n}(\phi_i)$ for $i = 1, 2, \dots, J$ ($\nu_{J+1} = \nu_1$). In this case, the *Wulff shape* $\mathcal{W}_\gamma = \bigcap_{\theta \in S^1} \{\mathbf{x} \in \mathbb{R}^2; \mathbf{x} \cdot \mathbf{n}(\theta) \leq \gamma(\mathbf{n}(\theta))\}$ is also a J -sided convex polygon with the outward normal vector of the i -th edge being ν_i : $\mathcal{W}_\gamma = \bigcap_{i=1}^J \{\mathbf{x} \in \mathbb{R}^2; \mathbf{x} \cdot \nu_i \leq \gamma(\nu_i)\}$. We define a set of normal vectors of \mathcal{W}_γ by $\mathcal{N}_\gamma = \{\nu_1, \nu_2, \dots, \nu_J\}$.

Admissibility. Following [7], we call \mathcal{P} \mathcal{W}_γ -essentially admissible if and only if the consecutive outward unit normal vectors $\mathbf{n}_i, \mathbf{n}_{i+1} \in \mathcal{N}$ ($\mathbf{n}_{N+1} = \mathbf{n}_1$) satisfy $\boldsymbol{\eta}/|\boldsymbol{\eta}| \notin \mathcal{N}_\gamma$, where $\boldsymbol{\eta} = (1 - \lambda)\mathbf{n}_i + \lambda\mathbf{n}_{i+1}$ for $\lambda \in (0, 1)$ and $i = 1, 2, \dots, N$. Note that \mathcal{P} is a \mathcal{W}_γ -essentially admissible convex polygon if and only if $\mathcal{N} \supseteq \mathcal{N}_\gamma$ holds. We call \mathcal{P} \mathcal{W}_γ -admissible if and only if \mathcal{P} is a \mathcal{W}_γ -essentially admissible polygon and $\mathcal{N} = \mathcal{N}_\gamma$ holds. In other words, \mathcal{P} is a \mathcal{W}_γ -admissible convex polygon if and only if $\mathbf{n}_i = \nu_i$ holds for all $i = 1, 2, \dots, N = J$.

Gradient of the total interfacial energy. Let \mathcal{P} be a \mathcal{W}_γ -essentially admissible N -sided convex polygon with the N -tuple of hight functions $h = (h_1, h_2, \dots, h_N)$. Then the total interfacial (crystalline) energy on \mathcal{P} is $\mathcal{E}_\gamma[h] = \sum_{i=1}^N \gamma(\mathbf{n}_i)d_i[h]$. For two N -tuples $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_N), \psi = (\psi_1, \psi_2, \dots, \psi_N) \in \mathbb{R}^N$, let us define the inner product on \mathcal{P} as $(\varphi, \psi)_2 = \sum_{i=1}^N \varphi_i\psi_i d_i[h]$. Furthermore, we define the rate of variation of $\mathcal{E}_\gamma[h]$ in the direction φ and the first variation $\delta\mathcal{E}_\gamma[h]/\delta h$ as follows:

$$\frac{\delta\mathcal{E}_\gamma[h]}{\delta\varphi} = \frac{d}{d\varepsilon}\mathcal{E}_\gamma[h + \varphi] \Big|_{\varepsilon=0} = \text{grad } \mathcal{E}_\gamma[h] \bullet \varphi = \left(\frac{\delta\mathcal{E}_\gamma[h]}{\delta h}, \varphi \right)_2.$$

Crystalline curvature. The first variation of $\mathcal{E}_\gamma[h]$ of \mathcal{P} at S_i is

$$\frac{\delta\mathcal{E}_\gamma[h]}{\delta\varphi} = \sum_{i=1}^N \gamma_i d_i[\varphi] = \sum_{i=1}^N d_i[\gamma]\varphi_i = \sum_{i=1}^N \frac{d_i[\gamma]}{d_i[h]}\varphi_i d_i[h], \quad \gamma = (\gamma_1, \gamma_2, \dots, \gamma_N),$$

where $\gamma_i = \gamma(\mathbf{n}_i)$ for all i . Hence we have $(\delta\mathcal{E}_\gamma[h]/\delta h)_i = d_i[\gamma]/d_i[h]$ for all i in this metric $(\cdot, \cdot)_2$. This quantity is called *crystalline curvature* on the i -th edge S_i , and we denote it

by $\Lambda_\gamma(\mathbf{n}_i) = d_i[\gamma]/d_i[h]$. The numerator $d_i[\gamma]$ is described as $d_i[\gamma] = l_\gamma(\mathbf{n}_i)$, where $l_\gamma(\mathbf{n})$ is the length of the j -th edge of \mathcal{W}_γ if $\mathbf{n} = \nu_j$ for some j , otherwise $l_\gamma(\mathbf{n}) = 0$. Therefore if $\mathcal{P} = \mathcal{W}_\gamma$, then the crystalline curvature is 1.

An area-preserving motion by crystalline curvature. The enclosed area \mathcal{A} of \mathcal{P} is given by $\mathcal{A}[h] = \sum_{i=1}^N h_i d_i[h]/2$. Then the rate of variation of $\mathcal{A}[h]$ in the direction φ is

$$\frac{\delta \mathcal{A}[h]}{\delta \varphi} = \left. \frac{d}{d\varepsilon} \mathcal{A}[h + \varphi] \right|_{\varepsilon=0} = \sum_{i=1}^N \varphi_i d_i[h].$$

By taking $\varphi_i = -(\delta \mathcal{E}_\gamma[h]/\delta h)_i = -\Lambda_\gamma(\mathbf{n}_i)$, we have $\delta \mathcal{A}[h]/\delta \varphi = -\sum_{i=1}^N \Lambda_\gamma(\mathbf{n}_i) d_i[h]$. Hence by taking $\varphi_i = \bar{\Lambda}_\gamma - \Lambda_\gamma(\mathbf{n}_i)$, we have $\delta \mathcal{A}[h]/\delta \varphi = 0$. Here

$$\bar{\Lambda}_\gamma = \frac{\sum_{i=1}^N \Lambda_\gamma(\mathbf{n}_i) d_i[h]}{\sum_{k=1}^N d_k} = \frac{\sum_{i=1}^N l_\gamma(\mathbf{n}_i)}{\mathcal{L}}$$

is the average of the crystalline curvature, and \mathcal{L} is the total length of \mathcal{P} , i.e., $\mathcal{L} = \sum_{i=1}^N d_i$. Thus we have the gradient flow of \mathcal{E}_γ along \mathcal{P} which encloses a fixed area:

$$V_i = \bar{\Lambda}_\gamma - \Lambda_\gamma(\mathbf{n}_i), \quad i = 1, 2, \dots, N, \quad (3.2)$$

where $V_i(t) = \dot{h}_i(t)$ is the normal velocity on \mathcal{S}_i in the direction \mathbf{n}_i at the time t . Here and hereafter, we denote \dot{u} by du/dt . From (3.1), the time derivative of $d_i(t) = d_i[h]$ is given by

$$\dot{d}_i = -(\cot \vartheta_i + \cot \vartheta_{i+1})V_i + V_{i-1} \operatorname{cosec} \vartheta_i + V_{i+1} \operatorname{cosec} \vartheta_{i+1} \quad (3.3)$$

for $i = 1, 2, \dots, N$. Note that (3.2) and (3.3) are equivalent each other.

Negative polygons. Applying the area-preserving crystalline curvature flow to deformation of negative crystals, we will introduce the concept of negative polygons. Enclosed region of \mathcal{W}_γ -admissible convex polygon \mathcal{P} is crystal, and then normal vector \mathbf{n} is direction from ice to gas, i.e., \mathcal{P} is convex to $-\mathbf{n}$ direction and crystalline curvature $\Lambda_\gamma(\mathbf{n})$ is positive everywhere. We define *negative polygon* of \mathcal{P} by $-\mathcal{P}$, and we denote it by $\bar{\mathcal{P}}$. Then vertices of negative polygon $\bar{\mathcal{P}}$ are labeled clockwise, $\bar{\mathcal{P}}$ is concave to $-\mathbf{n}$ direction, and crystalline curvature $\Lambda_\gamma(\mathbf{n})$ is negative everywhere. If \mathcal{P} is a regular polygon, then $\mathcal{P} = \bar{\mathcal{P}}$. In this sense, the negative Wulff shape $\bar{\mathcal{W}}_\gamma = -\mathcal{W}_\gamma$ is defined as $\bar{\mathcal{W}}_\gamma = \bigcap_{j=1}^J \{x \in \mathbb{R}^2; x \cdot (-\nu_j) \leq \gamma(\nu_j)\}$. See Figure 7.

Known results. The problem is stated as follows.

Problem 1 For a given \mathcal{W}_γ -essentially admissible closed curve \mathcal{P}_0 , find a family of \mathcal{W}_γ -essentially admissible curves $\{P(t)\}_{0 \leq t < T}$ satisfying (3.2) (or (3.3)) with $P(0) = \mathcal{P}_0$. Since (3.3) are the system of ordinary differential equations, the maximal existence time is positive $T > 0$.

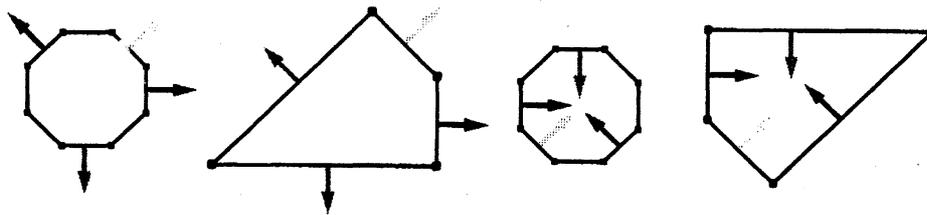


Figure 7: From left to right: \mathcal{W}_γ -essentially admissible polygon, the Wulff shape \mathcal{W}_γ , $\overline{\mathcal{W}_\gamma}$ -essentially admissible negative polygon, and the negative Wulff shape $\overline{\mathcal{W}_\gamma}$

What might happen to $\mathcal{P}(t)$ as t tends to $T \leq \infty$? For this question, we have the following three results. The first result is the case where motion is isotropic and polygon is admissible.

Theorem A Let the interfacial energy be isotropic $\gamma \equiv 1$. Assume the initial polygon \mathcal{P}_0 is an N -sided \mathcal{W}_γ -admissible convex polygon. Then a solution \mathcal{W}_γ -admissible polygon $\mathcal{P}(t)$ of Problem 1 exists globally in time keeping the area enclosed by the polygon constant \mathcal{A} , and $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the Hausdorff metric as t tends to infinity, where $\gamma_*(\mathbf{n}_i) \equiv \sqrt{2\mathcal{A}/\sum_{k=1}^N l_1(\mathbf{n}_k)}$ is constant. In particular, if \mathcal{P}_0 is centrally symmetric with respect to the origin, then we have an exponential rate of convergence.

This theorem is proved by Yazaki [16] by using the isoperimetric inequality and the theory of dynamical systems. We note that $\partial\mathcal{W}_{\gamma_*}$ is the circumscribed polygon of a circle with radius γ_* , and then this result is a semidiscrete version of Gage [4].

The second result is the case where motion is anisotropic and polygon is admissible.

Theorem B Let the crystalline energy be $\gamma > 0$. Assume the initial polygon \mathcal{P}_0 is an N -sided \mathcal{W}_γ -admissible convex polygon. Then a solution \mathcal{W}_γ -admissible polygon $\mathcal{P}(t)$ of Problem 1 exists globally in time keeping the area enclosed by the polygon constant \mathcal{A} , and $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the Hausdorff metric as t tends to infinity, where $\gamma_*(\mathbf{n}_i) = \gamma(\mathbf{n}_i)/W$, $W = \sqrt{|\mathcal{W}_\gamma|/\mathcal{A}}$ for all $i = 1, 2, \dots, N$ and $|\mathcal{W}_\gamma| = \sum_{k=1}^N \gamma(\mathbf{n}_k) l_\gamma(\mathbf{n}_k)/2$ is enclosed area of \mathcal{W}_γ .

This theorem is proved in Yazaki [17, Part I] by using the anisoperimetric inequality or Brunn and Minkowski's inequality and the theory of dynamical systems which is a similar technique as in Yazaki [16].

The last result is the case where motion is anisotropic and polygon is \mathcal{W}_γ -essentially admissible. The next theorem is proved in Yazaki [19].

Theorem C Let the crystalline energy be $\gamma > 0$. Assume the initial polygon \mathcal{P}_0 is an N -sided \mathcal{W}_γ -essentially admissible convex polygon. If the maximal existence time of a

solution \mathcal{W}_γ -essentially admissible polygon $\mathcal{P}(t)$ of Problem 1 is finite $T < \infty$, then there exists the i -th edge \mathcal{S}_i such that $\lim_{t \rightarrow T} d_i(t) = 0$ and $l_\gamma(\mathbf{n}_i) = 0$ hold. That is, the normal vector of vanishing edge does not belong to \mathcal{N}_γ , and $\inf_{0 < t < T} d_k(t) > 0$ holds for all $\mathbf{n}_k \in \mathcal{N}_\gamma$.

4 Conclusion

We will show our main results and discuss future works.

Main results. By Theorem B, any $\overline{\mathcal{W}}_\gamma$ -admissible negative polygon converges to the negative Wulff shape $\overline{\mathcal{W}}_\gamma$ as time tends to infinity. Figure 6 shows convergence of $\overline{\mathcal{W}}_\gamma$ -essentially admissible negative polygon (36-sided regular polygon) to the negative Wulff shape $\overline{\mathcal{W}}_\gamma$ in the case where $\overline{\mathcal{W}}_\Gamma$ is a regular hexagon. However, this numerical result is an open problem: For any \mathcal{W}_γ -essentially admissible convex polygon \mathcal{P}_0 , is T a finite value? If the answer of this question is yes, then we have the finite time sequence $T_1 < T_2 < \dots < T_M$ such that $\mathcal{P}(T_i)$ is \mathcal{W}_γ -essentially admissible for $i = 1, 2, \dots, M - 1$ and $\mathcal{P}(T_M)$ is \mathcal{W}_γ -admissible. In the general case where $V_i = g(\mathbf{n}_i, \Lambda_\gamma(\mathbf{n}_i))$ for all i under certain conditions of g , the answer of the above question is yes. See Yazaki [18]. Note that g does not include $\overline{\Lambda}_\gamma$.

Discussion. The initial shape of negative crystal is an apparently circular form. Nakaya asserted that the boundary is stepped structure rather than a smooth curve [10, Figure 26]. This corresponds to the case where the initial \mathcal{P}_0 is \mathcal{W}_γ -admissible non-strictly-convex polygon, when \mathcal{W}_γ is a regular hexagon. Although we can compute evolution of \mathcal{W}_γ -essentially admissible polygonal curves, the mathematical justification is our on-going research. Furthermore, in our future research, the assumptions (A1) and (A5) will be removed, i.e., We will consider influence of temperature in deformation of three dimensional negative crystals. Then they will be filled with super- or sub-saturated water vapor, depending on the position, and we will need evolution equations which describe the diffusion process and the surface kinetic process.

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