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Eventual Stability Criteria for Periodic Points of Michio Morishima's Example

-森嶋通夫の例における周期点集合の最終的大域一様漸近安定性-

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Abstract

In this article we begin with the Moroshima's example, which implies a kind of eventually asymptotical stability of solutions for a difference equation $x(n+1)=f(x(n))$ for $n\geq 0$. We define new definitions of eventual stability of periodic points in the meaning of the large in the same way as ones of Lakshmikantham et. al. and Yoshizawa. By applying the Liapunov's second method we give eventual stability criteria in the large of the difference equation. In order to illustrate the main results on eventual stability an example of a set of 2-periodic points for eventual stability is given with a numerical result and analytical estimation.

1 Introduction

In 1977 Morishima[3] gave results on the stability, oscillation and chaos of periodic points concerning the following difference equation.

$$x(n+1)=\frac{A(n)}{A(n)+B(n)} \quad n=0,1,\cdots$$

and

$$\begin{aligned} A(n) &= \max[\frac{a}{b}x(n)+\{1-(1+a)x(n)\},0], \\ B(n) &= \max[(1-x(n))\{\frac{a}{b}-\frac{x(n)(1-(1+a)x(n))}{(1-x(n))^2}\},0] \end{aligned}$$

Here a,b are positive parameters. His results[3] with $a=0.6, b=1$ were studied independently with Li-Yorke[2] in 1975.

Morishima[4] studied the chaotic behavior of orbits of

$$x(n+1)=f(x(n)), \tag{1.1}$$

where $f:[0,1]\to[0,1]$ is continuous, $x:\mathbf{Z}_+=\{0,1,2,\cdots\}\to[0,1]$ is the price of the commodity and also he discussed some type of stability of periodic points, where the stability is not globally uniformly asymptotically stable but every orbits of (1.1) has unstable subsequences in the beginning and the stable behavior from some iterations.

In this article we show results on the globally asymptotical stability for periodic points of (1.1) as well as we discuss the globally eventually asymptotical stability. See Lakshikantham-Leela[1], Yoshizawa[5] concerning the eventual stability for the case of ordinary differential equations.

2 Notations

Consider difference equation (1.1) in $I^m \subset \mathbf{R}^m$ with $I = [0,1]$ and positive integer m. Denote $x(n) = (x_1(n), x_2(n), \cdots, x_m(n))^T$ is a relative price vector of $m-$commodities, where $0 \leq x_j(n) \leq 1$ for $j = 1, 2, \cdots, m$ and $\sum_{j=1}^{m} x_j(n) = 1$ for $n = 0, 1, 2, \cdots$. See [3, 4] in detail. A function $f : I^m \to I^m$ is continuous.

Let k be a positive integer. Denote a set of $k-$peridic points by $P(k) = \{x^* \in I^m\}$, where $f^i(x^*) \neq f^j(x^*)$ for $1 \leq i \neq j \leq k$ and $f^k(x^*) = x^*$. Denote by $x(n; n_0, x_0)$ a solution of (1.1) for $n \geq n_0$ with $x(n_0; n_0, x_0) = x_0$ satisfying the initial condition $(n_0, x_0) \in \mathbf{Z}_+ \times I^m$. Denote by $\| x \|$ a norm of $x \in \mathbf{R}^m$. For $r > 0$ we denote the following neighborhoods: when a point $x_0 \in \mathbf{R}^m, B(x_0, r) = \{x \in I^m : \| x - x_0 \| < r\}$; when a subset $P \subset \mathbf{R}^m, S(P, r) = \cup_{x \in P} B(x, r)$.

A set of $k-$periodic points $P(k)$ is called eventually uniformly stable [EV-US] if for each $\varepsilon > 0$ there exist $N_0 \in \mathbf{Z}_+$ and $\delta > 0$ such that for every $x_0 \in S(P(k), \delta)$ and every $n_0 \geq N_0$, it holds that each solution x(n; n$_0, x_0) \in S(P(k), \varepsilon)$ for $n \geq n_0, i.e.,$

$$d(x(n; n_0, x_0), P(k)) < \varepsilon.$$

Here a distance between a point $x \in \mathbf{R}^m$ and a subset $A \subset \mathbf{R}^m$ is defined by $d(x, A) = \inf\{\| x - a \| : a \in A\}$. A set of $k-$periodic points $P(k)$ is called eventually uniformly attractive to finite coverings [EV-UA.FC] if each finite covering $\{C_q \subset I^m$ such that $\cup_{q=1}^{Q} C_q \supset I^m$ and each $\varepsilon > 0$, there exist $N_0 \in \mathbf{Z}_+$ and $T_0 \in \mathbf{Z}_+$ such that for every $1 \leq q \leq Q$, every $x_0 \in C_q$, every $n_0 \geq N_0$, and it holds that every solution $x(n; n_0, x_0) \in S(P(k), \varepsilon)$ for $n \geq n_0 + T_0, i.e.,$

$$d(x(n; n_0, x_0), P(k)) < \varepsilon.$$

The set of $k-$periodic points $P(k)$ is called eventually uniformly asymptotically stable to finite coverings [EV-UAS.FC] if $P(k)$ is [EV-US] and [EV-UA.FC].

3 Criteria of Eventual Stability

Assume that Eq.(1.1) has a set of k-periodic points

$$P(k) = \{x_1, x_2, \cdots, x_k\}$$

for $k = 1, 2, \cdots$. We show two criteria for eventually uniformly asymptotically stable of $P(k)$ by applying Liapunov's second method. In case $k = 1, P(1)$ is a set of fixed point.

Let a set of functions denote

$$CIP = \{a : I \to I \text{ is continuous, strictly increasing and positive definite function }\}$$

and $R_+ = [0, \infty)$.

In the following theorem we give eventually uniformly asymptotically stable to finite coverings of $P(k)$.

Theorem. $k-$periodic points $P(k)$ is eventually uniformly asymptotically stable to finite coverings under that there exists a function $V : \mathbf{Z}_+ \times I^m \to \mathbf{R}_+$ satisfying the following condition (a)-(b).

(a) For any $r > 0$ there exist a nonnegative integer $N_0 \geq 0$ and two functions $a_r, b_r \in CIP$ such that

$$a_r(d(x, P(k))) \leq V(n, x) \leq b_r(d(x, P(k))$$

for any $n \geq N_0$ and any $x \in I^m - S(P(k), r)$.

(b) Let $\Delta V(n, x) = V(n+1, f^k(x)) - V(n, x)$ for $(n, x) \in \mathbf{Z}_+ \times I^m$. For any $r > 0$ there exist a nonnegative integer $N_0 \geq 0$ and a function $c_r \in CIP$ such that

$$\Delta V(n, x) \leq -c_r(d(P(k), x))$$

for any $n \geq N_0$ and any $x \in I^m - S(P(k), r)$.

Outline of the proof is as follows (1) and (2).
(1) It is proved that the set $P(k)$ is [EV-US]. At first, we get the following inequalities.

$$\begin{aligned}
&\tilde{a_r}(d(x, P(k))) \leq V(n, x) \leq b_r(d(x, P(k)));\\
&\Delta V(n, x) \leq -\tilde{c_r}(d(x, P(k)));\\
&\tilde{a_r}(d) = \min[a_r(d), c_r(d)] \text{ for } d > 0, \tilde{c_r}(d) = \frac{1}{2}\tilde{a_r}(d)
\end{aligned}$$

For a sufficiently small $\alpha_1 > 0$ and any $p_\omega \in P(k)$ it can be seen that

$$\forall x \in B(p_\omega, \alpha_2) \Rightarrow F(x) \in B(p_\omega, \alpha_1). \tag{3.2}$$

For any $\varepsilon > 0$ define

$$\phi_\omega(\varepsilon) = \inf\{V(n, x) : \varepsilon \leq\| x - p_\omega \|\leq \alpha_1, \forall n \geq n_0\}. \tag{3.3}$$

We immediately get

$$V(n, x) < \phi_\omega(\varepsilon) \text{ for } \forall x \in B(p_\omega, \delta_\omega),\ \forall n_0 \geq N_0 \tag{3.4}$$

If not so, we can prove the above statement. We, secondly, have the following relations.

$$\begin{aligned}
&1 \leq \exists k(1) \leq k, 0 < \exists\delta \leq \delta_{k(1)} : \forall x \in B(p_{k(1)}, \delta_1), \forall n_0 \geq N_0;\\
&1 \leq \exists k(2) \leq k : \forall n \geq n_0 \Rightarrow x(n; n_0, x) \in B(p_{k(2)}, \varepsilon).
\end{aligned} \tag{3.5}$$

It can be seen that (1.1) is uniformly bonded as follows:

$$\forall\alpha > 0, \exists\beta(\alpha) > 0 : \forall n_0 \geq 0, \| x(n; n_0, x) \|< \beta(\alpha) \text{ for } \| x \|< \alpha, n \geq n_0. \tag{3.6}$$

(2)Assume that $P(k)$ is not [EV-UA.FC] as follows. There exist a real number $\varepsilon_1 > 0$ and a finite covering $\{C_q \subset I^m$ such that $\cup_{q=1}^{Q} C_q \supset I^m\}$ such that for any integers $N, T \in \mathbf{Z}_+$ there exist an initial point $x_0 \in C_{\tilde{q}}$ and integers $n_0(N, T) \geq N, n_1(N, T) \geq n_0(N, T) + T$, We assume that $d(x(n_1; n_0, x_0), P(k)) \geq \varepsilon_1$. Then we have a sequence $\{z_j :\| z_j \|\leq \alpha\}$ and $z = \lim_{j\to\infty} z_j \notin P(k)$. For a sufficiently small $\eta > 0$ we get a neighborhood $O(z, \eta)$ of z as follows.

$$S(P(k), \varepsilon_1) \cap cl(O(z, \eta)) = \emptyset. \tag{3.7}$$

From $V(n,x) \neq 0$ on $O(z,\eta)$, we can define $h(n,x) = V(n, f^k(x))/V(n,x)$ on $O(z,\eta)$. By $\tilde{c_r}(d)/\tilde{a_r}(d) = 1/2$, it can be seen that

$$h(n,x) \leq 1 - [\tilde{c_r}(d(x,P(k)))/\tilde{a_r}(d(x,P(k)))] = 1/2. \tag{3.8}$$

Then it leads to a contradiction. **Q.E.D.**

In case where $k = 1$ the above theorem leads to an eventual stability theorem of fixed point for (1.1).

Corollary. Eq.(1.1) has a fixed point x^*. The point x^* is eventually uniformly asymptotically stable to finite coverings under that there exists a function $V : \mathbf{Z}_+ \times I^m \to \mathbf{R}_+$ satisfying Condition (a)-(b).

(a) For any $r > 0$ there exist an integer $N_0 \geq 0$ and two functions $a_r, b_r \in CIP$ such that

$$a_r(\| x - x^* \|) \leq V(n,x) \leq b_r(\| x - x^* \|)$$

for any integers $n \geq N_0$ and any initial points $x \in I^m - \{x^*\}$.

(b) Let $\Delta V(n,x) = V(n+1, f(x)) - V(n,x)$ for $(n,x) \in \mathbf{Z}_+ \times I^m$. For any $r > 0$ there exist an integer $N_0 \geq 0$ and a function $c_r \in CIP$ such that

$$\Delta V(n,x) \leq -c_r(\| x - x^* \|)$$

for any $n \geq N_0$ and any $x \in I^m - \{x^*\}$.

4 Illustration of Theorem

We illustrate Theorem in the case $k = 2$ and $P(2) = \{0.5, 0.7\}$ in the space $\mathbf{R}$ with a numerical result. Consider Morishima's example as follows.

$$x(n+1) = f(x(n)) = \frac{A(n)}{A(n) + B(n)}.$$

Here $A(n) = \max[x + bE_1(x(n)), 0], B(n) = \max[1 - x + bE_2(x(n)), 0]$ and $a = 0.6$ and $E_1(x) = -x + \frac{1-x}{a}, E_2(x) = -\frac{xE_1(x)}{1-x}$. See [3] in detail. Then, in $b = 0.6$, we get

$$f(x) = \frac{1.8x^2 - 4.8x + 3}{9.6x^2 - 13.8x + 6}, \quad f'(x) = \frac{21.24x^2 - 36x + 12.6}{(9.6x^2 - 13.8x + 6)^2}.$$

Let

$$V(x) = d(x, P(2)) = \min[|x - 0.5|, |x - 0.7|]$$

for $x \in I$. We consider for each $r > 0$, $a_r(d) = b_r(d) = d$ $(d > 0)$, then $a_r, b_r \in CIP$ and it holds that Condition(a) of Theorem is satisfied. It can be seen that

$$\begin{aligned}
\Delta V(x) &= \min(|f^2(x) - 0.5|, |f^2(x) - 0.7|) - d(x, P(2)) \\
&= \min(|f^2(x) - f^2(0.5)|, |f^2(x) - f^2(0.7)|) - d(x, P(2)) \\
&= \min_{x^*=0.5,0.7} \left| \int_0^1 \frac{df^2}{dx}(x^* + \theta(x - x^*))(x - x^*)d\theta \right| - d(x, P(2))
\end{aligned}$$

$$
\begin{aligned}
&\leq \min_{x^*=0.5,0.7} \max_{x \in I} |\frac{df^2}{dx}(x)||x - x^*| - d(x, P(2)) \\
&= \max_{x \in I} |\frac{df^2}{dx}(x)| d(x, P(2)) - d(x, P(2)) \\
&= (\max_{x \in I} |f'(f(x))f'(x)| - 1) d(x, P(2)).
\end{aligned}
$$

It holds that $\Delta V(x) \leq \max_{x \in I}(f'(f(x))f'(x) - 1)V(x)$ and it expectd that $\Delta V(x) \leq -cV(x)$ for $x \in I$ with a real number $c > 0$. Putting $y(x) = f'(f(x))f'(x) - 1$, when $y(x) < 0$, then there exists a positive number c such that

$$
\Delta V(x) \leq -cV(x), \tag{4.9}
$$

then putting $C(d) = cd$, we have

$$
C \in CIP : \Delta V(x) \leq -C(V(x)).
$$

Therefore it holds that Condition (b) of Theorem is satisfied. By a numerical result on $y(x)$, it can be seen that there exists positive $c \leq 0.4$ satisfying (4.9). See Figure 1.

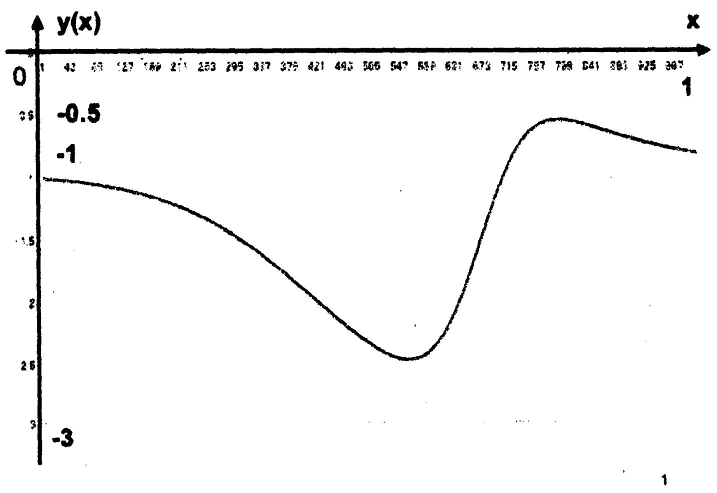

Figure 1: Let $y(x) = f'(f(x))f'(x) - 1$ for $0 \leq x \leq 1$. Then it holds that $y(x) \leq -0.4$ for $0 \leq x \leq 1$.

Putting

$$
p = 21.24x^2 - 36x + 12.6, \quad q = (9.6x^2 - 13.8x + 6)^2,
$$

then we have $f' = p/q$ and $(p/q)^2 - 1 < 0$. In fact

$$\begin{aligned}
&\frac{p^2 - q^2}{q^2} \\
&= \frac{(p-q)(p+q)}{q^2} \\
&= [(21.24x^2 - 36x + 12.6) - (9.6x^2 - 13.8x + 6)^2] \\
&\qquad \times [21.24x^2 - 36x + 12.6 + (9.6x^2 - 13.8x + 6)^2]/q^2 \\
&= [-92.16x^4 + 264.96x^3 - 284.4x^2 - 118.8x - 23.4] \\
&\times [21.24(x - (1.18)^{-1})^2 + 12.6 - (9/5.09) + (9.6x^2 - 13.8x + 6)^2]/q^2
\end{aligned}$$

and $12.6 - (9/5.09) > 0$, $264.96x^3 - 284.4x^2 = 264.96x^2(x - 284.4/264.96) < 0$ for $0 \leq x \leq 1$. Hence it holds that on $x \in [0,1]$

$$y(x) = f'(f(x))f'(x) - 1 \leq \frac{p^2}{q^2} - 1 < 0.$$

Since y is continuous and $[0,1]$ is compact, then there exists a positive number c such that $y(x) < -c < 0$ on $[0,1]$.

5 Conclusions

We considered a definition of [EV-UAS.FC] (eventually uniformly asymptotic stability to finite coverings) in the same way as theory of ordinary differential equations.

We proved a theorem for [EV-UAS.FC] of difference equation $x(n+1) = f(x(n))$ by Liapunov's second method but including a computational result and also analytical estimation of ΔV.

We illustrated the eventual stability theorem by applying it to the Morishima's example.

References

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