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Kyoto University
Existence Theorems of Two Families of Vector Generalized Quasi-Optimization Problems with Applications

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ABSTRACT

In this paper, we apply Himmelberg's fixed point theorem to establish existence theorems of two families of vector generalized quasi-optimization problems. We apply our results to establish existence theorems of systems of generalized vector-quasi-equilibrium problems. Systems of weak loose quasi-saddle point problem.

1 Introduction

Recently, Lin [7] considered simultaneous vector quasi-equilibrium problem and proved existence results for its solution. By using these results, he derived existence results for a solution of vector quasi-saddle point problem.

In the recent past, systems of scalar (vector) equilibrium problems, systems of scalar (vector) generalized equilibrium problems, systems of scalar (vector) quasi-equilibrium problems, and systems of scalar (vector) generalized quasi-equilibrium problems are used as tools to solve Nash equilibrium problem (for vector-valued functions) and Debreu type equilibrium problem (for vector-valued functions), respectively, see for example [1, 2, 3, 4, 5] and references therein.
Very recently, Ansari et al. [6] considered systems of simultaneous generalized vector quasi-equilibrium problem and proved existence results for its solution by scalarization method. By using these results, they derived existence existence results of a solution of system of vector quasi-saddle point problem.

Let $I$ be any index set. For each $i \in I$, let $E_i$, $V_i$ and $Z_i$ be real locally convex topological vector spaces (in short, t.v.s.). For each $i \in I$, let $X_i \subset E_i$ be a nonempty convex set and $Y_i \subset V_i$ a nonempty convex set. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $S_i : X \times Y \to X_i$ be a multivalued map with nonempty values and $T_i : X \times Y \to Y_i$ be a multivalued map with nonempty values. Let $C_i : X \times Y \to Z_i$ be a multivalued map such that for each $(x, y) \in X \times Y$, $C_i(x, y)$ is a cone and $intC_i(x, y) \neq \emptyset$. Let $F_i : X \times Y \times X_i \to Z_i$ be a multivalued map with nonempty values and $G_i : X \times Y \times Y_i \to Z_i$ be a multivalued map with nonempty values.

Throughout this paper, we use these notation unless otherwise specified.

We first consider two families of vector generalized quasi-optimization problems:

Find a $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{x}_i \in S_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x}, \overline{y})$, $F_i(\overline{x}, \overline{y}, \overline{x}_i) \cap wMin_{C_i(\overline{x}, \overline{y})} F_i(\overline{x}, \overline{y}, S_i(\overline{x}, \overline{y})) \neq \emptyset$ and $G_i(\overline{x}, \overline{y}, \overline{y}_i) \cap wMin_{C_i(\overline{x}, \overline{y})} G_i(\overline{x}, \overline{y}, T_i(\overline{x}, \overline{y})) \neq \emptyset$.

For the special case of above problems is systems of simultaneous generalized vector quasi-equilibrium problem for multivalued maps.

Find a $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{x}_i \in S_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x}, \overline{y})$, $F_i(\overline{x}, \overline{y}, x_i) \cap (-intC_i(\overline{x}, \overline{y})) = \emptyset$ for all $x_i \in S_i(\overline{x}, \overline{y})$ and $G_i(\overline{x}, \overline{y}, y_i) \cap (-intC_i(\overline{x}, \overline{y})) = \emptyset$ for all $y_i \in T_i(\overline{x}, \overline{y})$.

If $F_i$ and $G_i$ are single-valued maps. will be reduced to find a $(\overline{x}, \overline{y}) \in X \times Y$ such that for each $i \in I$, $\overline{x}_i \in S_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x}, \overline{y})$, $f_i(\overline{x}, \overline{y}, x_i) \notin (-intC_i(\overline{x}, \overline{y}))$ for all $x_i \in S_i(\overline{x}, \overline{y})$ and $g_i(\overline{x}, \overline{y}, y_i) \notin (-intC_i(\overline{x}, \overline{y}))$ for all $y_i \in T_i(\overline{x}, \overline{y})$.

This problem is a generalization of in Ansari et al. [5].

In section 4, we consider the following systems of weak loose quasi-saddle point problem.

Find $\overline{x} = (\overline{x}_i)_{i \in I} \in X$ and $\overline{y} = (\overline{y}_i)_{i \in I} \in Y$ such that for each $i \in I$, $\overline{x}_i \in S_i(\overline{x}, \overline{y})$, $\overline{y}_i \in T_i(\overline{x}, \overline{y})$, $L_i(\overline{x}_i, \overline{y}_i) \cap wMax_{C_i(\overline{x}, \overline{y})} L_i(S_i(\overline{x}, \overline{y}), \overline{y}_i) \neq \emptyset$ and $L_i(\overline{x}_i, \overline{y}_i) \cap wMin_{C_i(\overline{x}, \overline{y})} L_i(\overline{x}_i, T_i(\overline{x}, \overline{y})) \neq \emptyset$, where $L_i : X_i \times Y_i \to Z_i$. 
In this paper, we prove existence theorems of two families of vector generalized quasi-optimization problems by Himmelberg's fixed point theorem. Then we apply our results to study existence theorem of systems of weak loose quasi-saddle point problem and systems of generalized vector quasi-equilibrium problems. These results improved and generalized some main results in [5].

2 Preliminaries

Throughout this paper, all topological spaces are assumed to be Hausdorff.

Definition 2.1. Let $Z$ be a real t.v.s., $D$ a convex cone in $Z$ with $\text{int}D \neq \emptyset$, and $A$ a nonempty subset of $Z$. Let $y_1, y_2 \in A$, we denote $y_1 \leq y_2$, if $y_2 - y_1 \in D$; $y_1 < y_2$, if $y_2 - y_1 \in \text{int}D$.

A point $\bar{y} \in A$ is called a vector minimal point of $A$ if for any $y \in A$, $y - \bar{y} \notin -D \setminus \{0\}$. A point $\bar{y} \in A$ is called a weakly vector minimal point of $A$ if for any $y \in A$, $y - \bar{y} \notin -\text{int}D$.

The set of vector minimal (resp. weakly vector minimal) points if $A$ is denoted by $\text{Min}_D A$ (resp. $w\text{Min}_D A$).

3 Existence Results for a Solution of Two Families of Vector Generalized Quasi-Optimization Problems

Theorem 3.1. For each $i \in I$, let $S_i$ be a continuous compact multivalued maps with nonempty closed convex values and $T_i$ be a continuous compact multivalued maps with nonempty closed convex values. For each $i \in I$, assume the following conditions are satisfied:

(i) $C_i(x, y)$ is a closed convex pointed cone with apex at the origin and $\text{int}C_i(x, y) \neq \emptyset$;

(ii) the map $W_i: X \times Y \to Z_i$ defined by $W_i(x, y) = Z_i \setminus \text{int}C_i(x, y)$ is u.s.c.;

(iii) $F_i$ is a continuous multivalued map with nonempty compact values such that for any fixed $(x, y) \in X \times Y$, $F_i(x, y, u_i)$ is properly quasiconvex in $u_i$; and
(iv) $G_i$ is a continuous multivalued map with nonempty compact values such that for any fixed $(x, y) \in X \times Y$, $G_i(x, y, v_i)$ is properly quasiconvex in $v_i$.

Then there exists a $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x}, \bar{y})$,

$$F_i(\bar{x}, \bar{y}, \bar{x}_i) \cap \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y)) \neq \emptyset$$

and $G_i(\bar{x}, \bar{y}, \bar{y}_i) \cap \mathop{\text{Min}}_{C_i(x, y)} G_i(x, y, T_i(x, y)) \neq \emptyset$.

In particular, if for each $i \in I$, for all $x \in X$ and $y \in Y$, $F_i(x, y, x_i) \subset C_i(x, y)$ and $G_i(x, y, y_i) \subset C_i(x, y)$. Then there exists a $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x}, \bar{y})$, $F_i(\bar{x}, \bar{y}, x_i) \cap (\text{int} C_i(\bar{x}, \bar{y})) = \emptyset$ for all $x_i \in S_i(\bar{x}, \bar{y})$ and $G_i(\bar{x}, \bar{y}, y_i) \cap (\text{int} C_i(\bar{x}, \bar{y})) = \emptyset$ for all $y_i \in T_i(\bar{x}, \bar{y})$.

**Proof.** For each $i \in I$, since $S_i$ and $T_i$ are compact, there exist compact subsets $D_i \subseteq X_i$ and $M_i \subset Y$ such that $S_i(X \times Y) \subseteq D_i$ and $T_i(X \times Y) \subseteq M_i$. For each $i \in I$ and for all $(x, y) \in X \times Y$, define two multivalued maps $\Phi_i : X \times Y \to D_i$ and $\Psi_i : X \times Y \to M_i$ by

$$\Phi_i(x, y) = \{u_i \in S_i(x, y) : F_i(x, y, u_i) \cap \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y)) \neq \emptyset\}$$

and

$$\Psi_i(x, y) = \{v_i \in T_i(x, y) : G_i(x, y, v_i) \cap \mathop{\text{Min}}_{C_i(x, y)} G_i(x, y, T_i(x, y)) \neq \emptyset\}.$$ Since $S_i : X \times Y \to X_i$ is a compact multivalued map with nonempty closed values, $S_i$ has nonempty compact values. Since $F_i : X \times Y \times X_i \to Z_i$ is u.s.c. with compact values, $F_i(x, y, S_i(x, y))$ is a nonempty compact set for each $i \in I$, $\emptyset \neq \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y)) \subset \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y))$.

Then there exists $k_i \in \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y))$ such that $k_i \in F_i(x, y, u_i)$ for some $u_i \in S_i(x, y)$. Therefore, $\Phi_i(x, y) \neq \emptyset$ for each $i \in I$ and for all $(x, y) \in X \times Y$. Suppose there exist some $(x, y) \in X \times Y$ and some $i \in I$ such that $\Phi_i(x, y)$ is not a convex subset of $S_i(x, y)$. Then there exist $v_i^1, v_i^2 \in \Phi_i(x, y)$ and $t \in [0, 1]$ such that

$$tv_i^1 + (1-t)v_i^2 \notin \Phi_i(x, y). \quad (1)$$

We have $v_i^1 \in S_i(x, y)$, $v_i^2 \in S_i(x, y)$,

$$F_i(x, y, v_i^1) \cap \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y)) \neq \emptyset,$$

and $F_i(x, y, v_i^2) \cap \mathop{\text{Min}}_{C_i(x, y)} F_i(x, y, S_i(x, y)) \neq \emptyset$.

Thus, there exists $a_i^1 \in F_i(x, y, v_i^1)$ such that for each $b_i \in F_i(x, y, S_i(x, y))$,
$b_i - a_i^1 \notin -intC_i(x, y) \tag{2}$

and there exists $a_i^2 \in F_i(x, y, v_i^2)$ such that for each $b_i \in F_i(x, y, S_i(x, y))$,

$$b_i - a_i^2 \notin -intC_i(x, y). \tag{3}$$

Since $S_i : X \times Y \to X_i$ is a multivalued map with nonempty convex values,

$$tv_i^1 + (1 - t)v_i^2 \in S_i(x, y). \tag{4}$$

By (1) and (4), we have

$$F_i(x, y, tv_i^1 + (1 - t)v_i^2) \cap w\text{Min}_{C_i(x, y)}F_i(x, y, S_i(x, y)) = \emptyset. \tag{5}$$

Then for each $c \in F_i(x, y, tv_i^1 + (1 - t)v_i^2)$, there exists $d_c \in F_i(x, y, S_i(x, y))$ such that $d_c - c \notin -intC_i(x, y)$.

By (2), (3) and conditions (iii), there exists $z_{a_i^1a_i^2} \in F_i(x, y, tv_i^1 + (1 - t)v_i^2)$ such that either

$$a_i^1 - z_{a_i^1a_i^2} \in C_i(x, y) \tag{7}$$

or $a_i^2 - z_{a_i^1a_i^2} \in C_i(x, y)$. \tag{8}

Without lost of generality, we may assume that (7) is true, then by (5), there exists $d_{z_{a_i^1a_i^2}} \in F_i(x, y, S_i(x, y))$ such that

$$d_{z_{a_i^1a_i^2}} - a_i^1 \in -intC_i(x, y). \tag{9}$$

By (7), $z_{a_i^1a_i^2} - a_i^1 \in -C_i(x, y)$ and (9), we have

$$d_{z_{a_i^1a_i^2}} - a_i^1 = (d_{z_{a_i^1a_i^2}} - z_{a_i^1a_i^2}) + (z_{a_i^1a_i^2} - a_i^1) \in (-intC_i(x, y)) + (-C_i(x, y)) \subset -intC_i(x, y). \tag{10}$$

By (2) and (10), we have a contraction. Therefore, for each $i \in I$ and for all $(x, y) \in X \times Y$, $\Phi_i(x, y)$ is a convex subset of $S_i(x, y)$.

For each $(x, y, u_i) \in \overline{Gr(\Phi_i)}$, there exists $(x^\alpha, y^\alpha, u_i^\alpha) \in Gr\Phi_i$ and $(x^\alpha, y^\alpha, u_i^\alpha) \to (x, y, u_i)$. One has $u_i^\alpha \in S_i(x^\alpha, y^\alpha)$ and

$$F_i(x^\alpha, y^\alpha, u_i^\alpha) \cap w\text{Min}_{C_i(x^\alpha, y^\alpha)}F_i(x^\alpha, y^\alpha, S_i(x^\alpha, y^\alpha)) \neq \emptyset. \tag{11}$$

Since $u_i^\alpha \in S_i(x^\alpha, y^\alpha)$ and $S_i$ is u.s.c. with closed values, $u_i \in S_i(x, y)$. By (11), there exists $\{b_i^\alpha\}$ in $Z_i$ such that

$$b_i^\alpha \in F_i(x^\alpha, y^\alpha, u_i^\alpha) \cap w\text{Min}_{C_i(x^\alpha, y^\alpha)}F_i(x^\alpha, y^\alpha, S_i(x^\alpha, y^\alpha)) \text{ for each } \alpha. \tag{12}$$

Let $K = \{(x^\alpha, y^\alpha, u_i^\alpha) : \alpha \in \Lambda\} \cup \{(x, y, u_i)\}$. Then $K$ is a compact set. By conditions (iii), $F_i(K)$ is a compact set in $Z_i$. By (12), there exists a subnet $\{b_i^\beta\}$ of $\{b_i^\alpha\}$ such that $b_i^\beta \to b_i \in F_i(K)$.

Since $b_i^\beta \in F_i(x^\beta, y^\beta, u_i^\beta)$ and $F_i$ is closed, $b_i \in F_i(x, y, u_i)$. Since $b_i^\beta \in F_i(x^\beta, y^\beta, u_i^\beta)$, $b_i \in F_i(x, y, u_i)$.
We need to show $b_i \in wMin_{C_i(x,y)} F_i(x, y, S_i(x, y))$.

For each $c_i \in F_i(x, y, S_i(x, y))$, we have $d_i \in S_i(x, y)$ such that

$$c_i \in F_i(x, y, d_i).$$

Since $S_i$ is l.s.c. and $d_i \in S_i(x, y)$, there is a net $\{d_i^\alpha\}$ such that $d_i^\alpha \in S_i(x^\alpha, y^\alpha)$ and $d_i^\alpha \to d_i$. Since $F_i$ is l.s.c., and $c_i \in F_i(x, y, d_i)$, there is a net $\{c_i^\alpha\}$ such that

$$c_i^\alpha \in F_i(x^\alpha, y^\alpha, d_i^\alpha)$$

and $c_i^\alpha \to c_i$. (13)

By (12) and (13),

$$c_i^\alpha - b_i^\alpha \notin -intC_i(x^\alpha, y^\alpha)$$

$$\Leftrightarrow b_i^\alpha - c_i^\alpha \in Z_i \setminus intC_i(x^\alpha, y^\alpha) = W_i(x^\alpha, y^\alpha)$$

By condition (ii), $W_i$ is a closed map and then $b_i - c_i \in W_i(x, y)$. Therefore, $c_i - b_i \notin (-intC_i(x, y))$ for all $c_i \in F_i(s, y, S_i(x, y))$ and

$$b_i \in wMin_{C_i(x,y)} F_i(x, y, S_i(x, y)).$$

By (14) and $b_i \in F_i(x, y, u_i)$, $b_i \in F_i(x, y, u_i) \cap wMin_{C_i(x,y)} F_i(x, y, S_i(x, y))$. Since $u_i \in S_i(x, y)$, $u_i \in \Phi_i(x, y)$ and $(x, y, u_i) \in Gr \Phi_i$. Therefore, $\Phi_i : X \times Y \to D_i$ is a closed map for each $i \in I$, it follows that $\Phi_i$ is u.s.c.

Since $\Phi_i$ is closed, $\Phi_i(x, y)$ is a closed set for each $(x, y) \in X \times Y$ and each $i \in I$. Similarly, for each $i \in I$, $\Psi_i$ is u.s.c. and $\Psi(x, y)$ is a closed set for each $(x, y) \in X \times Y$ and each $i \in I$.

For each $i \in I$, define the multivalued map $A_i : X \times Y \to D_i \times M_i$ by

$$A_i(x, y) = (\Phi_i(x, y), \Psi_i(x, y))$$

for all $(x, y) \in X \times Y$.

Then for each $i \in I$, $A_i$ is u.s.c. with nonempty compact convex values. Let $D = \prod_{i \in I} D_i$ and $M = \prod_{i \in I} M_i$. The multivalued map $A : X \times Y \to D \times M$ defined by $A(x, y) = \prod_{i \in I} A_i(x, y)$ is u.s.c. with nonempty compact convex values. By Himmelberg fixed point theorem [6], there exists a point $(\bar{x}, \bar{y}) \in D \times M$ such that $(\bar{x}, \bar{y}) \in A(\bar{x}, \bar{y})$. This means for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x}, \bar{y})$, $F_i(\bar{x}, \bar{y}, \bar{x}_i) \cap wMin_{C_i(x,y)} F_i(\bar{x}, \bar{y}, S_i(\bar{x}, \bar{y})) \neq \emptyset$ and $G_i(\bar{x}, \bar{y}, \bar{y}_i) \cap wMin_{C_i(x,y)} G_i(\bar{x}, \bar{y}, T_i(\bar{x}, \bar{y})) \neq \emptyset$.

Then there exists $b \in F_i(\bar{x}, \bar{y}, \bar{x}_i)$ such that for each $c \in F_i(\bar{x}, \bar{y}, S_i(\bar{x}, \bar{y}))$,

$$c - b \notin -intC_i(\bar{x}, \bar{y})$$

If $F_i(x, y, x_i) \subseteq C_i(x, y)$, it is easy to see that Therefore, $F_i(\bar{x}, \bar{y}, x_i) \cap (-intC_i(\bar{x}, \bar{y})) = \emptyset$ for all $x_i \in S_i(\bar{x}, \bar{y})$ and $G_i(\bar{x}, \bar{y}, y_i) \cap (-intC_i(\bar{x}, \bar{y})) = \emptyset$ for all $y_i \in T_i(\bar{x}, \bar{y})$.

Remark 3.1. Theorem 3.1 is still true if condition (iii) is replaced by
(iii') $F_i$ is a continuous multivalued map with compact values and for any fixed $(x, y) \in X \times Y$, $F_i(x, y, u_i)$ is $C(x, y)$ quasiconvex in $u_i$.

With the same arguments as Theorem 3.1, we have the following theorem.

**Theorem 3.2.** In theorem 3.1, if the condition (iii) of Theorem 3.1 is replaced by

(iii') $F_i : X \times Y \times X_i \to Z_i$ is a continuous multivalued map with nonempty compact values such that for any fixed $(x, y) \in X \times Y$, $F_i(x, y, u_i)$ is properly quasiconcave in $u_i$.

Then there exists a $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x}, \bar{y})$, $F_i(\bar{x}, \bar{y}, \bar{x}_i) \cap \text{wMax}_{C_i(x, y)} F_i(\bar{x}, \bar{y}, S_i(\bar{x}, \bar{y})) \neq \emptyset$ and $G_i(\bar{x}, \bar{y}, \bar{y}_i) \cap \text{wMin}_{C_i(x, y)} G_i(\bar{x}, \bar{y}, T_i(\bar{x}, \bar{y})) \neq \emptyset$.

**Corollary 3.1.** If conditions (iii) and (iv) of Theorem 3.1 is replaced by (iii)' and (iv)' respecteitly, where

(iii)' $f_i : X \times Y \times X_i \to Z_i$ is a continuous function such that for all $x = (x_i)_{i \in I} \in X$ and $y \in Y$, $f_i(x, y, x_i) \in C_i(x, y)$ and for any fixed $(x, y) \in X \times Y$, the map $u_i \mapsto f_i(x, y, u_i)$ is properly quasiconvex.

(iv)' $g_i : X \times Y \times Y_i \to Z_i$ is a continuous function such that for all $x \in X$ and $y = (y_i)_{i \in I} \in Y$, $g_i(x, y, y_i) \in C_i(x, y)$ and for any fixed $(x, y) \in X \times Y$, the map $u_i \mapsto g_i(x, y, u_i)$ is properly quasiconvex.

Then there exists a $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x}, \bar{y})$, $f_i(\bar{x}, \bar{y}, \bar{x}_i) \notin (-\text{int}C_i(\bar{x}, \bar{y}))$ for all $x_i \in S_i(\bar{x}, \bar{y})$ and $g_i(\bar{x}, \bar{y}, \bar{y}_i) \notin (-\text{int}C_i(\bar{x}, \bar{y}))$ for all $y_i \in T_i(\bar{x}, \bar{y})$.

**Corollary 3.2.** In Theorem 3.1, if we assume that (i), (ii) and

(iii) $F_i : X \times Y \times X_i \to Z_i$ is a continuous multivalued map with nonempty compact values such that for all $x \in X$ and $y \in Y$, $F_i(x, y, x_i) \subset C_i(x, y)$, and for any fixed $(x, y) \in X \times Y$, $F_i(x, y, u_i)$ is properly quasiconvex in $u_i$.
Then there exists a \((\bar{x}, \bar{y}) \in X \times Y\) such that for each \(i \in I\), \(\bar{x}_i \in S_i(\bar{x}, \bar{y})\), \(\bar{y}_i \in T_i(\bar{x}, \bar{y})\) and \(F_i(\bar{x}, \bar{y}, x_i) \cap (-\text{int}C_i(\bar{x}, \bar{y})) = \emptyset\) for all \(x_i \in S_i(\bar{x}, \bar{y})\).

**Corollary 3.3.** In Theorem 3.1, if we assume (i) (ii) and

(iii) \(G_i : X \times Y \times Y_i \rightarrow Z_i\) is a continuous multivalued map with nonempty compact values such that for any fixed \((x, y) \in X \times Y\), \(G_i(x, y, v_i)\) is properly quasiconvex in \(v_i\).

Then there exists a \((\bar{x}, \bar{y}) \in X \times Y\) such that for each \(i \in I\), \(\bar{x}_i \in S_i(\bar{x}, \bar{y})\), \(\bar{y}_i \in T_i(\bar{x}, \bar{y})\), and \(G_i(\bar{x}, \bar{y}, \bar{y}_i) \cap w\text{Min}_{C_i(\bar{x}, \bar{y})} G_i(\bar{x}, \bar{y}, T_i(\bar{x}, \bar{y})) \neq \emptyset\).

4 Applications to Systems of Loose Quasi-Saddle Point Problem and Constrained Competitive Nash-Type Equilibrium Problems

**Theorem 4.1.** Let \(I, F_i, V_i, Z_i, X_i, Y_i, X, Y, S_i\) and \(T_i\) be the same as in Theorem 3.1. Suppose that conditions (i), (ii) of theorem 3.1 are true. Suppose that

(iii) \(L_i : X_i \times Y_i \rightarrow Z_i\) is a continuous multivalued map with nonempty compact values;

(a) for any fixed \(y_i \in Y_i\), \(L_i(x_i, y_i)\) is properly quasiconcave in \(x_i\); and

(b) for any fixed \(x_i \in X_i\), \(L_i(x_i, y_i)\) is properly quasiconvex in \(y_i\).

Then there exists a \(\bar{x} = (\bar{x}_i)_{i \in I} \in X\) and \(\bar{y} = (\bar{y}_i)_{i \in I} \in Y\) such that for each \(i \in I\), \(\bar{x}_i \in S_i(\bar{x}, \bar{y})\), \(\bar{y}_i \in T_i(\bar{x}, \bar{y})\), \(L_i(\bar{x}_i, \bar{y}_i) \cap \text{wMax}_{C_i(\bar{x}, \bar{y})} L_i(S_i(\bar{x}, \bar{y}), \bar{y}_i) \neq \emptyset\) and \(L_i(\bar{x}_i, \bar{y}_i) \cap \text{wMin}_{C_i(\bar{x}, \bar{y})} L_i(\bar{x}_i, T_i(\bar{x}, \bar{y})) \neq \emptyset\).

**Proof.** For each \(i \in I\), let \(F_i(x, y, u_i) = L_i(u_i, y_i)\) and \(G_i(x, y, v_i) = L_i(x_i, v_i)\).

Then Theorem 4.1 follows from Theorem 3.2.

If \(L_i\) is a single valued map, we have the following systems of vector quasi-saddle point problem.
For each $i \in I$, let $S_i : X \to X_i$ be a continuous compact multivalued map with nonempty closed convex values and $T_i : Y \to Y_i$ be a continuous compact multivalued map with nonempty closed convex values. For each $i \in I$, assume the following conditions are satisfied.

(i) $C_i : X \to Z_i$ is a multivalued map such that for each $x \in X$, $C_i(x)$ is a closed convex pointed cone with apex at the origin and $intC_i(x) \neq \emptyset$;

(ii) the map $W_i : X \to Z_i$ defined by $W_i(x) = Z_i \setminus intC_i(x)$ is u.s.c.;

(iii) $L_i : X_i \times Y_i \to Z_i$ is a continuous map such that

(a) for any fixed $y_i \in Y_i$, $L_i(x_i, y_i)$ is properly quasiconcave in $x_i$; and

(b) for any fixed $x_i \in X_i$, $L_i(x_i, y_i)$ is properly quasiconvex in $y_i$.

Then there exists a $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{y})$, $L_i(\bar{x}_i, \bar{y}_i) - L_i(x_i, \bar{y}_i) \notin (-intC_i(\bar{x}))$ for all $x_i \in S_i(\bar{x})$. and $L_i(\bar{x}, y_i) - L_i(\bar{x}_i, \bar{y}_i) \notin (-intC_i(\bar{x}))$ for all $y_i \in T_i(\bar{y})$.

References


