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Kyoto University
Iterative methods for infinite families of nonexpansive mappings in Banach spaces

1 Introduction

Throughout this paper, let $E$ be a real Banach space with norm $\| \cdot \|$ and let $\mathbb{N}$ be the set of all positive integers. Let $C$ be a nonempty closed convex subset of $E$. Then, a mapping $T: C \to C$ is called nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|$$

for all $x, y \in C$. Browder [4] considered a sequence $\{x_n\}$ as follows:

$$x \in C, \quad x_n = \alpha_n x + (1 - \alpha_n)Tx_n \quad (\forall n \in \mathbb{N}), \quad (1.1)$$

where $\{\alpha_n\} \subset (0, 1)$ and he proved the first strong convergence theorem in the framework of a Hilbert space. Later, Reich [29], Takahashi and Ueda [51], Shioji and Takahashi [39], Nakajo [21] and others also proved strong convergence theorems of Browder’s type in Hilbert spaces or Banach spaces. On the other hand, Halpern [9] considered the following process: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n \quad (\forall n \in \mathbb{N}), \quad (1.2)$$

where $\{\alpha_n\} \subset [0, 1)$. Wittmann [52] proved a strong convergence theorem of Halpern’s type in the framework of a Hilbert space and then, several authors [2, 10, 11, 12, 13, 14, 17, 21, 33, 35, 38, 39, 40, 50] proved strong convergence theorems of Halpern’s type in Hilbert spaces or Banach spaces. Recently, Moudafi [20] and Xu [53] considered the following process by the viscosity approximation method: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n \quad (\forall n \in \mathbb{N}), \quad (1.3)$$

where $\{\alpha_n\} \subset [0, 1)$ and $f: C \to C$ is a contraction.

In this article, for an infinite family $\{T_n\}$ of nonexpansive mappings of $C$ into itself such that $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$, we consider a sequence $\{x_n\}$ generated by

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n \quad (\forall n \in \mathbb{N}),$$
where \(\{\alpha_n\} \subset (0,1)\) and \(f : C \to C\) is a contraction. Then, we give the conditions of \(\{\alpha_n\}\) and \(\{T_n\}\) under which \(\{x_n\}\) converges strongly to a common fixed point of \(\cap_{n=1}^{\infty} F(T_n)\). We also consider a sequence \(\{x_n\}\) generated by

\[
x_1 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \quad (\forall n \in \mathbb{N}),
\]

where \(\{\alpha_n\} \subset [0,1)\) and \(f : C \to C\) is a contraction. Then, we give the conditions of \(\{\alpha_n\}\) and \(\{T_n\}\) under which \(\{x_n\}\) converges strongly to a common fixed point of \(\cap_{n=1}^{\infty} F(T_n)\). Using these results, we improve and extend well-known strong convergence theorems.

2 Preliminaries

Let \(E\) be a real Banach space with norm \(\| \cdot \|\) and let \(E^*\) denote the dual of \(E\). We denote the value of \(y^* \in E^*\) at \(x \in E\) by \(\langle x, y^* \rangle\). The duality mapping \(J\) from \(E\) into \(2^{E^*}\) is defined by

\[
Jx = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}
\]

for every \(x \in E\). Let \(U = \{x \in E : \|x\| = 1\}\). The norm of \(E\) is said to be Gâteaux differentiable if for each \(x, y \in U\), the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists. In the case, \(E\) is called smooth. The norm of \(E\) is said to be uniformly Gâteaux differentiable if for each \(y \in U\), the limit (2.1) is attained uniformly for \(x \in U\). We know that if \(E\) is smooth, then the duality mapping \(J\) is single valued. Further, if the norm of \(E\) is uniformly Gâteaux differentiable, then \(J\) is norm to weak* uniformly continuous on each bounded subset of \(E\). Let \(C\) be a closed convex subset of \(E\). A mapping \(T : C \to C\) is said to be nonexpansive if \(\|Tx - Ty\| \leq \|x - y\|\) for all \(x, y \in C\). We denote by \(F(T)\) the set of all fixed points of \(T\). Let \(J\) denote the identity operator on \(E\). An operator \(A \subset E \times E\) with domain \(D(A) = \{x \in E : Ax \neq \emptyset\}\) and range \(R(A) = \bigcup\{Az : z \in D(A)\}\) is said to be accretive if for each \(x_i \in D(A)\) and \(y_i \in Ax_i, i = 1, 2\), there exists \(j \in J(x_1 - x_2)\) such that \(\langle y_1 - y_2, j \rangle \geq 0\). If \(A\) is accretive, then we have

\[
\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|
\]

for all \(r > 0\) and \(y_i \in Ax_i, i = 1, 2\). If \(A\) is accretive, then we can define, for each \(r > 0\), a nonexpansive single valued mapping \(J_r : R(I + rA) \to D(A)\) by \(J_r = (I + rA)^{-1}\). It is called the resolvent of \(A\). We also define the Yosida approximation \(A_r\) by \(A_r = (I - J_r)/r\). We know that \(A_r x \in AJ_r x\) for all \(x \in R(I + rA)\) and \(\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}\) for all \(x \in D(A) \cap R(I + rA)\). We also know that for an accretive operator \(A\), \(A^{-1} 0 = F(J_r)\) for all \(r > 0\), where \(A^{-1} 0 = \{u \in E : 0 \in Au\}\). An accretive operator \(A\) is said to be \(m\)-accretive if \(R(I + rA) = E\) for all \(r > 0\). A closed convex subset \(C\) of a Banach space \(E\) is said to have normal structure if for each bounded closed convex subset of \(K\) of \(C\) which contains at least two points, there exists an element \(x\) of \(K\) which is not a diametral point of \(K\), i.e.,

\[
\sup\{\|x - y\| : y \in K\} < \delta(K),
\]

where \(\delta(K)\) is the diameter of \(K\). It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure; see [44] for more details. The following result was proved by Kirk [18].
Theorem 2.1 (Kirk [18]). Let $E$ be a reflexive Banach space and let $C$ be a nonempty bounded closed convex subset of $E$ which has normal structure. Let $T$ be a nonexpansive mapping of $C$ into itself. Then $F(T)$ is nonempty.

A closed convex subset $C$ of a Banach space $E$ is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a nonempty bounded closed convex subset of $K$ of $C$ into itself has a fixed point in $K$. If $C$ is a closed convex subset of a reflexive Banach space which has normal structure, from Theorem 2.1, $C$ has the fixed point property for nonexpansive mappings.

We denote by $N$ the set of all natural numbers and let $\mu$ be a mean on $N$, i.e., a continuous linear functional on $\ell^\infty$ satisfying $\|\mu\| = 1 = \mu(1)$. We know that $\mu$ is a mean on $N$ if and only if

$$\inf_{n\in N} a_n \leq \mu(f) \leq \sup_{n\in N} a_n$$

for each $f = (a_1, a_2, \ldots) \in \ell^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. Let $f = (a_1, a_2, \ldots) \in \ell^\infty$ with $a_n \to a$ and let $\mu$ be a Banach limit on $N$. Then $\mu(f) = \mu_n(a_n) = a$; see [44] for more details. Further, we know the following result [51].

Theorem 2.2 (Takahashi and Ueda [51]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ with a uniformly Gâteaux differentiable norm, let $\{x_n\}$ be a bounded sequence of $E$ and let $\mu$ be a mean on $N$. Let $z \in C$. Then

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if $\mu_n \langle y - z, J(x_n - z) \rangle \leq 0$ for all $y \in C$, where $J$ is the duality mapping of $E$.

Let $C$ be a nonempty subset of a Banach space $E$. Let $D$ be a subset of $C$ and let $P$ be a retraction of $C$ onto $D$, i.e., $Px = x$ for each $x \in D$. Then $P$ is said to be sunny [28] if for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$,

$$P(Px + t(x - Px)) = Px.$$ 

A subset $D$ of $C$ is said to be a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction $P$ of $C$ onto $D$. We know that if $E$ is smooth and $P$ is a retraction of $C$ onto $D$, then $P$ is sunny and nonexpansive if and only if for each $x \in C$ and $z \in D$,

$$\langle x - Px, J(z - Px) \rangle \leq 0. \tag{2.2}$$

For more details, see [44].

3 Conditions for infinite families

Let $E$ be a Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_n\}$ and $T$ be families of nonexpansive mappings of $C$ into itself such that $\emptyset \neq F(T) = \cap_{n=1}^\infty F(T_n)$, where $F(T_n)$ is the set of all fixed points of $T_n$ and $F(T)$ is the set of all common fixed points of $T$. Then, $\{T_n\}$ is said to satisfy the condition (I) with $T$ if for each bounded sequence $\{z_n\}$ in $C$,

$$\lim_{n \to \infty} \|z_n - T_n z_n\| = 0$$
implies that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ for all $T \in T$. In particular, if $T = \{T\}$, i.e., $T$ consists of one mapping $T$, then $\{T_n\}$ is said to satisfy the condition (I) with $T$. $\{T_n\}$ is said to satisfy the condition (II) if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n \to \infty} \|z_{n+1} - T_nz_n\| = 0$$

implies that $\lim_{n \to \infty} \|z_n - T_mz_n\| = 0$ for all $m \in \mathbb{N}$. $\{T_n\}$ is said to satisfy the condition (III) if for every bounded subset $B$ of $C$,

$$\sum_{n=1}^{\infty} \sup \{\|T_n x - T_{n+1} x\| : x \in B\} < \infty.$$

**Proposition 3.1.** Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. Then, $\{T_n\}$ with $T_n = T$ for all $n \in \mathbb{N}$ satisfies the condition (I) with $T$ and the condition (III).

**Proof.** Put $T_n = T$ for all $n \in \mathbb{N}$. Then, it is obvious that $\{T_n\}$ satisfies the condition (I) with $T$ and the condition (III). \qed

**Theorem 3.2** ([24]). Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ and let $S$ and $T$ be nonexpansive mappings of $C$ into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ with $a \leq b$. Then, $\{T_n\}$ with $T_n = \gamma_n S + (1 - \gamma_n) T$ for all $n \in \mathbb{N}$ satisfies the condition (I) with $S + T$. Further, $\{T_n\}$ with $T_n = \gamma_n S + (1 - \gamma_n) T$ for all $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$ satisfies the condition (I) with $\frac{S + T}{2}$ and the condition (III).

The following lemma is related to Edelstein and O'Brien [6, Theorem 1].

**Lemma 3.3** ([48]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers with $0 < a \leq \beta_n \leq b < 1$ and let $B$ be a nonempty bounded subset of $C$. Define a mapping $S_n$ of $C$ into itself by

$$S_n x = S(\beta_n)x = (1 - \beta_n)x + \beta_n T x$$

and put $a_n = \sup_{x \in B} \|TS^n x - S^n x\|$ for all $n \in \mathbb{N}$, where $S^n = S_n S_{n-1} \cdots S_1$. Then, $a_n \to 0$. In particular, for any $m \in \mathbb{N}$,

$$\lim_{n \to \infty} \sup_{x \in B} \|S_m S^n x - S^n x\| = 0.$$

The following lemma was also proved by Takahashi [48].

**Lemma 3.4** ([48]). Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. For a nonempty bounded subset $B$ of $C$ and $n \in \mathbb{N}$, define a mapping $f_n$ of $[0, 1]^n$ into $(-\infty, \infty)$ by

$$f_n(\beta_n, \beta_{n-1}, \ldots, \beta_1) = \sup_{x \in B} \|TU^n x - U^n x\|$$

for all $(\beta_n, \beta_{n-1}, \ldots, \beta_1) \in [0, 1]^n$, where $U^n = S(\beta_n) S(\beta_{n-1}) \cdots S(\beta_1)$ and

$$S(\beta_k)x = (1 - \beta_k)x + \beta_k T x$$

for all $x \in C$ and $k \in \{1, 2, \ldots, n\}$. Then, $f_n$ is continuous.
Using Lemmas 3.3 and 3.4 we obtain the following theorem.

**Theorem 3.5 ([48]).** Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. For any $n \in \mathbb{N}$ and $\beta_n \in \mathbb{R}$ with $0 < a \leq \beta_n \leq b < 1$, define $S_n : C \to C$ as follows:

$$S_n x = (1 - \beta_n)x + \beta_n Tx \quad \text{for all } x \in C.$$  

Then, $\{S_n\}$ satisfies the condition (I) with $T$ and the condition (II).

We know the following lemma for resolvents of accretive operators; see [44].

**Lemma 3.6.** Let $E$ be a Banach space and let $A \subset E \times E$ be an accretive operator. Let $r, \lambda > 0$ and $D(A) \subset R(I + \lambda A)$. Then,

$$\frac{1}{\lambda} \|J_r x - J_{\lambda}J_r x\| \leq \frac{1}{r} \|x - J_r x\|$$

for every $x \in R(I + rA)$.

Using Lemma 3.6, we also have the following theorem.

**Theorem 3.7 ([48]).** Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $A \subset E \times E$ be an accretive operator such that

$$\overline{D(A)} \subset C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$$

and $A^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \in (0, \infty)$ and $\lim_{n \to \infty} \lambda_n = \infty$. Define $S_n = J_{\lambda_n}$ for any $n \in \mathbb{N}$. Then, $\{S_n\}$ satisfies the condition (I) with $J_1$ and the condition (II), where $J_1 = (I + A)^{-1}$. Moreover, $\{T_n\}$ with $T_n = J_{\lambda_n}$ ($\forall n \in \mathbb{N}$) such that $\{\lambda_n\} \subset (0, \infty)$, $\lim \inf_{n \to \infty} \lambda_n > 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ satisfies the condition (I) with $\{J_1\}$ and the condition (III).

Let $C$ be a nonempty closed convex subset of $E$. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of $C$ into itself and let $\beta_1, \beta_2, \ldots$ be real numbers such that $0 \leq \beta_i \leq 1$ for every $i \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, Takahashi [43] (see also [34, 45, 49]) introduced a mapping $W_n$ of $C$ into itself as follows:

$$U_{n,n+1} = I,$$

$$U_{n,n} = \beta_n S_n U_{n,n+1} + (1 - \beta_n)I,$$

$$U_{n,n-1} = \beta_{n-1} S_{n-1} U_{n,n} + (1 - \beta_{n-1})I,$$

$$\vdots$$

$$U_{n,k} = \beta_k S_k U_{n,k+1} + (1 - \beta_k)I,$$

$$\vdots$$

$$U_{n,2} = \beta_2 S_2 U_{n,3} + (1 - \beta_2)I,$$

$$W_n = U_{n,1} = \beta_1 S_1 U_{n,2} + (1 - \beta_1)I.$$  

Such a mapping $W_n$ is called the $W$-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$. We know that if $E$ is strictly convex, $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $0 < \beta_i < 1$ for every $i = 2, 3, \ldots, n$ and $0 < \beta_1 \leq 1$, then $F(W_n) = \cap_{i=1}^{n} F(S_i)$; see [45, 49]. We also have
that if $E$ is strictly convex, $\cap_{n=1}^{\infty}F(S_n) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0, 1)$, then, $\lim_{n \to \infty} U_{n,k}x$ exists for every $x \in C$ and $k \in \mathbb{N}$; see [34]. So, we can define a mapping $W$ of $C$ into itself as follows:

$$Wx = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1}x$$

for every $x \in C$. Such a $W$ is called the $W$-mapping generated by $S_1, S_2, \ldots$ and $\beta_1, \beta_2, \ldots$. We have that if $E$ is strictly convex, $\cap_{n=1}^{\infty}F(S_i) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0, 1)$, then, $F(W) = \cap_{i=1}^{\infty}F(S_i)$; see [34]. We know the following result for the $W$-mappings.

**Theorem 3.8 ([24]).** Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of $C$ into itself with $\cap_{n=1}^{\infty}F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \ldots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0, 1)$. Let $W_n$ be the $W$-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$ for every $n \in \mathbb{N}$ and let $W$ be the $W$-mapping generated by $S_1, S_2, \ldots$ and $\beta_1, \beta_2, \ldots$. Then, $\{T_n\}$ with $T_n = W_n \ (\forall n \in \mathbb{N})$ satisfies the condition (I) with $W$ and the condition (III).

### 4 Strong convergence theorem of Browder's type

We can prove a strong convergence theorem of Browder's type for a countable family of nonexpansive mappings in a Banach space.

**Theorem 4.1 ([48]).** Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let $C$ be a nonempty closed convex subset of $E$ which has the fixed point property for nonexpansive mappings. Let $T$ be a nonexpansive mapping of $C$ into itself and let $\{T_n\}$ be a family of nonexpansive mappings of $C$ into itself which satisfies $\emptyset \neq F(T) = \bigcap_{i=1}^{\infty}F(T_i)$. Further, suppose that $\{T_n\}$ satisfies the condition (I) with $T$. Define a sequence $\{x_n\}$ in $C$ as follows:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n, \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$ and $f$ is a contraction of $C$ into itself. Then, $\{x_n\}$ converges strongly to $u \in F(T)$, where $u = P_{F(T)} f(u)$ and $P_{F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(T)$.

We have the following result for nonexpansive mappings by Proposition 3.1 and Theorem 4.1.

**Theorem 4.2.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)T x_n \ (\forall n \in \mathbb{N})$, where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(T)$.

We also get the following result for nonexpansive mappings by Theorems 3.2 and 4.1.

**Theorem 4.3.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $S$ and $T$ be nonexpansive mappings of $C$ into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)(\gamma_n S x_n + (1 - \gamma_n)T x_n) \ (\forall n \in \mathbb{N})$, where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$ and $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ with $a \leq b$. Then, $\{x_n\}$ converges strongly to $P_{F(S) \cap F(T)}x$, where $P_{F(S) \cap F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(S) \cap F(T)$.
We have the following result for accretive operators from Theorems 3.7 and 4.1.

**Theorem 4.4.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $A \in C \times E$ be an accretive operator with $D(A) \subset C \subset \cap_{\lambda > 0} R(I + \lambda A)$ and $A^{-1}0 \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)J_{\lambda_n} x_n \ (\forall n \in \mathbb{N})$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$. If $\lim_{n \to \infty} \alpha_n = \infty$, $\{x_n\}$ converges strongly to $P_{A^{-1}0} x$, where $P_{A^{-1}0}$ is a sunny nonexpansive retraction of $C$ onto $A^{-1}0$.

We get the following result for the $W$-mappings from Theorems 3.8 and 4.1.

**Theorem 4.5.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of $C$ into itself with $F := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \ldots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \mathbb{N}$ for some $b \in (0, 1)$. Let $W_n$ be the $W$-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_1, \beta_2, \ldots, \beta_n$ for every $n \in \mathbb{N}$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n) W_n x_n \ (\forall n \in \mathbb{N})$, where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$, where $P_F$ is a sunny nonexpansive retraction of $C$ onto $F$.

5 Strong convergence theorem of Halpern's type

In this section, we prove two strong convergence theorems of Halpern's type for a countable family of nonexpansive mappings in a Banach space.

**Theorem 5.1** ([48]). Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let $C$ be a nonempty closed convex subset of $E$ which has the fixed point property for nonexpansive mappings. Let $T$ be a nonexpansive mapping of $C$ into itself and let $\{T_n\}$ be a family of nonexpansive mappings of $C$ into itself which satisfy $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Further, suppose that $\{T_n\}$ satisfies the condition (I) with $T$ and the condition (II). Let $\{x_n\}$ be a sequence in $C$ as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $f$ is a contraction of $C$ into itself. Then, $\{x_n\}$ converges strongly to $u \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, where $u = Pf(u)$ and $P$ is a sunny nonexpansive retraction of $C$ onto $F(T)$.

Using Theorems 3.5 and 5.1, we obtain the following result:

**Theorem 5.2** ([48]). Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let $C$ be a nonempty closed convex subset of $E$ which has the fixed point property for nonexpansive mappings and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty and let $f$ be a contraction of $C$ into itself. Define a sequence $\{x_n\}$ of $C$ as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - \beta_n)x_n + \beta_n T x_n) \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1.$$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of $T$. 

Theorem 5.2 improves and extends Suzuki’s result [42]. Using Theorems 3.7 and 5.1, we also obtain the following result which was proved by Takahashi [47].

**Theorem 5.3 (47).** Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let $C$ be a nonempty closed convex subset of $E$ which has the fixed point property for nonexpansive mappings. Let $A \subseteq E \times E$ be an accretive operator with $A^{-1}0 \neq \emptyset$ satisfying

$$D(A) \subseteq C \subseteq \bigcap_{t>0} R(I + tA),$$

where $D(A)$ is the closure of $D(A)$ and let $f$ be a contraction of $C$ into itself. Let $\{x_n\}$ be a sequence of $C$ defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J_{t_n}x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad t_n \to \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to $u \in A^{-1}0$, where $u = Pf(u)$ and $P$ is a sunny nonexpansive retraction of $C$ onto $A^{-1}0$.

**Theorem 5.4 (24).** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $\{T_n\}$ and $T$ be families of nonexpansive mappings of $C$ into itself which satisfy $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty}F(T_n)$. Further, suppose that $\{T_n\}$ satisfies the condition (I) with $T$ and the condition (III). Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n(\beta_n x + (1 - \beta_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\beta_n\} \subset [0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$ and \(\prod_{n=1}^{\infty}(1 - \alpha_n)(1 - \beta_n) = 0\). If $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(T)$.

Using Proposition 3.1 and Theorem 5.4, we obtain the following theorem:

**Theorem 5.5.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\beta_n x + (1 - \beta_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\beta_n\} \subset [0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1 - \alpha_n)(1 - \beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(T)$.

We have the following result [17] for nonexpansive mappings by Theorems 3.2 and 5.4.

**Theorem 5.6.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $S$ and $T$ be nonexpansive mappings of $C$ into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)(\gamma_n S + (1 - \gamma_n)T)(\beta_n x + (1 - \beta_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$. If $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}| + |\gamma_n - \gamma_{n+1}|) < \infty$, then $\{x_n\}$ converges strongly to $P_{F(S) \cap F(T)}x$, where $P_{F(S) \cap F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(S) \cap F(T)$. 

**Theorem 5.7.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ whose norm is uniformly Gâteaux differentiable and let $S$ and $T$ be nonexpansive mappings of $C$ into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)(\gamma_n S + (1 - \gamma_n)T)(\beta_n x + (1 - \beta_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$. If $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}| + |\gamma_n - \gamma_{n+1}|) < \infty$, then $\{x_n\}$ converges strongly to $P_{F(S) \cap F(T)}x$, where $P_{F(S) \cap F(T)}$ is a sunny nonexpansive retraction of $C$ onto $F(S) \cap F(T)$.
where \( \{\alpha_n\} \subset [0,1) \) and \( \{\beta_n\} \subset [0,1) \) satisfy \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0 \), \( \prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = 0 \) and \( \sum_{n=1}^{\infty} \left( |\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}| \right) < \infty \) and \( \{\gamma_n\} \subset [a,b] \) for some \( a, b \in (0,1) \) with \( a \leq b \) satisfies \( \sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty \). Then, \( \{x_n\} \) converges strongly to \( P_{F(S) \cap F(T)}x \), where \( P_{F(S) \cap F(T)} \) is a sunny nonexpansive retraction of \( C \) onto \( F(S) \cap F(T) \).

We have the following result [21] for accretive operators from Theorems 3.7 and 5.4.

**Theorem 5.7.** Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( E \) whose norm is uniformly Gâteaux differentiable and let \( A \subset E \times E \) be an accretive operator with \( D(A) \subset C \subset \cap_{\lambda > 0} R(I + \lambda A) \) and \( A^{-1} \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated as follows: \( x_1 = x \in C \) and

\[
    x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{\lambda_n}(\beta_n x + (1 - \beta_n)x_n) \quad (\forall n \in \mathbb{N}),
\]

where \( \{\alpha_n\} \subset [0,1) \) and \( \{\beta_n\} \subset [0,1) \) satisfy \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0 \), \( \prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = 0 \) and \( \sum_{n=1}^{\infty} \left( |\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}| \right) < \infty \) and \( \{\lambda_n\} \subset (0,\infty) \) satisfies \( \lim_{n \to \infty} \lambda_n > 0 \) and \( \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty \). Then, \( \{x_n\} \) converges strongly to \( P_{A^{-1}0}x \), where \( P_{A^{-1}0} \) is a sunny nonexpansive retraction of \( C \) onto \( A^{-1}0 \).

We get the following result [34] for \( W \)-mappings by Theorems 3.8 and 5.4.

**Theorem 5.8.** Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( E \) whose norm is uniformly Gâteaux differentiable. Let \( S_1, S_2, \ldots \) be infinite nonexpansive mappings of \( C \) into itself with \( F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset \) and let \( \beta_1, \beta_2, \ldots \) be real numbers with \( 0 < \beta_i < 1 \) for every \( i \in \mathbb{N} \) for some \( b \in (0,1) \). Let \( W_n \) be the \( W \)-mapping generated by \( S_n, S_{n-1}, \ldots, S_1 \) and \( \beta_n, \beta_{n-1}, \ldots, \beta_1 \) for every \( n \in \mathbb{N} \). Let \( \{x_n\} \) be a sequence generated as follows: \( x_1 = x \in C \) and

\[
    x_{n+1} = \alpha_n x + (1 - \alpha_n)W_n(\gamma_n x + (1 - \gamma_n)x_n) \quad (\forall n \in \mathbb{N}),
\]

where \( \{\alpha_n\} \subset [0,1) \) and \( \{\gamma_n\} \subset [0,1) \) satisfy \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \gamma_n = 0 \), \( \prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \gamma_n) = 0 \) and \( \sum_{n=1}^{\infty} \left( |\alpha_n - \alpha_{n+1}| + |\gamma_n - \gamma_{n+1}| \right) < \infty \). Then, \( \{x_n\} \) converges strongly to \( P_Fx \), where \( P_F \) is a sunny nonexpansive retraction of \( C \) onto \( F \).

**References**


