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On the nonlinear ratio ergodic theorem for order preserving operators in Lebesgue space

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Abstract

We present a pointwise nonlinear ergodic theorem of ratio type for a special class of nonlinear operators in Lebesgue space.

We will outline our work on convergence almost everywhere of ratio type processes for order preserving operators in Lebesgue space L_p $(1 \leq p \leq \infty)$. All papers on this subject (nonlinear ergodic theorems) dealt only with questions of weak and strong convergence, while the problem of almost sure convergence (which plays a dominant role in the linear ergodic theorems) has been ignored till the appearance of Krengel's example (1987) which shows that the pointwise ergodic theorems of Hopf-type and Akcoglu-type in the linear case fail to extend to the nonlinear case (nonexpansive operators). However, the question concerning the possibility of positive results for some specific class of nonlinear operators remains still open. We are particularly interested in this possibility of positive results and present a ratio result for a special class of nonlinear operators. Indeed, it seems to be significant to find those conditions under which pointwise nonlinear ergodic theorems of Cesàro type or its ratio type hold in L_p.

Now let $L_p = L_p(\Omega, \Xi, \mu)$, $1 \leq p \leq \infty$, be the usual Lebesgue spaces, where (Ω, Ξ, μ) is a $\sigma-$finite measure space. An operator T in L_p is said to be L_p-norm decreasing if $\|Tf\|_p \leq \|f\|_p$ holds for all $f \in L_p$. T is called order preserving in L_p if f, $g \in L_p$ and $f \leq g$ imply $Tf \leq Tg$. T is called nonexpansive in L_p if $\|Tf - Tg\|_p \leq \|f - g\|_p$ holds for all f, $g \in L_p$. We say that T is positively homogeneous if $T(cf) = cTf$ for all $f \in L_p$ and for any constant $c \geq 0$. Next, for a real number $\alpha > -1$ and each integer $n \geq 0$, let A_n^α denote the (C, α) coefficient of order α, which is defined by the generating function

$$\frac{1}{(1-\lambda)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha \lambda^n \qquad (0 < \lambda < 1)$$

$A_0^\alpha = 1$. We also let $A_0^{-1} = 1$ and $A_n^{-1} = 0$ $(n \geq 1)$. Then, for $\alpha > -1$, we have $A_n^\alpha > 0$, $A_n^0 = 1$ $(n \geq 0)$, $A_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$ $(n \to \infty)$ and

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} = \sum_{k=0}^{n} A_k^{\alpha-1} = \frac{(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n)}{n!}$$

Before proceeding to the details of argument, let us take a general view of pointwise ergodic theorems in connection with the problem just mentioned above. After the first appearance of Birkhoff's ergodic theorem (1931), the pointwise ergodic theorem for measure preserving transformations had providently and extensively been studied by Khinchine (1933), Kolmogorov (1937), Hopf (1937), Wiener (1939), Yosida (1939-1940), Kakutani (1939), Doob (1940),Pitt (1942), Dunford (1943), Hurewicz (1944), Hille (1945), Halmos

(1946) and Zygmund (1951). Among others, the sufficient conditions given by Dunford for the validity of pointwise ergodic theorems concerning general linear operators are quite essential in the theory of pointwise ergodic theorems, which are based upon Banach's convergence theorem. This fact says that convergence almost everywhere of operator averages depends essentially on the realization of Dunford's conditions. In this direction, the (first) simplest classical pointwise ergodic theorem was given by Hopf for positive linear operators in L_1.

THE HOPF THEOREM 1 (1954). If T is a positive linear operator in L_1 (where $\mu(\Omega) = 1$) with $T1 = 1$ and $\|T\|_1 = 1$, then $C_n^{(1)}[T]f = \frac{1}{n+1} \sum_{k=0}^{n} T^k f$ converges a.e. for all $f \in L_1$.

THE DUNFORD-SCHWARTZ THEOREM (1956). Let T be a linear operator in L_1 with $\|T\|_1 \leq 1$ and $\|T\|_\infty \leq 1$. Then for every p with $1 \leq p < \infty$ and every $f \in L_p$, the limit $\lim_{n\to\infty} C_n^{(1)}[T]f$ exists almost everywhere.

THE HOPF THEOREM 2 (1960). Let T be a positive linear contraction ($\|T\|_1 \leq 1$) in L_1 and suppose that there is a strictly positive $g \in L_1^+$ with $Tg = g$. Then $C_n^{(1)}[T]f$ converges a.e. for all $f \in L_1$.

THE CHACON-ORNSTEIN THEOREM (1960). Let T be a positive linear contraction in L_1. Then the ratio $\frac{C_n^{(1)}[T]f}{C_n^{(1)}[T]g}$ converges a.e. on $\{\sum_{k=0}^{\infty} T^k g > 0\}$ for all $f \in L_1$ and all $g \in L_1^+$ with $g > 0$.

THE AKCOGLU THEOREM (1966). If T is a positive linear contraction in L_p $(1 < p < \infty)$, then $C_n^{(1)}[T]f$ converges a.e. for all $f \in L_p$.

These theorems just cited above are now recognized as the most fundamental theorems in the theory of pointwise linear ergodic theorems. Here it should be noticed that the pointwise convergence of $C_n^{(1)}[T]f$ $(f \in L_1)$ does not hold in general for positive linear contractions in L_1. We are particularly interested in the question whether there exists the analogy between the linear case and the nonlinear case or not. In this connection, Baillon ([2]) proved the following weak nonlinear ergodic theorem:

THE BAILLON THEOREM (1978). Let C be a bounded closed convex subset of L_p $(1 < p < \infty)$ and let T be a nonexpansive self-mapping of C. Then for every $f \in C$, $C_n^{(1)}[T]f$ converges weakly to a T−fixed point in C.

In 1987, Krengel ([4]) constructed a remarkable example which shows that the pointwise (linear) ergodic theorems of Hopf-type and Akcoglu-type fail to extend to nonlinear (nonexpansive) operators. Especially Krengel' example signifies that the analogy between the linear case and the nonlinear case does not exist as far as pointwise ergodic theorems as convergence almost everywhere of Cesàro-type averages. But, applying Landau-type's Tauberian theorem for Dirichlet series([3]) (which is quite different from the standard argument in ergodic theory), we can make a positive approach to the problem against Krengel's (negative) example.

THEOREM 1 ([7]). Let T be an order preserving operator in L_p $(1 \le p \le \infty)$ with $T(0) = 0$, let $1 \le \alpha < \infty$ and let $f,\ f^* \in L_p^+$. Assume that E is a set in Ξ with $\mu(E) = 0$ such that for any $\omega \in \Omega - E$, (the abscissa of convergence)

$$a_\omega(\alpha; f) = \limsup_{n\to\infty} \frac{\log\ [\sum_{k=0}^{n} A_k^{\alpha-1}(T^k f)(\omega)]}{\log\ A_n^\alpha} \le 1$$

Assume that for any $\omega \in \Omega - (E \cup E_0)$ $(E_0 = \{f^* = \infty\})$, the analytic function

$$G_\omega(z) = \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z} - \frac{f^*(\omega)}{z-1} \quad (\mathrm{Re}(z) > 1, z \in \mathbf{C})$$

has an analytic or just continuous extension (also called $G_\omega(z)$) to the closed half-plane $\{\mathrm{Re}(z) \ge 1\}$. Finally assume that for each $\omega \in \Omega-(E\cup E_0)$, there exists a constant $M_\omega \ge 1$ such that (*) $G_\omega(z) = O(|z|^{M_\omega})$ $(\mathrm{Re}(z) \ge 1)$. Then

(**) $\quad \lim_{n\to\infty} \frac{\sum_{k=0}^{n} A_k^{\alpha-1}(T^k f)(\omega)}{A_n^\alpha} = f^*(\omega)$ holds for almost all $\omega \in \Omega$.

It is worthwhile to emphasize that the growth condition (*) in Theorem 1 is one of some sufficient conditions for the validity of (**). So it may be affirmed that the possibility of positive results remains still without assuming the growth condition (*). In fact, making heavy use of Landau's technique in celebrated Landau-Wiener-Ikehara's Tauberian theorem for Dirichlet series ([3]), we have

THEOREM 2 ([8]). Let T be an order preserving operator in L_p $(1 \le p \le \infty)$ with $T(0) = 0$. Let $0 < \alpha < \infty$ and $f,\ f^* \in L_p$. Assume that E is a set in Ξ with $\mu(E) = 0$ such that for any $\omega \in \Omega - E$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}$ converges (absolutely) for each $z \in \mathbf{C}$ with $\mathrm{Re}(z) > 1$. Assume that for any $\omega \in \Omega - (E \cup E_0)$ $(E_0 = \{f^* = \infty\})$, the analytic function $G_\omega(z)$ has an analytic or just continuous extension (also called $G_\omega(z)$) to the closed half-plane $\{\mathrm{Re}(z) \ge 1\}$. Then (**) holds for almost all $\omega \in \Omega$.

Now, using an argument similar to that used in the proof of Theorem 2, we can get the ratio-type result as planned:

THEOREM 3. Let T be an order preserving operator in L_p $(1 \le p \le \infty)$ with $T(0) = 0$. Let $0 < \alpha < \infty$ and $f,\ g,\ h \in L_p^+$ $(g > 0)$ be such that

$$\sum_{k=0}^{n} A_k^{\alpha-1}(T^k g) > 0 \quad a.e. \text{ for all integers } n \ge 0 \text{ and } \sum_{n=0}^{\infty} A_n^{\alpha-1}(T^n g) = \infty \quad a.e.$$

Assume that there is a μ−null set $E \in \Xi$ such that for all $\omega \in \Omega - E$, (the abscissa of convergence)

$$a_\omega(\alpha; f, g) = \limsup_{n\to\infty} \frac{\log\ [\sum_{k=0}^{n} A_k^{\alpha-1}(T^k f)(\omega)]}{\log\ [\sum_{k=0}^{n} A_k^{\alpha-1}(T^k g)(\omega)]} \le 1$$

and such that for all $\omega \in \Omega-(E\cup E_0)$ $(E_0 = \{h = \infty\})$

$$G_\omega(z) = \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(g(\omega))^{-1}[A_n(\omega; g)]^z} - \frac{h(\omega)}{z-1} \quad (\mathrm{Re}(z) > 1,\ A_n(\omega; g) = \textstyle\sum_{k=0}^{n} A_k^{\alpha-1}(T^k g)(\omega))$$

has an analytic or just continuous extension (also called $G_\omega(z)$) to the closed half-plane $\{\mathrm{Re}(z) \ge 1\}$. Then

$$\lim_{n\to\infty} \frac{\sum_k^n A_k^{\alpha-1}(T^k f)(\omega)}{\sum_k^n A_k^{\alpha-1}(T^k g)(\omega)} = \frac{h(\omega)}{g(\omega)} \text{ holds for almost all } \omega \in \Omega.$$

Here we want to take notice of the following facts (which are indeed needed for the proof of Theorem 3):

(1) For each fixed $\omega \in \Omega - E$, the generalized Dirichlet series
$\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{[A_n(\omega;\, g)]^z}$ converges for any $z \in \mathbf{C}$ with $\mathrm{Re}(z) > a_\omega(\alpha; f,\, g)$.

(2) For any fixed $\omega \in \Omega - E$, it holds that

$$\int_1^\infty \frac{1}{\nu^{z+1}}\Big[\sum_{0\le k\le n,\ (g(\omega))^{-1}A_n(\omega;\, g)\le\nu} A_k^{\alpha-1}(T^k f)(\omega)\Big]\, d\nu = \frac{1}{z}\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(g(\omega))^{-1}[A_n(\omega;\, g]^z}$$

$(\mathrm{Re}(z) > 1)$

(3) Let $\omega \in \Omega - (E \cup E_0)$ and set $H_\omega(y) = e^{-y}S_\omega(e^y)$ $(y \ge 0)$, where

$$S_\omega(\nu) = \begin{cases} \sum_{n\ge0,\ (g(\omega))^{-1}A_n(\omega;\, g)\le\nu} A_n^{\alpha-1}(T^n f)(\omega) & (\nu \ge 1) \\ 0 & (\nu < 1) \end{cases}$$

Then for some $a > 0$

$$\lim_{y\to\infty} \int_{-\infty}^{ay} H_\omega(y - \tfrac{\nu}{a})K_1(\nu)\, d\nu = h(\omega)\int_{-\infty}^{\infty} K_1(\nu)\, d\nu$$

($K_\varrho(t) = \frac{\sin^2(\varrho t)}{\varrho t^2}$ $(\varrho > 0)$ is the Fejér kernel).

Finally we make mention of Wittmann's result ([6]) because of its importance.

THE WITTMANN THEOREM (1991). Let T be order preserving, integral preserving ($\int Tf\, d\mu = \int f\, d\mu,\ f \in L_1$), positively homogeneous and L_∞–nonexpansive in L_1. Let $f \in L_1$ and define a process $\{S_n f : n \ge 0\}$ inductively by $S_0 f = f,\ S_{n+1}f = f + T(S_n f)$ $(n \ge 0)$. Then $\frac{1}{n+1}S_n f$ converges a.e. for all $f \in L_1$ and moreover, $\frac{1}{n+1}S_n f$ converges strongly in L_p $(1 \le p < \infty)$.

The convergence properties of Wittmann's nonlinear averages $\frac{1}{n+1}S_n f$ are deeper than those of the Cesàro averages $C_n^{(1)}[T]f$. Moreover, Garsia's proof of Hopf's maximal ergodic theorem easily generalizes to the nonlinear averages $\frac{1}{n+1}S_n f$ (but not to $C_n^{(1)}[T]f$). In particular, the above Wittmann theorem can not be expected for the Cesàro-type averages in the nonlinear situation (see [4], [5]). If the operator T is linear in the Wittmann theorem, then $\|T\|_1 \le 1$, $\|T\|_\infty \le 1$ and $\frac{1}{n+1}S_n f = C_n^{(1)}[T]f,\ n \ge 0$. (In this case, T is positive). Thus by the Dunford-Schwartz theorem we see that $\frac{1}{n+1}S_n f = C_n^{(1)}[T]f$ converges a.e. for all $f \in L_p$ $(1 \le p < \infty)$.

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