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<td>GOTO, Shun'ichi</td>
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Kyoto University
UNIQUENESS AND EXISTENCE FOR SPIRAL CRYSTAL GROWTH

北に道教育大学 札幌校 後藤俊一 (GOTO, Shun'ichi)
Hokkaido University of Education

In [2] Ohtsuka studied on a crystal growth of spirals and proposed us to use a level set method. Since the conventional level set method (see [1]) could not express spiral curves having the orientation, he modified the conventional method by using a sheet structure function.

Let $\Omega$ be a bounded domain of $\mathbb{R}^2$ with the smooth boundary and let $B_{\rho_j}(a_j)$ be $N$ screw dislocations in $\Omega$, which are disks small enough with $a_j \in \Omega$ and $\rho_j > 0$ so that $\overline{B_{\rho_j}(a_j)} \subset \Omega$ and $B_{\rho_j}(a_j) \cap B_{\rho_k}(a_k) = \emptyset$ if $j \neq k$. We denote

$$ W = \Omega \setminus \left( \bigcup_{j=1}^{N} \overline{B_{\rho_j}(a_j)} \right). $$

The level set equation for his spiral crystal growth on $W$ is

$$ u_t - |\nabla(u - \theta)| \left\{ \text{div} \frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} + C \right\} = 0 \quad \text{in } W, $$

with the boundary condition of Neumann type

$$ \langle \nu, \nabla(u - \theta) \rangle = 0 \quad \text{on } \partial W. $$

Here $C$ is a constant, $\nu$ is the unit normal vector of $\partial W$ and $\theta(x) = \sum_{j=1}^{N} m_j \arg(x - a_j)$ for $m_j \in \mathbb{Z} \setminus \{0\}$. We note that $\theta(x)$ is a multi-valued function, but $\nabla \theta$ is single-valued. When $\Gamma_t$ is the spiral curve, it must be defined

$$ \Gamma_t = \{ x \in \overline{W} : u(t, x) - \theta(x) \equiv 0 \mod 2\pi m \mathbb{Z} \}. $$

Here $m$ is the greatest common divisor of $|m_j|$. Ohtsuka proved the following results.
**Comparison Theorem.** Let $u$ and $v$ be a viscosity subsolution and a supersolution of (1) and (2), respectively. If $u^*(0,\cdot) \leq v_*(0,\cdot)$, then we have $u^*(t,x) \leq v_*(t,x)$ for any $t > 0$.

**Existence Theorem.** For any given $u_0 \in C(\overline{W})$ there exists a unique global-in-time viscosity solution $u \in C([0,\infty) \times \overline{W})$ of (1) and (2) with initial data $u(0,\cdot) = u_0$.

This note is a short remark for the Ohtsuka's theory, that is, we would like to consider the uniqueness of $\Gamma_t$. It means that, for a given initial spiral $\Gamma_0$, we choose $u_0$ an initial function satisfying

$$\Gamma_0 = \{x \in \overline{W} : u_0(x) - \theta(x) \equiv 0 \mod 2\pi m \mathbb{Z}\},$$

the Existence Theorem says that there exists a unique solution $u$, but we can choose $v_0$ an another initial function satisfying

$$\Gamma_0 = \{x \in \overline{W} : v_0(x) - \theta(x) \equiv 0 \mod 2\pi m \mathbb{Z}\}$$

and we get a unique solution $v$. Our question is

$$\{x \in \overline{W} : u(t,x) - \theta(x) \equiv 0 \mod 2\pi m \mathbb{Z}\}$$

and

$$\{x \in \overline{W} : v(t,x) - \theta(x) \equiv 0 \mod 2\pi m \mathbb{Z}\}$$

are tracing the same spiral curve?

The paper [1] solved this uniqueness problem for the case of closed curves. The key step is to construct the order changing function satisfying $u_0(x) \leq G(v_0(x))$, when, generally, $u_0$ and $v_0$ are not maked order each other. Since $G$ is nondecreasing, if $v(t,x)$ is a viscosity supersolution, then $G(v(t,x))$ is also a viscosity supersolution. By using the Comparison Theorem we see that $u(t,x) \leq G(v(t,x))$, which leads us to compair the level sets of $u$ and $v$.

We try to extend this key idea to the spiral case. Applying the Ohtsuka's method in [2] we first introduce the covering space of $\overline{W}$ like

$$\mathfrak{X} = \{(x,\xi) \in \overline{W} \times \mathbb{R}^N : \xi = (\xi_1, \cdots, \xi_N), \ (\cos \xi_j, \sin \xi_j) = \frac{x - a_j}{|x - a_j|}\}$$
and assume that
\[
\left\{ (x, \xi) \in \mathcal{X} : u_0(x) - \sum_{j=1}^{N} m_j \xi_j > 0 \right\} = \left\{ (x, \xi) \in \mathcal{X} : v_0(x) - \sum_{j=1}^{N} m_j \xi_j > 0 \right\}.
\]

We construct an order changing function $G$ with
\[
u_0(x) - \sum_{j=1}^{N} m_j \xi_j \leq G \left( v_0(x) - \sum_{j=1}^{N} m_j \xi_j \right) \text{ for } (x, \xi) \in \mathcal{X}.
\]

The important properties for $G$ are nondecreasing and satisfying the periodical condition
\[(\#) G(s) = G(s + 2\pi m_j) - 2\pi m_j. \] Basically, $G$ is modified from
\[
G_1(s) = \sup \left\{ (\tilde{u}_0(y, \eta))_+ : (y, \eta) \in \mathcal{X}, \tilde{v}_0(y, \eta) \leq s \right\}.
\]

Here $\tilde{u}_0(y, \eta) = u_0(y) - \sum_{j=1}^{N} m_j \eta_j$, $\tilde{v}_0(y, \eta) = v_0(y) - \sum_{j=1}^{N} m_j \eta_j$ and $(a)_+ = \max\{a, 0\}$.

Finally, we obtain

**Invariance Lemma.** Let $\nu$ be a viscosity supersolution with initial data $\nu(0, \cdot) = \nu_0$ and define
\[
(3) \quad w(t, x) = G(v(t, x) - \theta(x)) + \theta(x)
\]
in the sense of some meaning in the covering space (because $\theta(x)$ is multi-valued). Then we have $w$ is a viscosity supersolution with $w(0, \cdot) = w_0$.

The meaning of the definition (3) is the following: We denote that
\[
\mathcal{L} = \bigcup_{j=1}^{N} \mathcal{L}_j, \quad \mathcal{L}_j = \left\{ x \in \overline{W} : \frac{x - a_j}{|x - a_j|} = (-1, 0) \right\}
\]
and $\Theta_j(x) = \text{Arg}(x - a_j)$ is the principal value of the argument which is a function from $\overline{W} \setminus \mathcal{L}_j$ to $(-\pi, \pi)$. Then, $\Theta(x) = \sum_{j=1}^{N} m_j \Theta_j(x)$ is a single-valued function with a jump discontinuity on $\mathcal{L}$. However, since $G$ is periodic like $(\#)$, we see that
\[
g(x) = \begin{cases} G(f(x) - \Theta(x)) + \Theta(x) & \text{if } x \in \overline{W} \setminus \mathcal{L}, \\ \lim_{y \to x} \{G(f(y) - \Theta(y)) + \Theta(y)\} & \text{if } x \in \mathcal{L} \end{cases}
\]
is continuous on $\mathfrak{L}$.

We must discuss here about the construction of an initial function $u_0$ for a given $\Gamma_0$, which gives us the existence result on the growth of $\Gamma_t$. The author hopes it will be stated in a forthcoming paper.

This research was started by Maki Nakagawa as the master's thesis [3] in a simple case, which is supervised by the author. After that, Takeshi Ohtsuka and the author have revised and completed it.

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